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Three-to-one internal resonances in a general cubic non-linear continuous system

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Abstract

A general continuous system with an arbitrary cubic non-linearity is considered. The non-linearity is expressed in terms of an arbitrary cubic operator. Three-to-one internal resonance case is considered. A general approximate solution is presented for the system. Amplitude and phase modulation equations are derived. Steady state solutions and their stability are discussed in the general sense. The sufficiency condition for such resonances to occur is derived. Finally the algorithm is applied to a beam resting on a non-linear elastic foundation.

1. Introduction

Vibrations of continuous systems are modelled in terms of partial differential equations. Such systems contain different types of non-linearities. One common non-linearity observed is a cubic non-linearity. Systems with such non-linearities possess some characteristic features. Three-to-one internal resonances occur frequently and energy is easily transferred from the excited mode to the specific mode with 3:1 internal resonance.

A fairly general treatment of the problem is considered by using the formerly developed operator notation. The linear as well as the non-linear part of the equation of motion are represented in terms of arbitrary operators. Damping and harmonic external excitation are added to the system. An approximate solution of the problem is found using the method of multiple scales, a perturbation technique. A three-to-one internal resonance case is treated in the analysis. Amplitude and phase modulation equations are derived with coefficients defined in a general sense in terms of integrals of the cubic non-linearity operator. Steady state solutions of the system as

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well as their stability are examined. The algorithm developed is applied to a beam problem resting on a non-linear elastic foundation. The fundamental mode is externally excited and due to the 3:1 internal resonance, some of the energy is transferred to the third mode. Force responses and frequency responses are plotted for these modes of vibrations.

The concept of analyzing non-linear vibrations of continuous systems using a general operator notation was presented in Ref. [1] for the first time. The aim of that study was to compare the advantages of direct-perturbation method to the discretization perturbation method in a general equation having arbitrary quadratic and cubic non-linearities. The finite mode analysis in Ref. [1] was generalized to an infinite mode analysis in Ref. [2]. The primary resonances were considered in those analysis. Later, subharmonic, superharmonic and combination type of resonances were examined using the same model [3]. The general operator notation was also used to compare different versions of method of multiple scales [4]. Primary resonances of the odd non-linearity models (cubic, quintic) were also considered in Refs. [5,6]. For a single partial differential equation, 3:1 internal resonances are considered for the first time in this study using the arbitrary non-linear operators. For coupled systems, internal resonances were considered assuming an interaction between the modes of different equations [7–9]. The general operator notation developed and used by Pakdemirli and co-workers [1–9] has been adopted by others also (see Refs. [10,11] for example).

The main concern here is to solve a fairly general system with cubic non-linearity having 3:1 internal resonances only. For applying mode truncations, one should first determine all internal resonances of the specific problem and be sure that the specific natural frequencies selected would not yield any other resonances than the ones assumed. The beam vibration problem considered in this work is given as an application of the general algorithm developed and hence not studied in detail. Only a specific parameter value for which 3:1 internal resonance occurs is treated. For a more detailed analysis on beam vibrations having cubic non-linearities and internal resonances, the reader is referred to Refs. [11–15] for example.

2. Equation of motion

The general model considered is as follows

$$\ddot{w} + \hat{\mu}\dot{w} + \mathbf{L}(w) + \varepsilon\mathbf{C}(w, w, w,) = \hat{F}\cos\Omega t, \tag{1}$$

where w is the response, $\mathbf{L}$ is an arbitrary spatial linear operator and $\mathbf{C}(w, w, w)$ is an arbitrary spatial cubic non-linear operator. In continuous systems (string, cable, beam etc.), if stretching effects are taken into consideration, one common cubic non-linearity is $w'' \int_D w'^2 \, dx$ where prime denotes differentiation with respect to the spatial variable. ε is a small parameter. Under the primary resonance assumption, damping and external excitation amplitude are ordered such that they counter the effect of non-linearity

$$\hat{\mu} = \varepsilon\mu, \quad \hat{F} = \varepsilon F. \tag{2}$$

The boundary conditions for Eq. (1) are assumed to be linear and homogenous

$$\mathbf{B}_1(w) = 0 \text{ at } x = 0, \quad \mathbf{B}_2(w) = 0 \text{ at } x = 1, \tag{3}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are arbitrary spatial linear operators. The cubic non-linearity possesses the property of being multilinear such that

$$\begin{aligned}\mathbf{C}(c_1w_1+c_2w_2,c_3w_3+c_4w_4,c_5w_5+c_6w_6)&=c_1c_3c_5\mathbf{C}(w_1,w_3,w_5)+c_1c_3c_6\mathbf{C}(w_1,w_3,w_6)\\&+c_1c_4c_5\mathbf{C}(w_1,w_4,w_5)+c_1c_4c_6\mathbf{C}(w_1,w_4,w_6)+c_2c_3c_5\mathbf{C}(w_2,w_3,w_5)+c_2c_3c_6\mathbf{C}(w_2,w_3,w_6)\\&+c_2c_4c_5\mathbf{C}(w_2,w_4,w_5)+c_2c_4c_6\mathbf{C}(w_2,w_4,w_6),\end{aligned} \tag{4}$$

$$\{\mathbf{C}(w_1,w_2,w_3)\neq\mathbf{C}(w_1,w_3,w_2)\neq\mathbf{C}(w_2,w_1,w_3)\neq\mathbf{C}(w_2,w_3,w_1)\neq\mathbf{C}(w_3,w_1,w_2)\neq\mathbf{C}(w_3,w_2,w_1)\quad\text{in general}\},$$

where c_i are arbitrary constants or time dependent coefficients.

Although the model is fairly general it has some limitations: viscoelastic effects, non-linear inertial effects as well as gyroscopic effects are excluded. Non-linear boundary conditions and parametric excitations are also excluded. The linear operator is assumed to be self-adjoint.

3. Perturbation analysis

Eqs. (1)–(3) will be solved approximately by the method of multiple scales [16,17]. The case of primary resonances of the external excitation and three-to-one internal resonances of the system will be considered. The approximate solution is

$$w(x,t;\varepsilon)=w_0(x,T_0,T_1)+\varepsilon w_1(x,T_0,T_1)+\cdots, \tag{5}$$

where $T_0=t$ is the usual fast time scale and $T_1=\varepsilon t$ is the slow time scale in the method of multiple scales. Derivatives are expressed in terms of the new time variables

$$\mathrm{d}/\mathrm{d}t=D_0+\varepsilon D_1+\cdots\quad \mathrm{d}^2/\mathrm{d}t^2=D_0^2+2\varepsilon D_0D_1+\cdots, \tag{6}$$

where $D_k=\partial/\partial T_k$. Substituting all into the original equations and separating at each order of ε yields

$$O(1): D_0^2w_0+\mathbf{L}(w_0)=0\quad \mathbf{B}_1(w_0)=0 \text{ at } x=0,\quad \mathbf{B}_2(w_0)=0 \text{ at } x=1, \tag{7}$$

$$\begin{aligned}&O(\varepsilon): D_0^2w_1+\mathbf{L}(w_1)=-2D_0D_1w_0-\mu D_0w_0-\mathbf{C}(w_0,w_0,w_0)+F\cos\Omega T_0\\&\mathbf{B}_1(w_1)=0 \text{ at } x=0,\quad \mathbf{B}_2(w_1)=0 \text{ at } x=1.\end{aligned} \tag{8}$$

At the first order, the solution can be expressed as follows

$$w_0(x,T_0,T_1)=\sum_{m=1}^{\infty}(A_m(T_1)\mathrm{e}^{\mathrm{i}\omega_mT_0}+cc)Y_m(x), \tag{9}$$

where cc stands for complex conjugates of the preceding terms. Inserting this into the $O(1)$ equation yields the boundary value problems

$$\mathbf{L}(Y_m)-\omega_m^2Y_m=0,\quad \mathbf{B}_1(Y_m)=0 \text{ at } x=0,\quad \mathbf{B}_2(Y_m)=0 \text{ at } x=1,\quad m=1,2,\ldots, \tag{10}$$

where ω_m are the natural frequencies and Y_m are the mode shapes of the problem. Substituting (9) to the right-hand side of $O(\varepsilon)$ equation and arranging, one has

$$
\begin{aligned}
D_0^2 w_1 + \mathbf{L}(w_1) = & -\sum_{n=1}^{\infty} \mathrm{i}\omega_n (2D_1 A_n + \mu A_n) \mathrm{e}^{\mathrm{i}\omega_n T_0} Y_n \\
& - \sum_{m,p,q=1}^{\infty} \{A_m A_p A_q \mathrm{e}^{\mathrm{i}(\omega_m+\omega_p+\omega_q)T_0} \mathbf{C}(Y_m, Y_p, Y_q) \\
& + A_m A_p \bar{A}_q \mathrm{e}^{\mathrm{i}(\omega_m+\omega_p-\omega_q)T_0} [\mathbf{C}(Y_m, Y_p, Y_q) + \mathbf{C}(Y_m, Y_q, Y_p) \\
& + \mathbf{C}(Y_q, Y_p, Y_m)]\} + \frac{F}{2} \mathrm{e}^{\mathrm{i}\Omega T_0 + CC},
\end{aligned} \tag{11}
$$

$$\mathbf{B}_1(w_1) = 0 \text{ at } x = 0, \quad \mathbf{B}_2(w_1) = 0 \text{ at } x = 1.$$

It is clear that internal resonances occur when

$$\omega_m + \omega_p + \omega_q \approx \omega_n, \quad \omega_m + \omega_p - \omega_q \approx \pm\omega_n. \tag{12}$$

Hence, up to four modes can interact with each other in special circumstances. For a special choice of parameter, the interactions of the first four modes in a non-linear beam problem have been investigated in Ref. [14]. In the absence of resonances given in Eq. (12), the solvability condition would yield

$$2D_1 A_n + \mu A_n = 0, \tag{13}$$

or $A_n = A_{on} \mathrm{e}^{-\mu T_1/2}$. It is clear that the modes that are not directly excited by an external excitation, or indirectly excited through an internal resonance would decay in time (see Ref. [17] p. 462 for a detailed discussion).

In the following part of the analysis, it will be assumed that the system possesses two mode interactions through a three-to-one internal resonance only. Under this assumption, the mode that is directly excited (ω_1) and indirectly excited through internal resonance (ω_2) will be considered only. Note that the numbers are assigned to the modes arbitrarily and do not necessarily represent the first and second modes of vibration. Eq. (11) takes the special form

$$
\begin{aligned}
& D_0^2 w_1 + \mathbf{L}(w_1) = -\mathrm{i}\omega_1 (2D_1 A_1 + \mu A_1) \mathrm{e}^{\mathrm{i}\omega_1 T_0} \; Y_1 - \mathrm{i}\omega_2 (2D_1 A_2 + \mu A_2) \mathrm{e}^{\mathrm{i}\omega_2 T_0} Y_2 \\
& - \left(A_1^3 \mathrm{e}^{3\mathrm{i}\omega_1 T_0} + 3A_1^2 \bar{A}_1 \mathrm{e}^{\mathrm{i}\omega_1 T_0}\right) \mathbf{C}(Y_1, Y_1, Y_1) \\
& - \left(A_2 \bar{A}_1^2 \, \mathrm{e}^{\mathrm{i}(\omega_2 - 2\omega_1)T_0} + A_1^2 A_2 \, \mathrm{e}^{\mathrm{i}(\omega_2 + 2\omega_1)T_0} + 2A_1 \bar{A}_1 A_2 \, \mathrm{e}^{\mathrm{i}\omega_2 T_0}\right) \\
& \times [\mathbf{C}(Y_1, Y_1, Y_2) + \mathbf{C}(Y_1, Y_2, Y_1) + \mathbf{C}(Y_2, Y_1, Y_1)] \\
& - \left(A_1 A_2^2 \mathrm{e}^{\mathrm{i}(\omega_1 + 2\omega_2)T_0} + A_2^2 \bar{A}_1 \mathrm{e}^{\mathrm{i}(2\omega_2 - \omega_1)T_0} + 2A_1 A_2 \bar{A}_2 \mathrm{e}^{\mathrm{i}\omega_1 T_0}\right) \\
& \times [\mathbf{C}(Y_1, Y_2, Y_2) + \mathbf{C}(Y_2, Y_1, Y_2) + \mathbf{C}(Y_2, Y_2, Y_1)] \\
& - \left(A_2^3 \mathrm{e}^{3\mathrm{i}\omega_2 T_0} + 3A_2^2 \bar{A}_2 \mathrm{e}^{\mathrm{i}\omega_2 T_0}\right) \; \mathbf{C}(Y_2, Y_2, Y_2) + \frac{F}{2} \mathrm{e}^{\mathrm{i}\Omega T_0} + cc,
\end{aligned} \tag{14}
$$

$$\mathbf{B}_1(w_1) = 0 \;\text{ at }\; x = 0, \;\; \mathbf{B}_2(w_1) = 0 \;\text{ at }\; x = 1$$

The following detuning parameters of $O(1)$ are defined for primary resonances of the external excitation and 3:1 internal resonances between the natural frequencies

$$\Omega = \omega_1 + \varepsilon\sigma, \quad \omega = 3\omega_1 + \varepsilon\rho. \tag{15}$$

The solution for this order is assumed to be of the form

$$w_1(x, T_0, T_1) = \phi_1(x, T_1)e^{i\omega_1 T_0} + \phi_2(x, T_1)e^{i\omega_2 T_0} + W(x, T_0, T_1) + cc, \tag{16}$$

where $W(x, T_0, T_1)$ corresponds to the solution of non-secular terms. Substituting (16) into Eq. (14), using (15) yields

$$\begin{aligned}\mathbf{L}(\phi_1)\text{-}\omega_1^2\phi_1 = &-i\omega_1(2D_1A_1 + \mu A_1)Y_1 - 3A_1^2\bar{A}_1 \quad \mathbf{C}(Y_1, Y_1, Y_1)\\ &- A_2\bar{A}_1^2 e^{i\rho T_1}[\mathbf{C}(Y_1, Y_1, Y_2) + \mathbf{C}(Y_1, Y_2, Y_1) + \mathbf{C}(Y_2, Y_1, Y_1)]\\ &- 2A_1A_2\bar{A}_2 \ [\mathbf{C}(Y_1, Y_2, Y_2) + \mathbf{C}(Y_2, Y_1, Y_2) + \mathbf{C}(Y_2, Y_2, Y_1)] + \frac{F}{2}e^{i\sigma T_1},\end{aligned} \tag{17}$$

$$\mathbf{B}_1(\phi_1) = 0 \text{ at } x = 0, \quad \mathbf{B}_2(\phi_1) = 0 \text{ at } x = 1,$$

$$\begin{aligned}\mathbf{L}(\phi_2)\text{-}\omega_2^2\phi_2 = &-i\omega_2(2D_1A_2 + \mu A_2)Y_2 - A_1^3e^{-i\rho T_1}\mathbf{C}(Y_1, Y_1, Y_1)\\ &- 2A_1\bar{A}_1A_2[\mathbf{C}(Y_1, Y_1, Y_2) + \mathbf{C}(Y_1, Y_2, Y_1) + \mathbf{C}(Y_2, Y_1, Y_1)]\\ &- 3A_2^2\bar{A}_2\mathbf{C}(Y_2, Y_2, Y_2),\end{aligned} \tag{18}$$

$$\mathbf{B}_1(\phi_2) = 0 \text{ at } x = 0, \quad \mathbf{B}_2(\phi_2) = 0 \text{ at } x = 1$$

for secular terms. Since the homogeneous part of Eqs. (17) and (18) possesses non-trivial solutions, the non-homogeneous equations have a solution only if a solvability condition is satisfied [16]. For the present problem the solvability conditions are

$$i\omega_1(2D_1A_1 + \mu A_1) + \alpha_1A_1^2\bar{A}_1 + \alpha_2A_2\bar{A}_1^2e^{i\rho T_1} + \alpha_3A_1A_2\bar{A}_2 - \frac{1}{2}fe^{i\sigma T_1} = 0, \tag{19}$$

$$i\omega_2(2D_1A_2 + \mu A_2) + \alpha_4A_1^3\,e^{-i\rho T_1} + \alpha_5A_1\bar{A}_1A_2 + \alpha_6A_2^2\bar{A}_2 = 0, \tag{20}$$

where

$$\alpha_1 = 3\int_0^1 Y_1\,\mathbf{C}(Y_1, Y_1, Y_1)\,dx, \tag{21}$$

$$\alpha_2 = \int_0^1 Y_1[\mathbf{C}(Y_1, Y_1, Y_2) + \mathbf{C}(Y_1, Y_2, Y_1) + \mathbf{C}(Y_2, Y_1, Y_1)]\,dx, \tag{22}$$

$$\alpha_3 = 2\int_0^1 Y_1[\mathbf{C}(Y_1, Y_2, Y_2) + \mathbf{C}(Y_2, Y_1, Y_2) + \mathbf{C}(Y_2, Y_2, Y_1)]\,dx, \tag{23}$$

$$\alpha_4 = \int_0^1 Y_2\mathbf{C}(Y_1, Y_1, Y_1)\,dx, \tag{24}$$

$$\alpha_5 = 2\int_0^1 Y_2\,[\mathbf{C}(Y_1, Y_1, Y_2) + \mathbf{C}(Y_1, Y_2, Y_1) + \mathbf{C}(Y_2, Y_1, Y_1)]\,dx, \tag{25}$$

$$\alpha_6 = 3\int_0^1 Y_2\,[\mathbf{C}(Y_2, Y_2, Y_2)\,\mathrm{d}x, \tag{26}$$

$$f = \int_0^1 F Y_1\,\mathrm{d}x. \tag{27}$$

Note that the constants of the complex amplitude modulation equations are defined in terms of arbitrary cubic operators in a general form. For the specific cubic operator and mode shapes, the integrals can be evaluated either analytically or for more involved problems numerically. For the mode shapes, $\int_0^1 Y_1^2\,\mathrm{d}x = 1$, $\int_0^1 Y_2^2\,\mathrm{d}x = 1$ normalization conditions are applied.

4. Approximate solutions and their stability

In this section, the real amplitude and phase modulation equations will be developed from the complex amplitude modulation equations. The approximate solution will be generated. The steady state solutions of amplitude–phase modulation equations and their stability will be discussed.

Substituting the polar forms

$$A_1 = \tfrac{1}{2}a_1(T_1)\mathrm{e}^{\mathrm{i}\beta_1(T_1)}, \quad A_2 = \tfrac{1}{2}a_2(T_1)\mathrm{e}^{\mathrm{i}\beta_2(T_1)}, \tag{28}$$

into Eqs. (19) and (20), separating real and imaginary parts, one has

$$\begin{aligned}
&\omega_1 a_1' + \frac{\mu}{2}\omega_1 a_1 + \frac{\alpha_2}{8}a_1^2 a_2 \sin\lambda - \frac{1}{2}f\sin\gamma = 0,\\
&-a_1\omega_1\beta_1' + \frac{\alpha_1}{8}a_1^3 + \frac{\alpha_2}{8}a_1^2 a_2\cos\lambda + \frac{\alpha_3}{8}a_1 a_2^2 - \frac{1}{2}f\cos\gamma = 0,\\
&\omega_2 a_2' + \frac{\mu}{2}\omega_2 a_2 - \frac{\alpha_4}{8}a_1^3\sin\lambda = 0,\\
&-a_2\omega_2\beta_2' + \frac{\alpha_4}{8}a_1^3\cos\lambda + \frac{\alpha_5}{8}a_1^2 a_2 + \frac{\alpha_6}{8}a_2^3 = 0,
\end{aligned} \tag{29}$$

where

$$\gamma = \sigma T_1 - \beta_1, \quad \lambda = \beta_2 - 3\beta_1 + \rho T_1. \tag{30}$$

Steady state solutions correspond to $a_1' = a_2' = \gamma' = \lambda' = 0$ or substituting $a_1' = a_2' = 0$, $\beta_1' = \sigma, \beta_2' = 3\sigma - \rho$, one has

$$\begin{aligned}
&\frac{\mu}{2}\omega_1 a_1 + \frac{\alpha_2}{8}a_1^2 a_2\sin\lambda - \frac{1}{2}f\sin\gamma = 0,\\
&-a_1\omega_1\sigma + \frac{\alpha_1}{8}a_1^3 + \frac{\alpha_2}{8}a_1^2 a_2\cos\lambda + \frac{\alpha_3}{8}a_1 a_2^2 - \frac{1}{2}f\cos\gamma = 0,\\
&\frac{\mu}{2}\omega_2 a_2 - \frac{\alpha_4}{8}a_1^3\sin\lambda = 0,\\
&-a_2\omega_2(3\sigma - \rho) + \frac{\alpha_4}{8}a_1^3\cos\lambda + \frac{\alpha_5}{8}a_1^2 a_2 + \frac{\alpha_6}{8}a_2^3 = 0.
\end{aligned} \tag{31}$$

From $(31)_3$ above, for 3:1 internal resonance to occur, in addition to the condition of one natural frequency being approximately equal to the three times the other frequency (necessary condition), the following sufficiency condition should also hold

$$\alpha_4 \neq 0 \quad \text{or} \quad \int_0^1 Y_2\, \mathbf{C}(Y_1, Y_1, Y_1)\, \mathrm{d}x \neq 0. \tag{32}$$

This general condition has been derived for the first time by the authors. It is one of the proofs of advantages gained by attacking the general cubic non-linearity problem, rather than working on special cases.

To determine the stability of the system, Eqs. (29) are rewritten by eliminating β_1' and β_2'

$$\begin{aligned}
a_1' &= -\frac{\mu}{2}a_1 - \frac{\alpha_2}{8\omega_1}a_1^2 a_2 \sin\lambda + \frac{f}{2\omega_1}\sin\gamma = F_1(a_1, a_2, \gamma, \lambda),\\
a_2' &= -\frac{\mu}{2}a_2 + \frac{\alpha_4}{8\omega_2}a_1^3 \sin\lambda = F_2(a_1, a_2, \gamma, \lambda),\\
\gamma' &= \sigma - \frac{\alpha_1}{8\omega_1}a_1^2 - \frac{\alpha_2}{8\omega_1}a_1 a_2 \cos\lambda - \frac{\alpha_3}{8\omega_1}a_2^2 + \frac{f}{2a_1\omega_1}\cos\gamma = F_3(a_1, a_2, \gamma, \lambda),\\
\lambda' &= \rho + \left(\frac{\alpha_4 a_1^3}{8a_2\omega_2} - \frac{3\alpha_2 a_1 a_2}{8\omega_1}\right)\cos\lambda + \left(\frac{\alpha_5}{8\omega_2} - \frac{3\alpha_1}{8\omega_1}\right)a_1^2\\
&\quad + \left(\frac{\alpha_6}{8\omega_2} - \frac{3\alpha_3}{8\omega_1}\right)a_2^2 + \frac{3f}{2a_1\omega_1}\cos\gamma = F_4(a_1, a_2, \gamma, \lambda).
\end{aligned} \tag{33}$$

The Jacobian matrix is constructed to determine the stability of fixed points

$$\left.\begin{bmatrix}
\dfrac{\partial F_1}{\partial a_1} & \dfrac{\partial F_1}{\partial a_2} & \dfrac{\partial F_1}{\partial \gamma} & \dfrac{\partial F_1}{\partial \lambda}\\
\dfrac{\partial F_2}{\partial a_1} & \dfrac{\partial F_2}{\partial a_2} & \dfrac{\partial F_2}{\partial \gamma} & \dfrac{\partial F_2}{\partial \lambda}\\
\dfrac{\partial F_3}{\partial a_1} & \dfrac{\partial F_3}{\partial a_2} & \dfrac{\partial F_3}{\partial \gamma} & \dfrac{\partial F_3}{\partial \lambda}\\
\dfrac{\partial F_4}{\partial a_1} & \dfrac{\partial F_4}{\partial a_2} & \dfrac{\partial F_4}{\partial \gamma} & \dfrac{\partial F_4}{\partial \lambda}
\end{bmatrix}\right|_{\substack{a_1 = a_{10}\\ a_2 = a_{20}\\ \gamma = \gamma_0\\ \lambda = \lambda_0}}. \tag{34}$$

By evaluating the eigenvalues of the Jacobian matrix, stability is determined. Eigenvalues should not have positive real parts to maintain stability.

The approximate solution for the problem is

$$w(x, t, \varepsilon) = a_1 \cos(\Omega t - \gamma) Y_1(x) + a_2 \cos(3\Omega t + \lambda - 3\gamma) Y_2(x) + O(\varepsilon), \tag{35}$$

where a_1, a_2, γ and λ are governed by Eqs. (33).

5. Application to a beam problem

The algorithm will be applied to a simply supported Euler–Bernoulli beam resting on a non-linear elastic foundation (Fig. 1). Only a special case of 3:1 internal resonance will be considered here. For further reading on beam vibrations with cubic non-linearities and internal resonances, see Refs. [11–15]

$$\ddot{w} + \varepsilon\mu\dot{w} + w^{iv} + k_1 w + \varepsilon k_2 w^3 = \varepsilon F \cos\Omega t,$$
$$w(0,t) = w''(0,t) = w(1,t) = w''(1,t) = 0. \quad (36)$$

The operators are

$$\mathbf{L}(w) = w^{iv} + k_1 w \quad \mathbf{C}(w,w,w) = k_2 w^3. \quad (37)$$

The eigenvalue problems in Eq. (10) yield

$$Y_n^{iv} - \beta_n^4 Y_n = 0 \quad Y_n(0) = Y_n''(0) = Y_n(1) = Y_n''(1) = 0, \quad (38)$$

where $\beta_n^4 = \omega_n^2 - k_1, \omega_n$ being the natural frequency. Solving the problem

$$Y_n(x) = \sqrt{2}\sin n\pi x, \quad \omega_n = \sqrt{n^4\pi^4 + k_1}, \quad n = 1, 2, 3, \ldots . \quad (39)$$

By exciting the fundamental mode, the second natural frequency can not be excited through a 3:1 internal resonance since $\alpha_4 = 0$ (see Eq. 32). Hence an interaction of the fundamental mode with the third mode is considered. For $k_1 = 870$, $\omega_1 = 31.1$ and $\omega_2 = 93.6$. The mode shapes are $Y_1 = \sqrt{2}\sin\pi x$ and $Y_2 = \sqrt{2}\sin 3\pi x$ (subscript 1 refers to the externally excited mode and 2 to the other mode excited internally). Note that for the specific choice of $k_1 = 693\pi^4/152$ or in a close neighborhood, the first four natural frequencies were interacted through an internal resonance [14]. For the special choice, however, since the first and third modes are finely tuned, only a 3:1 internal resonance is possible.

The next step is to calculate the coefficients of amplitude–phase modulation equations by evaluating the integrals for the specific mode shapes and operators (i.e., Eqs. (21)–(26))

$$\alpha_1 = \frac{9}{2}k_2, \quad \alpha_2 = -\frac{3}{2}k_2, \quad \alpha_3 = 6k_2, \quad \alpha_4 = -\frac{k_2}{2}, \quad \alpha_5 = 6k_2, \quad \alpha_6 = \frac{9}{2}k_2. \quad (40)$$

Fixed points are the roots of Eq. (31) and their stability are governed by Eqs. (33) and (34). The roots are found by the solve subroutine of Matlab numerically.

In Fig. 2, the force response graph is given for parameter values $\omega_1 = 31.1$, $\omega_2 = 93.6$, $\rho = 0.3$, $\mu = 0.01$, $k_2 = 1$ and $\sigma = 0.04$. The third mode can be activated for $f \geqslant 0.5$. Solid lines correspond to stable solutions and dashed lines correspond to unstable solutions. The frequency response graphs of each mode are given in Figs. 3a and b for $f = 1$, σ varying and all other parameters

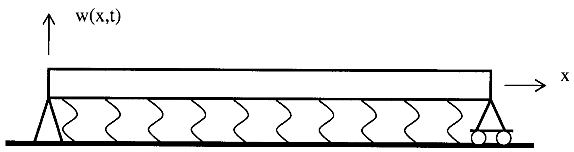

Fig. 1. An Euler–Bernoulli beam resting on a non-linear elastic foundation.

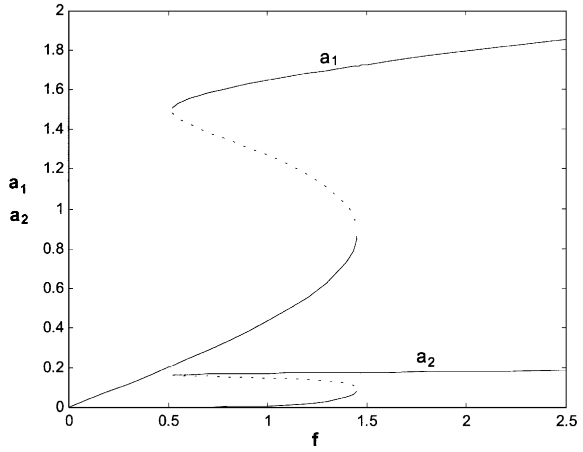

Fig. 2. Force response curves for the externally excited (a_1) and internally excited (a_2) modes (solid: stable; dashed: unstable solutions).

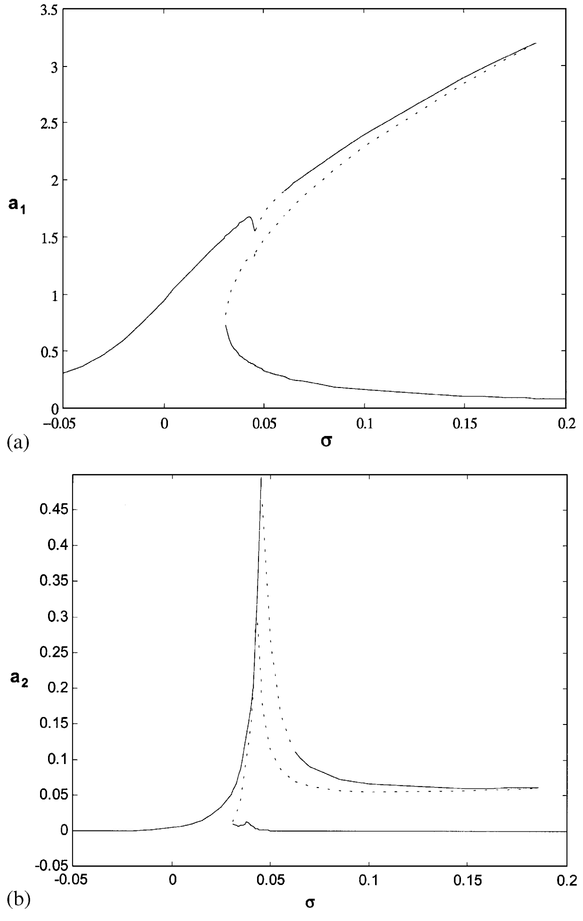

Fig. 3. (a) Frequency response curve for the externally excited mode. (b) Frequency response curve for the mode excited with a 3:1 internal resonance.

remaining the same. The dynamics of the problem is complicated with stable and unstable solutions co-existing for a fixed external frequency.

The approximate solution for the problem can be written from Eqs. (35) and (39)

$$w(x,t,\varepsilon) = a_1 \cos(\Omega t - \gamma)\sqrt{2}\sin \pi x + a_2 \cos(3\Omega t + \lambda - 3\gamma)\sqrt{2}\sin 3\pi x + O(\varepsilon). \tag{41}$$

A critical question may be the validity of this solution on large time scales. Boertjens and Horssen [18] investigated a quadratic non-linearity beam problem and showed that the two-term approximations are of order ε approximations of the exact solution on time scales of order $1/\varepsilon$. The system they have considered was a conservative system without damping and their result was indeed critical for conservative systems. For systems with damping, the transient solution would die out and the present two-term expansion would produce results compatible with experiments.

6. Concluding remarks

A cubic non-linearity system is expressed in a general form. Three-to-one internal resonances are investigated for the system. Approximate solutions are derived. The amplitude and phase modulations of the solution are derived. The coefficients of modulation equations are expressed in general integral forms of the mode shapes and operators. Steady state solutions and their stability are discussed. The sufficiency condition for such resonances to appear is derived. Finally the algorithm developed is applied to a beam vibration problem resting on a non-linear elastic foundation.

Within the limitation of the model discussed at the end of Section 2, the algorithm developed can be applied to a wide range of problems having cubic non-linearities with 3:1 internal resonances.

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