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Letter to the Editor

Mathematical modelling of a beam-like flexible structure in slewing motion assuming non-linear curvature

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1. Introduction

We present the study, via perturbation techniques, of several cases of resonant response of a non-linear mathematical model of a flexible beam-like structure in slewing motion. Non-linear curvature is assumed for the beam. Potential application in aerospace and robotic engineering is envisioned.

Fig. 1 shows the system to be analyzed here. For the non-linear beam-like flexible structure, cubic geometric non-linear terms are considered in the equations of motion. The harmonic excitation on the flexible structure will be provided by the prescribed angular displacement θ and its derivatives. This angular displacement is also designated as slewing motion [1].

In this study, the behavior of the angular displacement will be known beforehand and will not be influenced by the flexible structure dynamics. Thus, the system is said to be under ideal excitation. If the excitation and the system dynamics were both unknown and coupled, the whole system would be said to be non-ideal (see Ref. [2]; for a review of the theory see Refs. [3–5]).

For the prescribed excitation profile, $\theta(t)$, we propose one that makes the flexible structure oscillate harmonically between the extremes θ_A (initial condition) and θ_C, according to Fig. 1, with amplitude C and frequency Ω.

In general, for non-linear systems such as the one presented here, unlike the linear case, excitations with frequencies multiple or fractional of some of the natural frequencies of the associated linear system may also lead to resonant conditions [6]. For this reason, many critical

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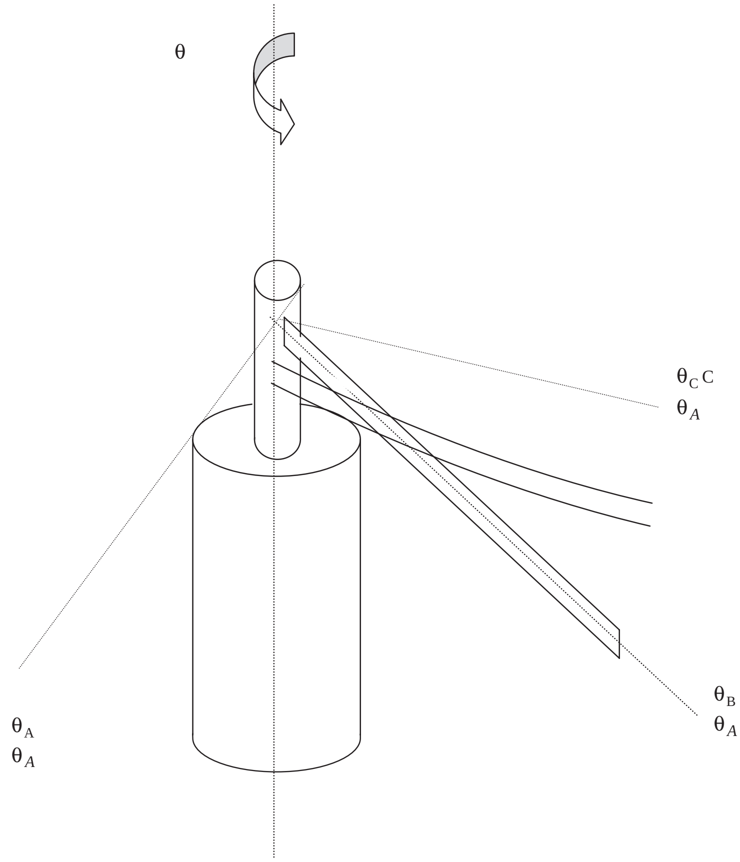

Fig. 1. The analyzed model. The beam is represented with no deflection and deflected in the position θ_A.

conditions are carefully analyzed in this paper. When the excitation frequency is near any of the natural frequencies of the associated linear system, one speaks of a primary resonance. In the cases in which the excitation frequency has values near the frequencies associated with the non-linear part of the governing equations (sub-harmonics and super-harmonics), one speaks of secondary resonances, which are all the others except the primary ones.

The aspect and the construction of the frequency response curves for non-linear systems, such as our case, differ from those for linear ones. This will allow for the occurrence of some particular phenomena, as abrupt variations in the amplitudes of vibration in some situations when the frequency (or the amplitude) of the excitation is slightly changed. This is known as the jump phenomenon. The regions where the jump phenomenon occurs are regions associated with bifurcations of solutions and some peculiar behaviors known as *chaotic* may be displayed by the vibrating system (see for instance Ref. [7]).

In this paper, to consider the case of primary resonance, where the excitation amplitude must be of the same order as the non-linear terms and structural damping, the weakly non-linear system will be investigated in the neighborhood of the beam free vibration.

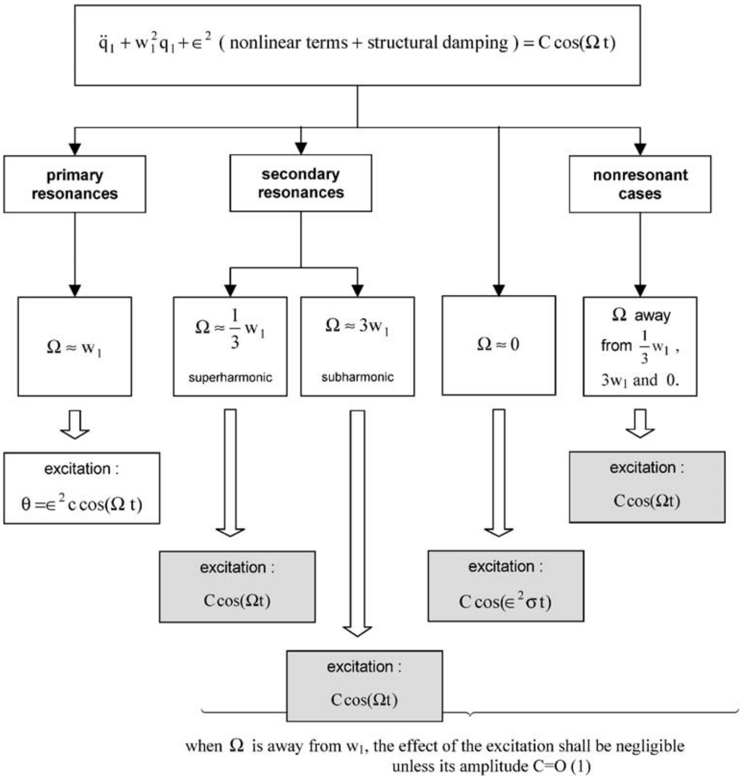

Fig. 2. The different orders to be considered for the excitation along the investigations regarding the forced oscillations of one degree of freedom systems.

For the study of secondary resonances and non-resonant cases, the excitation amplitude shall be $O(1)$. In this case, the weakly non-linear system shall be investigated in the neighborhood of the linear beam forced vibration. Fig. 2 presents a route to the different considerations to be made in this investigation.

The analysis developed in this work will provide the first important steps for obtaining the frequency response curves and the modulation equations for amplitude and phase of the system to be pursued in further researches. In experimental prototypes or in real systems, the hub to which the beam is clamped and which represents the source of the excitation can be understood to be some kind of actuator (for instance, a dc motor for robotic applications).

2. Discusion of the governing equations of motion

Next we will derive special cases of the governing equations of motion taking into account several resonance possibilities. We will consider particular cases with and without structural damping.

2.1. Secondary resonance and non-resonant cases—excitation: amplitude O(1) and frequency O(1), mathematical model without structural damping

The governing equations of motion shall be obtained through an energy method. For this reason, one needs to know the expressions for the kinetic and potential energies of the system to be modelled.

The kinetic energy of the rotating beam (where θ represents the angular displacement), assuming transversal (v) and longitudinal (u) displacements for each point along the beam (non-linear curvature), is given by

$$T = \frac{\rho}{2}\int_0^L \{(-\dot{u} + \dot{\theta}v)^2 + [\dot{v} + \dot{\theta}(x-u)]^2\}\, \mathrm{d}x, \tag{1}$$

where ρ is the mass per unit of length and L is the length of the undeflected beam.

The potential energy is given by

$$V = \frac{1}{2}\int_0^L \left(EAe^2 + EI\phi'^2 \frac{\mathrm{d}x}{\mathrm{d}l}\right) \mathrm{d}x, \tag{2}$$

where E is the Young's modulus, A is the area and I is the moment of inertia of the cross-section of the beam, l is the arc length along the beam, and e is the axial strain. In Eq. (2), $\phi = v'$.

By introducing expressions (1) and (2) in the expression due to the extended Hamilton's Principle [9], given by

$$\delta \int_{t_1}^{t_2} L\, \mathrm{d}t = \delta \int_{t_1}^{t_2} (T - V)\, \mathrm{d}t,$$

one easily obtains the following dimensional governing equations of motion for the flexible structure variables $v(x,t)$ and $u(x,t)$:

$$\rho(\dot{\theta}^2 v + 2\dot{\theta}\dot{u} - \ddot{\theta}x + \ddot{\theta}u - \ddot{v}) - EI(v^{iv} + u^{iv}v' + 4u'''v'' + 6u''v''' + 3u'v^{iv} - \tfrac{5}{2}v''^3 - 10v'v''v''' - \tfrac{5}{2}v'^2v^{iv}) - EA(u''v' + u'v'' - \tfrac{3}{2}v'^2v'') = 0,$$
$$\rho(-\ddot{\theta}v - 2\dot{\theta}\dot{v} - \ddot{u} - \dot{\theta}^2 x + \dot{\theta}^2 u) - EI(v'v^{iv}) - EA(-u'' + v'v'') = 0.$$

The variables in the dimensional governing equations of motion are made dimensionless using the relations

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{T_c}, \quad \tilde{l} = \frac{1}{L}, \quad \ddot{\tilde{\theta}}(t) = \frac{T_c^2\ddot{\theta}(t)}{\epsilon}, \quad \tilde{v}(x,t) = \frac{v(x,t)}{\epsilon L}$$

and

$$\tilde{u}(x,t) = \frac{u(x,t)}{\epsilon^2 L},$$

where $\sim$ denotes a dimensionless variable (this notation is dropped later), T_c is the characteristic time defined as $T_c = (L^2/\beta^2)\sqrt{\rho/EI}$ with $\beta = 1.8780$ (the first natural frequency for a cantilever beam), and $\epsilon \ll 1$ is a small dimensionless parameter.

Note that we are approximating the variable v in the non-dimensional governing equations through the assumed modes method [10] by

$$v(x,t) \cong \sum_{i=1}^{N} q_i(t)\varphi_i(x),$$

where $\varphi_i(x)$ represents the linear, free-vibration modes for the beam when $\dot{\theta}(t) \equiv 0$. $q_i(t)$ represents the generalized co-ordinates describing the motion of the beam. This leads to the following non-dimensional discretized perturbed governing equations of motion (the algebraic details may be seen in Ref. [1]):

$$\ddot{q}_\ell + w_\ell^2 q_\ell + \alpha_\ell \ddot{\theta} + \epsilon^2 \left[\dot{\theta}^2 \sum_{i=1}^{N} \beta_{i\ell} q_i - \sum_{i=1}^{N}\sum_{j=1}^{N} (\mathscr{P}_{ij\ell}\dot{\theta} q_i \dot{q}_j - \lambda_{ij\ell}\ddot{\theta} q_i q_j) + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \Lambda_{ijk\ell} q_i(\dot{q}_j\dot{q}_k + q_j \ddot{q}_k) + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \Gamma_{ijk\ell} q_i q_j q_k \right] = 0 \tag{3}$$

and boundary conditions

$$\varphi_\ell(0) = 0, \quad \varphi'_\ell(0) = 0, \quad \varphi''_\ell(1) = 0 \ \text{and} \ \varphi'''_\ell(1) = 0. \tag{4}$$

In Eq. (3), variable $u\ (x,t)$ is eliminated through the relation given by

$$u(x,t) = \frac{1}{2}\int v'^2 \, \mathrm{d}x + O(\epsilon) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j R_{ij}(x) + O(\epsilon). \tag{5}$$

By considering only one mode of vibration in Eq. (3) (for instance, the first flexural mode), the perturbed governing equation of motion for $q_1\ (t)$ is given by

$$\ddot{q}_1 + w_1^2 q_1 + \alpha_1 \ddot{\theta} + \epsilon^2[\beta_{11}\dot{\theta}^2 q_1 - \mathscr{P}_{111}\dot{\theta} q_1 \dot{q}_1 - \lambda_{111}\ddot{\theta} q_1^2 + \Lambda_{1111} q_1 \dot{q}_1^2 + \Lambda_{1111} q_1^2 \ddot{q}_1 + \Gamma_{1111} q_1^3] = 0. \tag{6}$$

The coefficients in Eq. (6) are presented in Appendix A. The terms in Eq. (6) that indicate the coupling between the space-temporal variable v (or, in the discretization, the temporal variable q_1 and its derivatives) and the prescribed variable θ (and its derivatives) are represented by

$$\epsilon^2[\beta_{11}\dot{\theta}^2 q_1 - \mathscr{P}_{111}\dot{\theta} q_1 \dot{q}_1 - \lambda_{111}\ddot{\theta} q_1^2]. \tag{7}$$

By analyzing the ideal system problem, the harmonic prescribed profiles of θ (and its derivatives) will be considered according to

$$\theta = C\sin(\Omega t) = C\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{8}$$

$$\dot{\theta} = C\Omega\cos(\Omega t) = C\frac{\Omega}{2}(\mathrm{e}^{\mathrm{i}\Omega t} + \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{9}$$

$$\ddot{\theta} = -C\Omega^2\sin(\Omega t) = -C\frac{\Omega^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{10}$$

where, at first, the amplitude C shall be considered equal to 1 (therefore, $O(1)$).

2.2. Secondary resonances and non-resonant cases—excitation: amplitude $O(1)$ and frequency $O(1)$, mathematical model with structural damping

Structural damping, μ, is now included in the mathematical model presented in Eq. (6), to make the model more realistic:

$$\mu^* = \mu\mu_c, \tag{11}$$

where μ^* is the non-dimensional structural damping; μ is the dimensional structural damping (kg/m s); μ_c = characteristic damping $= L^2/\beta^2\sqrt{EI\rho}$ (m s/kg). By including the structural damping (according to Eq. (11)) in Eq. (6) of the same order of the non-linear terms ($O(\epsilon^2)$), the new governing equation of motion obtained is

$$\begin{aligned}\ddot{q}_1 + w_1^2 q_1 + \alpha_1\ddot{\theta} + \epsilon^2[\mu\dot{q}_1 + \beta_{11}\dot{\theta}^2 q_1 - \mathscr{P}_{111}\dot{\theta}q_1\dot{q}_1 \\ - \lambda_{111}\ddot{\theta}q_1^2 + \Lambda_{1111}q_1\dot{q}_1^2 + \Lambda_{1111}q_1^2\ddot{q}_1 + \Gamma_{1111}q_1^3] = 0,\end{aligned} \tag{12}$$

where the non-dimensional structural damping, for convenience, is represented without the (*).

2.3. Primary resonance—excitation: amplitude $O(\epsilon^2)$ and fequency $O(1)$

The harmonic prescribed profiles to the angular displacement θ (and its derivatives), as presented previously in Eqs. (8)–(10), will now be rescaled according to

$$\theta = \epsilon^2 c\sin(\Omega t) = \epsilon^2 c\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{13}$$

$$\dot{\theta} = \epsilon^2 c\Omega\cos(\Omega t) = \epsilon^2 c\frac{\Omega}{2}(\mathrm{e}^{\mathrm{i}\Omega t} + \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{14}$$

$$\ddot{\theta} = -\epsilon^2 c\Omega^2\sin(\Omega t) = -\epsilon^2 c\frac{\Omega^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t}), \tag{15}$$

where the amplitude C is $O(\epsilon^2)$ and given by $C = \epsilon^2 c$.

This approach is more appropriate for the study of primary resonance of weakly non-linear systems [6] and assures that the amplitude of the associated linear system, q_{10}, is not unbounded when $\Omega \approx w_n$ (where w_n represents each one of the linear natural frequencies associated to a primary resonance of the weakly non-linear system).

In other words, we try to assure that the non-linear terms are not as important as the linear terms, for, as shall be seen further on, the $O(\epsilon^0)$ solution (or the solution of the associated linear system, q_{10}) appears on the right side of the $O(\epsilon^2)$ equation for the perturbed part of the solution (q_{11}).

2.4. Primary resonance—excitation: amplitude $O(\epsilon^2)$ and frequency $O(1)$, mathematical model without structural damping

By utilizing relations (13)–(15), Eq. (6) is transformed into

$$\ddot{q}_1 + w_1^2 q_1 + \epsilon^2[\alpha_1\ddot{\theta} + \Lambda_{1111}q_1\dot{q}_1^2 + \Lambda_{1111}q_1^2\ddot{q}_1 + \Gamma_{1111}q_1^3] = 0, \tag{16}$$

where all terms of order greater than ϵ were neglected. The coupling terms, as described in Eq. (7), are included in the higher order neglected terms and really are not important when $C = \epsilon^2 c$.

The system behavior here shall be studied in the neighborhood of its linear undamped free vibration condition.

2.5. Primary resonance—excitation: amplitude $O(\epsilon^2)$ and frequency $O(1)$, mathematical model with structural damping

By utilizing relations (13)–(15), Eq. (12) is transformed into

$$\ddot{q}_1 + w_1^2 q_1 + \epsilon^2[\mu\dot{q}_1 + \alpha_1\ddot{\theta} + \Lambda_{1111} q_1 \dot{q}_1^2 + \Lambda_{1111} q_1^2 \ddot{q}_1 + \Gamma_{1111} q_1^3] = 0. \tag{17}$$

3. Application of the multiple scale method (MSM): the search for an analytic solution

In this section, we will analyze the solutions by using the classical method of Multiple Scales [8].

3.1. Secondary resonances and non-resonant cases—excitation: amplitude O(1) and frequency O(1), mathematical model without structural damping

For the solution of the perturbed system represented by Eq. (6), the following uniform expansion is proposed [8]:

$$q_1 = q_{10}(T_0, T_1) + \epsilon^2 q_{11}(T_0, T_1). \tag{18}$$

By substituting Eq. (18) into Eq. (6) and collecting terms of same order of ϵ, one obtains

(*a*) *Order* ϵ^0:

$$\frac{\partial^2 q_{10}}{\partial T_0^2} + w_1^2 q_{10} = \alpha_1 \frac{\Omega^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega T_0} - \mathrm{e}^{-\mathrm{i}\Omega T_0}). \tag{19}$$

(*b*) *Order* ϵ^2:

$$\begin{aligned}\frac{\partial^2 q_{11}}{\partial T_0^2} + w_1^2 q_{11} &= -2\left(\frac{\partial^2 q_{10}}{\partial T_0\,\partial T_1}\right) - \frac{\beta_{11}\Omega^2}{4}(\mathrm{e}^{2\mathrm{i}\Omega T_0} + 2 + \mathrm{e}^{-2\mathrm{i}\Omega T_0})q_{10} \\ &\quad + \mathscr{P}_{111}\frac{\Omega}{2}(\mathrm{e}^{\mathrm{i}\Omega T_0} + \mathrm{e}^{-\mathrm{i}\Omega T_0})q_{10}\frac{\partial q_{10}}{\partial T_0} - \frac{\lambda_{111}\Omega^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega T_0} - \mathrm{e}^{\mathrm{i}\Omega T_0})q_{10}^2 \\ &\quad - \Lambda_{1111} q_{10}\left(\frac{\partial q_{10}}{\partial T_0}\right)^2 - \Lambda_{1111}(q_{10}^2)\frac{\partial^2 q_{10}}{\partial T_0^2} - \Gamma_{1111} q_{10}^3 = 0.\end{aligned} \tag{20}$$

The solution of Eq. (19) is given by

$$q_{10} = A(T_1)\mathrm{e}^{\mathrm{i}w_1 T_0} + \bar{A}(T_1)\mathrm{e}^{-\mathrm{i}w_1 T_0} + \left(\frac{\alpha_1\Omega^2}{2\mathrm{i}(w_1^2 - \Omega^2)}\right)\mathrm{e}^{\mathrm{i}\Omega T_0} - \left(\frac{\alpha_1\Omega^2}{2\mathrm{i}(w_1^2 - \Omega^2)}\right)\mathrm{e}^{-\mathrm{i}\Omega T_0}. \tag{21}$$

From here on, for the sake of convenience, $A\ (T_1)$ and $\bar{A}(T_1)$ shall be represented simply by A and $\bar{A}$. Let it also be

$$B = \left(\frac{\alpha_1\Omega^2}{2i(w_1^2 - \Omega^2)}\right).$$

Substituting Eq. (21) into Eq. (20) yields

$$\frac{\partial^2 q_{11}}{\partial T_0^2} + w_1^2 q_{11} = -2\mathrm{i}w_1\left(\frac{\partial A}{\partial T_1}\right)\mathrm{e}^{\mathrm{i}w_1T_0} + 2\mathrm{i}w_1\left(\frac{\partial \bar{A}}{\partial T_1}\right)\mathrm{e}^{-\mathrm{i}w_1T_0} - \frac{\beta_{11}\Omega^2}{4}\kappa_1 + \mathscr{P}_{111}\frac{\Omega}{2}\kappa_2 - \frac{\lambda_{111}\Omega^2}{2\mathrm{i}}\kappa_3 - \Lambda_{1111}(\kappa_4+\kappa_5) - \Gamma_{1111}\kappa_6, \tag{22}$$

where

$$\kappa_1 = A\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0} + \bar{A}\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0} + B\mathrm{e}^{\mathrm{i}3\Omega T_0} - B\mathrm{e}^{\mathrm{i}\Omega T_0} + 2A\mathrm{e}^{\mathrm{i}w_1T_0} + 2\bar{A}\mathrm{e}^{-\mathrm{i}w_1T_0} + 2B\mathrm{e}^{\mathrm{i}\Omega T_0} - B\mathrm{e}^{-\mathrm{i}\Omega T_0} + A\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0} + \bar{A}\mathrm{e}^{-\mathrm{i}(w_1+2\Omega)T_0} - B\mathrm{e}^{-\mathrm{i}3\Omega T_0},$$

$$\begin{aligned}\kappa_2 = {}& \mathrm{i}w_1A^2\mathrm{e}^{\mathrm{i}(2w_1+\Omega)T_0} + \mathrm{i}w_1A^2\mathrm{e}^{\mathrm{i}(2w_1-\Omega)T_0} + \mathrm{i}AB(w_1+\Omega)\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0} + 2\mathrm{i}\Omega AB\mathrm{e}^{\mathrm{i}w_1T_0}\\ & - \mathrm{i}AB(w_1-\Omega)\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0} - \mathrm{i}w_1\bar{A}^2\mathrm{e}^{-\mathrm{i}(2w_1-\Omega)T_0} - \mathrm{i}w_1\bar{A}^2\mathrm{e}^{-\mathrm{i}(2w_1+\Omega)T_0} + \mathrm{i}\Omega B^2\mathrm{e}^{\mathrm{i}\Omega T_0}\\ & - \mathrm{i}\bar{A}B(w_1-\Omega)\mathrm{e}^{-\mathrm{i}(w_1+2\Omega)T_0} + \mathrm{i}\bar{A}B(w_1-\Omega)\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0} + 2\mathrm{i}\Omega\bar{A}B\mathrm{e}^{-\mathrm{i}w_1T_0} + \mathrm{i}\Omega B^2\mathrm{e}^{\mathrm{i}3\Omega T_0}\\ & - \mathrm{i}\Omega B^2\mathrm{e}^{-\mathrm{i}3\Omega T_0} - \mathrm{i}\Omega B^2\mathrm{e}^{-\mathrm{i}\Omega T_0},\end{aligned}$$

$$\begin{aligned}\kappa_3 = {}& A^2\mathrm{e}^{\mathrm{i}(2w_1+\Omega)T_0} - A^2\mathrm{e}^{\mathrm{i}(2w_1-\Omega)T_0} + 2AB\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0} + 2AB\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0} - 4AB\mathrm{e}^{\mathrm{i}w_1T_0}\\ & + (2A\bar{A} - 3B^2)\mathrm{e}^{\mathrm{i}\Omega T_0} - (2A\bar{A} - 3B^2)\mathrm{e}^{-\mathrm{i}\Omega T_0} + 2\bar{A}B\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0} + 2\bar{A}B\mathrm{e}^{-\mathrm{i}(w_1+2\Omega)T_0}\\ & + B^2\mathrm{e}^{\mathrm{i}3\Omega T_0} - B^2\mathrm{e}^{-\mathrm{i}3\Omega T_0} - 4\bar{A}B\mathrm{e}^{-\mathrm{i}w_1T_0} + \bar{A}^2\mathrm{e}^{-\mathrm{i}(2w_1-\Omega)T_0} - \bar{A}^2\mathrm{e}^{-\mathrm{i}(2w_1+\Omega)T_0},\end{aligned}$$

$$\begin{aligned}\kappa_4 = {}& -w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0} + (w_1^2A^2\bar{A} - 2\Omega^2AB^2)\mathrm{e}^{\mathrm{i}w_1T_0} - (2w_1\Omega A^2B + w_1^2A^2B)\mathrm{e}^{\mathrm{i}(2w_1+\Omega)T_0}\\ & - (2w_1\Omega A^2B - w_1^2A^2B)\mathrm{e}^{\mathrm{i}(2w_1-\Omega)T_0} + (2w_1^2A\bar{A}B - \Omega^2B^3)\mathrm{e}^{\mathrm{i}\Omega T_0} - w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0}\\ & - (2w_1^2A\bar{A}B - \Omega^2B^3)\mathrm{e}^{-\mathrm{i}\Omega T_0} - (2w_1\Omega AB^2 + \Omega^2AB^2)\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0}\\ & - \Omega^2B^3\mathrm{e}^{\mathrm{i}3\Omega T_0} + (2w_1\Omega AB^2 - \Omega^2AB^2)\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0} + (w_1^2A\bar{A}^2 - 2\Omega^2\bar{A}B^2)\mathrm{e}^{-\mathrm{i}w_1T_0}\\ & + \Omega^2B^3\mathrm{e}^{-\mathrm{i}3\Omega T_0} + (2w_1\Omega\bar{A}^2B - w_1^2\bar{A}^2B)\mathrm{e}^{-\mathrm{i}(2w_1-\Omega)T_0} + (2w_1\Omega\bar{A}^2B + w_1^2\bar{A}^2B)\mathrm{e}^{-\mathrm{i}(2w_1+\Omega)T_0}\\ & + (2w_1\Omega\bar{A}B^2 - \Omega^2\bar{A}B^2)\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0} - (2w_1\Omega\bar{A}B^2 + \Omega^2\bar{A}B^2)\mathrm{e}^{-\mathrm{i}(w_1+2\Omega)T_0},\end{aligned}$$

$$\begin{aligned}\kappa_5 = {}& -w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0} + (2w_1^2AB^2 + 4AB^2\Omega^2 - 2w_1^2A^2\bar{A})\mathrm{e}^{\mathrm{i}w_1T_0} + \Omega^2B^3\mathrm{e}^{-\mathrm{i}3\Omega T_0}\\ & - \Omega^2B^3\mathrm{e}^{\mathrm{i}3\Omega T_0} + (3\Omega^2B^3 - 2\Omega^2A\bar{A}B)\mathrm{e}^{\mathrm{i}\Omega T_0} + (2w_1^2\bar{A}B^2 + 4\Omega^2\bar{A}B^2 - 3w_1^2A\bar{A}^2)\mathrm{e}^{-\mathrm{i}w_1T_0}\\ & - w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0} + (2\Omega^2A\bar{A}B - 3\Omega^2B^3)\mathrm{e}^{-\mathrm{i}\Omega T_0} - (w_1^2A^2B + \Omega^2A^2B)\mathrm{e}^{\mathrm{i}(2w_1+\Omega)T_0}\\ & + (2w_1^2A^2B + \Omega^2A^2B)\mathrm{e}^{\mathrm{i}(2w_1-\Omega)T_0} - (w_1^2AB^2 + 2\Omega^2AB^2)\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0}\\ & - (w_1^2AB^2 + 2\Omega^2AB^2)\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0} - (2w_1^2\bar{A}^2B + \Omega^2\bar{A}^2B)\mathrm{e}^{-\mathrm{i}(2w_1-\Omega)T_0}\\ & + (2w_1^2\bar{A}^2B + \Omega^2\bar{A}^2B)\mathrm{e}^{-\mathrm{i}(2w_1+\Omega)T_0} - (w_1^2\bar{A}B^2 + 2\Omega^2\bar{A}B^2)\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0}\\ & - (w_1^2\bar{A}B^2 + 2\Omega^2\bar{A}B^2)\mathrm{e}^{-\mathrm{i}(w_1+2\Omega)T_0},\end{aligned}$$

$$\begin{aligned}\kappa_6 = {} & A^3 e^{i3w_1T_0} + \bar{A}^3 e^{-i3w_1T_0} - B^3 e^{-i3\Omega T_0} + B^3 e^{i3\Omega T_0} + (3A^2\bar{A} - 6AB^2)e^{iw_1T_0} \\ & + (2A\bar{A}B - 3B^3)e^{i\Omega T_0} + (3B^3 - 2A\bar{A}B)e^{-i\Omega T_0} + (3A\bar{A}^2 - 6\bar{A}B^2)e^{-iw_1T_0} \\ & + (3A^2B)e^{i(2w_1+\Omega)T_0} - (3A^2B)e^{i(2w_1-\Omega)T_0} + (3AB^2)e^{i(w_1+2\Omega)T_0} + (3AB^2)e^{i(w_1-2\Omega)T_0} \\ & + (3\bar{A}^2B)e^{-i(2w_1-\Omega)T_0} - (3\bar{A}^2B)e^{-i(2w_1+\Omega)T_0} + (3\bar{A}B^2)e^{-i(w_1-2\Omega)T_0} + (3\bar{A}B^2)e^{-i(w_1+2\Omega)T_0}.\end{aligned}$$

In Section 4 the conditions will be shown for which the secular terms and the small divisors in Eq. (22) do not unbound the solution of Eq. (6). In Section 4 all the resonant cases for this system will be presented according to the adopted mathematical model.

3.2. Secondary resonances and non-resonant cases—excitation: amplitude $O(1)$ and frequency $O(1)$, mathematical model with structural damping

Following the same steps previously presented for the undamped case, in the right side of Eq. (20) shall now be introduced the term

$$-\mu\left(\frac{\partial q_{10}}{\partial T_0}\right). \tag{23}$$

In Eq. (22), the expression

$$-\mu(iAw_1e^{iw_1T_0} - i\bar{A}w_1e^{-iw_1T_0} + iB\Omega e^{i\Omega T_0} + iB\Omega e^{-i\Omega T_0}) \tag{24}$$

will be introduced, where the term $iAw_1e^{iw_1T_0}$ is associated with secular terms that compromise the desired periodic solution and the term $iB\Omega e^{i\Omega T_0}$ is associated with small divisor terms that equally compromise the wanted solution (when $\Omega \approx w_1$) and both shall be properly eliminated.

3.3. Primary resonance—excitation: amplitude $O(\epsilon^2)$ and frequency $O(1)$, mathematical model without structural damping

For the solution of the perturbed system represented by Eq. (16), it is proposed to use the same uniform expansion presented in Eq. (18). Substituting the expansion (18) into Eq. (16) and collecting terms of same order of ϵ, one obtains

(1) *Order* ϵ^0:

$$\frac{\partial^2 q_{10}}{\partial T_0^2} + w_1^2 q_{10} = 0. \tag{25}$$

(2) *Order* ϵ^2:

$$\begin{aligned}\frac{\partial^2 q_{11}}{\partial T_0^2} + w_1^2 q_{11} = {} & -2\left(\frac{\partial^2 q_{10}}{\partial T_0\,\partial T_1}\right) - \alpha_1 c\frac{\Omega^2}{2i}(e^{i\Omega T_0} - e^{-i\Omega T_0}) - \Lambda_{1111}q_{10}\left(\frac{\partial q_{10}}{\partial T_0}\right)^2 \\ & - \Lambda_{1111}(q_{10}^2)\frac{\partial^2 q_{10}}{\partial T_0^2} - \Gamma_{1111}q_{10}^3 = 0.\end{aligned} \tag{26}$$

The solution of Eq. (25) is given by

$$q_{10} = A(T_1)e^{iw_1T_0} + \bar{A}(T_1)e^{-iw_1T_0}. \tag{27}$$

Substituting Eq. (27) into Eq. (26) yields

$$\begin{aligned}\frac{\partial^2 q_{11}}{\partial T_0^2}+w_1^2 q_{11} = &-2\mathrm{i}w_1\left(\frac{\partial A}{\partial T_1}\right)\mathrm{e}^{\mathrm{i}w_1T_0}+2\mathrm{i}w_1\left(\frac{\partial \bar{A}}{\partial T_1}\right)\mathrm{e}^{-\mathrm{i}w_1T_0}-\alpha_1 c\frac{\Omega^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega T_0}-\mathrm{e}^{-\mathrm{i}\Omega T_0})\\ &-\Lambda_{1111}(-w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0}+w_1^2A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0}-w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0}+w_1^2A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0})\\ &-\Lambda_{1111}(-w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0}-2w_1^2A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0}-3w_1^2A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0}-w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0})\\ &-\Gamma_{1111}(A^3\mathrm{e}^{i3w_1T_0}+\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0}+3A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0}+3A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0}).\end{aligned} \quad (28)$$

The critical conditions related to the solution of Eq. (28) (resonant cases) shall be discussed in the following section.

3.4. Primary resonance—excitation: amplitude $O(\epsilon^2)$ and frequency $O(1)$, mathematical model with structural damping

The same discussion presented previously is still valid here. Eq. (25) is again obtained (and solution (27)). In Eq. (26), one must add now, to the right side, the term given by Eq. (23). In Eq. (28), one must add now, to the right side, the terms presented in

$$-\mu\left(\frac{\partial q_{10}}{\partial T_0}\right)=-\mu(\mathrm{i}w_1A\mathrm{e}^{\mathrm{i}w_1T_0}-\mathrm{i}w_1\bar{A}\mathrm{e}^{-\mathrm{i}w_1T_0}). \quad (29)$$

4. On location of the resonant cases

The equations from which the resonant cases considering Ω away from zero and away from w_1, otherwise will be investigated the secondary resonance cases, will be investigated are Eq. (22) for the model presented in Appendix A and Eq. (22) plus the terms presented in Eq. (24) added to the right side of the model presented in Section 3.1. The equations from which the primary resonance cases will be investigated are: Eq. (28) for the model presented in Section 3.3, and Eq. (28) plus the terms presented in Eq. (29) added to the right side for the model presented in Section 3.1 can be written in the form

$$\ddot{x}+w_1^2x=\text{Terms multiplied by } \mathrm{e}^{\pm\mathrm{i}\Xi T_0}, \quad (30)$$

where Ξ represents each one of the exponents presented in Table 1 (secondary resonances) and 2 (primary resonances).

The terms $\mathrm{e}^{\mathrm{i}(w_1+2\Omega)T_0}$ and $\mathrm{e}^{\mathrm{i}(w_1-2\Omega)T_0}$ that appear in Eq. (22) are not considered critical terms in the investigation regarding secondary resonances because the excitation frequency, Ω, is away from zero. In the same way, the terms $\mathrm{e}^{\mathrm{i}\Omega T_0}$, $\mathrm{e}^{-\mathrm{i}(w_1-2\Omega)T_0}$ and $\mathrm{e}^{\mathrm{i}(2w_1-\Omega)T_0}$ that appear in the same Eq. (22) are not critical in the analysis of secondary resonances because Ω is away from w_1.

Table 2 presents the critical situations related to the investigation of the primary resonance obtained through Eq. (28).

Table 1
Presentation of the terms that produce secular terms or small divisors in the desired periodic solution for the excitation frequency, Ω, away form zero and w_1 (secondary resonances)

Term in Eq. (17) multiplied by	Produces in the desired solution an effect of the kind	Critical condition	Situation
$e^{iw_1T_0}$	Secular term	Always	**1**
$e^{-iw_1T_0}$	None		
$e^{i3w_1T_0}$	None		
$e^{-i3w_1T_0}$	None		
$e^{i\Omega T_0}$	None		
$e^{-i\Omega T_0}$	None		
$e^{i3\Omega T_0}$	Small divisor	$\Omega \approx \frac{1}{3} w_1$	**2**
$e^{-i3\Omega T_0}$	None		
$e^{i(w_1+2\Omega)T_0}$	None		
$e^{-i(w_1+2\Omega)T_0}$	None		
$e^{i(w_1-2\Omega)T_0}$	None		
$e^{-i(w_1-2\Omega)T_0}$	None		
$e^{i(2w_1+\Omega)T_0}$	None		
$e^{-i(2w_1+\Omega)T_0}$	None		
$e^{i(2w_1-\Omega)T_0}$	None		
$e^{-i(2w_1+\Omega)T_0}$	Small divisor	$\Omega \approx 3\, w_1$	**3**

Table 2
Presentation of the terms that produce secular terms or small divisors in the desired periodic solution for the excitation frequency, Ω, away form zero and near w_1 (primary resonance)

Term in Eq. (23) multiplied by	Produces in the desired solution	Critical condition	Situation
$e^{iw_1T_0}$	Secular term	Always	**4**
$e^{iw_1T_0}$			
$e^{-iw_1T_0}$	None		
$e^{i3w_1T_0}$	None		
$e^{-i3w_1T_0}$	None		
$e^{i\Omega T_0}$	Small divisor	$\Omega \approx w_1$	**5**
$e^{-i\Omega T_0}$	None		

5. On the case $\Omega \approx 0$—excitation: amplitude $O(1)$and frequency $O(\varepsilon^2)$

For the case when the excitation frequency, Ω, is near zero, one has: $\Omega = 0 + \epsilon^2\sigma = \epsilon^2\sigma$ or $\Omega t = \Omega T_0 = \epsilon^2 \sigma T_0 = \sigma T_1$ and the prescribed harmonic profiles for θ (and its derivatives) are considered according to

$$\theta = C \sin(\epsilon^2 \sigma T_0) = C \frac{1}{2i} (e^{i\sigma T_1} - e^{-i\sigma T_1}), \tag{31}$$

$$\dot{\theta} = C\varepsilon^2 \sigma \cos(\epsilon^2 \sigma T_0) = C \frac{\epsilon^2 \sigma}{2} (e^{i\sigma T_1} + e^{-i\sigma T_1}), \tag{32}$$

$$\ddot{\theta} = -C\epsilon^4\sigma^2 \sin(\epsilon^2\sigma T_0) = -C\frac{\epsilon^4\sigma^2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\sigma T_1} - \mathrm{e}^{-\mathrm{i}\sigma T_1}). \tag{33}$$

5.1. The case $\Omega \approx 0$—excitation: amplitude $O(1)$ and frequency $O(\epsilon^2)$, mathematical model without structural damping

By utilizing Eq. (31) to Eq. (33), the governing equation of motion for the first flexural mode of vibration is given by

$$\ddot{q}_1 + w_1^2 q_1 + \epsilon^2[\Lambda_{1111}q_1\dot{q}_1^2 + \Lambda_{1111}q_1^2\ddot{q}_1 + \Gamma_{1111}q_1^3] = 0. \tag{34}$$

The rest of the procedures previously discussed here are identical. In this case, the order ε^0 solution will be given by

$$q_{10} = A(T_1)\mathrm{e}^{\mathrm{i}w_1 T_0} + \bar{A}(T_1)\mathrm{e}^{-\mathrm{i}w_1 T_0} \tag{35}$$

and the terms that will produce secular terms in the desired solution shall be collected from

$$\begin{aligned}\frac{\partial^2 q_{11}}{\partial T_0^2} + w_1^2 q_{11} = & -2\mathrm{i}w_1\left(\frac{\partial A}{\partial T_1}\right)\mathrm{e}^{\mathrm{i}w_1 T_0} + 2\mathrm{i}w_1\left(\frac{\partial \bar{A}}{\partial T_1}\right)\mathrm{e}^{-\mathrm{i}w_1 T_0} \\ & - \Lambda_{1111}(-w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0} + w_1^2A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0} + w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0} + w_1^2A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0} \\ & - w_1^2A^3\mathrm{e}^{\mathrm{i}3w_1T_0} - 2w_1^2A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0} - 3w_1^2A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0} - w_1^2\bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0}) \\ & - \Gamma_{1111}(A^3\mathrm{e}^{\mathrm{i}3w_1T_0} + \bar{A}^3\mathrm{e}^{-\mathrm{i}3w_1T_0} - 3A^2\bar{A}\mathrm{e}^{\mathrm{i}w_1T_0} + 3A\bar{A}^2\mathrm{e}^{-\mathrm{i}w_1T_0}) = 0. \end{aligned} \tag{36}$$

Now, the terms related to the production of small divisors in the wanted solution will not exist (Table 3).

5.2. The case $\Omega \approx 0$—excitation: amplitude $O(1)$ and frequency $O(\epsilon^2)$, mathematical model with structural damping

Considering the structural damping, Eq. (34) turns into

$$\ddot{q}_1 + w_1^2 q_1 + \epsilon^2[\mu\dot{q}_1 + \Lambda_{1111}q_1\dot{q}_1^2 + \Lambda_{1111}q_1^2\ddot{q}_1 + \Gamma_{1111}q_1^3] = 0 \tag{37}$$

Table 3
Presentation of the terms that produces secular terms or small divisors in the desired periodic solution for the excitation frequency $\Omega \approx 0$

Term in Eq. (30) multiplied by	Produces in the desired solution an	Critical condition	Situation
$\mathrm{e}^{\mathrm{i}w_1T_0}$ $\mathrm{e}^{\mathrm{i}w_1T_0}$	Secular term	Always	**6**
$\mathrm{e}^{-\mathrm{i}w_1T_0}$	None		
$\mathrm{e}^{\mathrm{i}3w_1T_0}$	None		
$\mathrm{e}^{-\mathrm{i}3w_1T_0}$	None		

Table 4
Critical cases of interest to be investigated (including the non-resonant case)

Case	Kind of resonance
I: situation 1 + situation 2	1:1/3
II: situation 6	1:0
III: situation 1 + situation 3	1:3
IV: situation 4 + situation 5	1:1
V: region away from any resonance	Non-resonant case

and the terms that will produce secular terms and small divisor terms in the wanted solution shall be collected from Eq. (36) plus the additional terms given by Eq. (29).

6. Some concluding remarks

Table 4 gives the critical cases to be investigated and all the possible kinds of resonance one can find in this dynamical system. Each one of the cases of interest shown in Table 4 must be investigated separately and will have its particular frequency response function, its particular amplitude and phase modulation equations and conditions for stability and chaos [8]. In future works, we will analyze each problem separately.

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Appendix A. Coefficients of the ordinary diferential equation for the temporal component Q_1 of $V(X, T)$

Here, we derive the coefficients for any number of admissible functions, ϕ_κ (where $\kappa = i$ or j or k or ℓ in the following expressions) [1]:

$$\phi(x) = \cosh(a_i Lx) - \cos(a_i Lx) - \alpha_i(\sinh(a_i Lx) - \sin(a_i Lx)),$$

$$\alpha_i = \frac{\cosh(a_i L) + \cos(a_i L)}{\sinh(a_i L) + \sin(a_i L)},$$

$$w_j^2 = \frac{w_n^2}{(a_1 L)^4}, \quad R_{ij}(x) = \int_0^x \phi_i'(\xi)\phi_j'(\xi)\,\mathrm{d}\xi = R_{ji}(x), \quad V_i(x) = -\int_x^1 \phi_i(\xi)\,\mathrm{d}\xi,$$

$$S_{ij}(x) = -\int_x^1 \left[\int_0^\eta \phi_i'(\xi)\phi_j'(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\eta, \quad W_{ij}(x) = -\int_x^1 \phi_i'(\xi)\phi_j(\xi)\,\mathrm{d}\xi, \quad \alpha_\ell = \int_0^1 x\phi_\ell\,\mathrm{d}x,$$

$$\beta_{i\ell} = \left[\int_0^1 \left(x\phi_i'\phi_\ell + \frac{1}{2}(x^2-1)\phi_i''\phi_\ell\right)\mathrm{d}x\right] - 1, \quad \mathscr{P}_{ij\ell} = \int_0^1 (2R_{ij}\phi_\ell - 2\phi_i'' V_j\phi_\ell - 2\phi_i'\phi_j\phi_\ell)\,\mathrm{d}x,$$

$$\lambda_{ij\ell} = \int_0^1 \left(-\frac{1}{2}R_{ij}\phi_\ell + \phi_i'' V_j\phi_\ell + \phi_i'\phi_j\phi_\ell\right)\mathrm{d}x, \quad \Lambda_{ijk\ell} = \int_0^1 (S_{jk}\phi_i''\phi_\ell + R_{jk}\phi_i'\phi_\ell)\,\mathrm{d}x,$$

$$\Gamma_{ijk\ell} = \int_0^1 \left[\frac{3}{(a_\ell L)^4}\phi_i'\phi_j''\phi_k'''\phi_\ell + \frac{3}{2(a_\ell L)^4}\phi_i''\phi_j''\phi''\phi_\ell + w_j^2(\phi_i'\phi_j\phi_k'\phi_\ell + W_{ij}\phi_k''\phi_\ell)\right]\mathrm{d}x.$$

In the cases dealt in this work, one considers only one admissible function and, therefore, one has $i = j = k = \ell = 1$.

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