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Letter to the Editor

Analysis of the simple harmonic oscillator with fractional damping

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The simple harmonic oscillator (SHO) with fractional damping is characterized by the following second order differential equation:

$$\ddot{x} + x = -\varepsilon(\dot{x})^{1/3}, \quad \varepsilon > 0, \tag{1}$$

where ε is a positive parameter. Note that this equation is of odd parity, i.e., if $x \to -x$, then the equation is invariant except for a non-essential overall negative sign. Also, observe that the damping term, on the right-side of Eq. (1), is equal to the velocity raised to the one-third power. This expression is always real valued since the cube root of a real number has a real value with the same sign as that number itself.

The purpose of this note is to show that all solutions to Eq. (1) are damped and oscillatory. The method of first order averaging [1–3] is then used to calculate an approximation to this set of solutions. Further, the analysis indicates that the system only executes a finite number of oscillations before it reaches its equilibrium state. An estimate of this number is made.

Eq. (1) can be written as a system of two first order differential equations [1,2]

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - \varepsilon y^{1/3}. \tag{2}$$

The corresponding trajectories in the $(x, y = dx/dt)$ phase space are determined by the solutions to the equation

$$\frac{dy}{dx} = -\left(\frac{x + \varepsilon y^{1/3}}{y}\right). \tag{3}$$

It follows from Eq. (2) that a single equilibrium or fixed point exists at $(\bar{x}, \bar{y}) = (0, 0)$. Note that Eq. (3) is invariant under the transformation

$$x \to -x, \quad y \to -y. \tag{4}$$

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Consequently, trajectories in the (x, y) phase space have inversion through the origin symmetry [2].

Consider the following Liapunov function for Eqs. (1) or (2) [4]:

$$V(x,y) \equiv \frac{y^2}{2} + \frac{x^2}{2}. \tag{5}$$

This function is the energy integral for the undamped SHO and has the property

$$V(x,y) > 0, \quad x \neq 0 \quad \text{and} \quad y \neq 0. \tag{6}$$

Taking the time derivative of $V(x, y)$ and replacing $\dot{x}$ and $\dot{y}$ by the expressions in Eq. (2) gives

$$\frac{dV}{dt} = y\dot{y} + x\dot{x} = -y(x + \varepsilon y^{1/3}) + xy = -\varepsilon y^{4/3}. \tag{7}$$

Since $\varepsilon > 0$, it follows that

$$\frac{dV}{dt} \leqslant 0, \tag{8}$$

from which it can be concluded that

$$\lim_{t\to\infty} \left(\frac{x^2+y^2}{2}\right) = 0. \tag{9}$$

Thus, all solutions, $x(t)$, of Eq. (1) eventually decrease to zero. The implication of this result is the fixed point at $(\bar{x}, \bar{y}) = (0, 0)$ is globally stable.

The method of first order averaging [1–3] can now be applied to Eq. (1) to calculate an analytical approximation to its oscillatory solutions. This solution is

$$x(t) = a(t)\cos[t + \phi(t)], \tag{10}$$

where the "amplitude" $a(t)$ and "phase" $\phi(t)$ are given as solutions to the following equations for the particular problem given by Eq. (1):

$$\frac{da}{dt} = -\left(\frac{\varepsilon}{2\pi}\right) a^{1/3} \int_0^{2\pi} (\sin\psi)^{4/3}\, d\psi, \tag{11}$$

$$\frac{d\phi}{dt} = -\left(\frac{\varepsilon}{2\pi a}\right) a^{1/3} \int_0^{2\pi} (\sin\psi)^{1/3} \cos\psi\, d\psi. \tag{12}$$

These calculations are made under the requirement $0 < \varepsilon \ll 1$. The integral, on the right-side of Eq. (12), is zero. This follows from the fact that the integration limits can be changed from $(0, 2\pi)$ to $(-\pi, \pi)$. Over this range of ψ values, the integrand is odd; consequently, the integral is zero. Therefore, it follows that

$$\phi(t) = \phi_0 = \textit{constant}. \tag{13}$$

It can be shown that [5]

$$(\sin\psi)^{4/3} = c_0 + c_1\cos 2\psi + c_2 \cos 4\psi + \cdots, \tag{14}$$

where $c_0 = 0.580$, $c_1 = -0.464$, etc. Using this result, the integral in Eq. (11) can be evaluated to obtain

$$\frac{da}{dt} = -(\varepsilon c_0) a^{1/3}. \tag{15}$$

The solution to Eq. (15), with the initial condition, $a(t) = a_0$, is

$$a(t) = a_0 \left(\frac{t^* - t}{t^*} \right)^{3/2}, \tag{16}$$

where the "characteristic time", t^*, is given by

$$t^* = \frac{3a^{2/3}}{2c_0\varepsilon}. \tag{17}$$

A closer examination of Eq. (16) shows that for times, $t > t^*$, the amplitude $a(t)$ is pure imaginary. Physically, $a(t)$ can always be selected to be non-negative and real [2,3]. Placing these two issues together means that Eq. (16) should be rewritten as

$$a(t) = \begin{cases} a_0 \left(\dfrac{t^* - t}{t^*} \right)^{3/2}, & 0 \leqslant t \leqslant t^*, \\ 0, & t > t^*. \end{cases} \tag{18}$$

Thus, the amplitude of the oscillator is zero for $t > t^*$ and as a result only a finite number of "cycles of oscillations" occur.

An estimate of the number of oscillation cycles, N, which take place before the amplitude goes to zero, can be calculated. First, for $0 < \varepsilon \ll 1$, the angular frequency of the free oscillations is [2,3]

$$\omega(\varepsilon) = 1 + O(\varepsilon). \tag{19}$$

Second, since the period is

$$T(\varepsilon) = \frac{2\pi}{\omega(\varepsilon)} = 2\pi + O(\varepsilon) \tag{20}$$

and because the oscillations stop after the time $t = t^*$, then

$$N \equiv \frac{t^*}{T} = \frac{3a_0^{2/3}}{4\pi c_0 \varepsilon}. \tag{21}$$

If the results of Eqs. (13) and (18) are substituted into Eq. (10), then the damped oscillations of Eq. (1) are represented by the relation

$$X(t) = \begin{cases} a_0 \left(\dfrac{t^* - t}{t^*} \right)^{3/2} \cos(t + \phi_0), & 0 \leqslant t \leqslant t^*, \\ 0, & t > t^*. \end{cases} \tag{22}$$

It is also of interest that the only other known non-linear, damped oscillator for which a finite number of oscillations occurs, is the Coulomb damped oscillator. An analysis of its solutions and related behavior is given by McLachlan [6] and Mickens [2].

The work presented here is an extension of previous studies on the so-called non-linear "fractional oscillators" [7,8]. The equations modelling such systems either have terms in their elastic forces which are fractional powers of the position x and/or also contain terms for the damping which have fractional powers.

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