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Letter to the Editor

Frequencies of beams carrying multiple masses: Rayleigh estimation versus eigenanalysis solutions

K.H. Low*

School of Mechanical and Production Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore

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1. Introduction

There have been extensive research works on the vibration analysis of beams or rods carrying concentrated masses at arbitrary locations. Approximate and exact analyses were used to obtain the natural frequencies [1–20]. The eigenfunction of the beam–mass systems was obtained by satisfying the differential equations of motion and by imposing the corresponding boundary and compatibility conditions associated to the masses [13–15]. The method of frequency determinant was then used to generate the frequency equation. It was, however, claimed that, with this method, the number of the beam equations increases as the number of attached masses increases. Therefore, the method of Laplace transform was suggested by introducing the Dirac delta function (δ) for the concentrated mass [1–3,5,6,12]. In other works, Gurgoze and Batan [11] concerned the numerical solution of the transcendental frequency equation. The characteristic equation was obtained by using Rayleigh–Ritz method [17] and free vibrations were analyzed by using the Laplace transform method [18]. Maurizi and Belles [19] compared two fundamental theories of beam vibrations. Ozkaya and Pakdemirli [20] obtained the frequencies for the clamped–clamped beam with mass and searched approximate solutions of free and forced non-linear vibrations using a perturbation method.

Low et al. [21] found that the results of experiments and the theory did not match well for beams of large slenderness ratio for centre-loaded beams. Different assumed shape functions to obtain the kinetic and potential energies of the three classical beams carrying a concentrated mass were presented [22,23]. A later work [24] showed that the correlation between theory and experiments was much improved when stretching effects were included.

In Ref. [25], the fundamental eigenvalue of beams carrying concentrated masses was predicted merely from the individual beam system carrying a single mass, by virtue of Dunkerley's formula. The time saving owing to the proposed method was illustrated in the parametric study. In another

*Tel.: +65-7910200; fax: +65-7910200.
E-mail address: mkhlow@ntu.edu.sg (K.H. Low).

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work [26,27], both the method of frequency determinant and the method of Laplace transform were considered and compared for the solution process and computation time saving.

In the present work, an Euler–Bernoulli type beam carrying multiple masses on various locations is again considered. The method of Rayleigh quotient is applied together with the respective shape function with a simple trigonometric function for a quick frequency estimation of the beam–mass system. In this paper, firstly, the solution methods for frequencies of three mass-loaded beams are presented with both the transcendental characteristic equation and the Rayleigh estimation. Secondly, parametric results by solving the eigenfrequency equation and Rayleigh's expression are presented and compared. Finally, the effectiveness and validity of Rayleigh's estimation is studied and discussed.

2. Model considered

As an example of multiple-mass loaded beams, let us consider a beam carrying three concentrated masses at $x = a_1$, $x = a_2$, and $x = a_3$, where x is the spatial co-ordinate along the beam length of l as shown in Fig. 1.

The differential equation associated with the present eigenvalue problem is known as [28,29]

$$\frac{d^4 V}{dx^4} - k^4 V = 0, \tag{1}$$

in which

$$k^4 = \frac{\rho A \omega^2}{EI}, \tag{2}$$

where ρ is the beam density, A is the cross-sectional area, E is Young's modulus, I is the moment of inertia of the beam cross-section with respect to the neutral axis of the beam, and ω represents the eigenfrequency of the beam with masses.

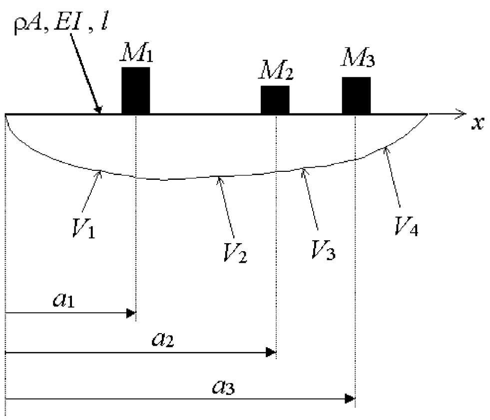

Fig. 1. Beam–mass system considered.

The general solutions of the ordinary differential equation (1) for the loaded beam system, as shown in Fig. 1, can be defined in different segments as [28,29]

$$\begin{aligned}
V_1(x) &= C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx,\\
V_2(x) &= C_5 \sin kx + C_6 \cos kx + C_7 \sinh kx + C_8 \cosh kx,\\
V_3(x) &= C_9 \sin kx + C_{10} \cos kx + C_{11} \sinh kx + C_{12} \cosh kx,\\
V_4(x) &= C_{13} \sin kx + C_{14} \cos kx + C_{15} \sinh kx + C_{16} \cosh kx, \qquad (3)
\end{aligned}$$

in which C_q ($q = 1$–16) are constants to be determined, while V_1, V_2, V_3 and V_4 are the left and right transverse displacements associated to the respective concentrated masses M_1, M_2 and M_3.

The compatibility conditions at the location of three concentrated masses in Fig. 1 are given as follows:

$$\begin{aligned}
&V_1(a_1) = V_2(a_1), \quad V_1'(a_1) = V_2'(a_1), \quad V_1''(a_1) = V_2''(a_1),\\
&V_1'''(a_1) - V_2'''(a_1) + \alpha_1 k^4 V_1(a_1) = 0,\\
&V_2(a_2) = V_3(a_2), \quad V_2'(a_2) = V_3'(a_2), \quad V_2''(a_2) = V_3''(a_2),\\
&V_2'''(a_2) - V_3'''(a_2) + \alpha_2 k^4 V_2(a_2) = 0,\\
&V_3(a_3) = V_4(a_3), \quad V_3'(a_3) = V_4'(a_3), \quad V_3''(a_3) = V_4''(a_3),\\
&V_3'''(a_3) - V_4'''(a_3) + \alpha_3 k^4 V_3(a_3) = 0, \qquad (4)
\end{aligned}$$

where primes denote differentiation with respect to the position variable x. The corresponding mass ratios have been defined by $\alpha_1 = M_1/(\rho A l)$, $\alpha_2 = M_2/(\rho A l)$ and $\alpha_3 = M_3/(\rho A l)$.

For a complete formulation of the boundary-value problem, the boundary conditions for the three beam ends considered in this work can now be specified as follows:

$$\begin{aligned}
&V = 0 \quad \text{and} \quad V' = 0 \quad \text{(clamped end)},\\
&V'' = 0 \quad \text{and} \quad V''' = 0 \quad \text{(free end)},\\
&V = 0 \quad \text{and} \quad V'' = 0 \quad \text{(pinned end)}. \qquad (5)
\end{aligned}$$

3. Frequency solutions

3.1. Eigenfrequency equations

Conditions specified in Eqs. (4) and (5) can be written in terms of C_q ($q = 1$–16) by virtue of Eq. (3)

$$\mathbf{BC} = \mathbf{0}, \qquad (6)$$

in which $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16}\}^{\mathrm{T}}$ and $\mathbf{B}$ is the 16×16 matrix associated to a particular beam type.

The frequency equation, $\det(\mathbf{B}) = 0$, is derived symbolically in this work by virtue of *Maple* software [30]. The characteristic equations are written in terms of eigenvalue β ($\beta_i = k_i l$ for mode i), position parameters η_j ($= a_j/l$), and mass ratios α_j ($= M_j/(\rho A l)$). The generated eigenfrequency (or characteristic) equations for different cases can then be numerically solved for the eigenvalues (or eigenfrequencies) by using the same software. Note that the beams with three different ends are considered here: clamped–clamped, clamped–free and pinned–pinned.

A similar frequency analysis was performed for single-mass-loaded beams with classical boundary conditions [9,16,25]. By solving the determinant of an 8×8 matrix, the eigenfrequency equation for each case was obtained and expressed explicitly in terms of eigenvalue β, position parameter η, and mass ratio α.

To begin with the simplest beam–mass model, the frequency equations for the three beams carrying a single mass are listed as follows [16,25]:

(i) *Clamped–clamped* (16 terms):

$$\begin{aligned}
&2(1-\cos\beta\cosh\beta)+\alpha\beta[\sin\beta\cosh\beta\cosh^2\beta\eta-\cos\beta\sinh\beta\cos^2\beta\eta\\
&\quad+\cos\beta\cosh\beta(\sin\beta\eta\cosh\beta\eta-\cos\beta\eta\sinh\beta\eta)\\
&\quad+\cos\beta\sinh\beta(\cos\beta\eta\cosh\beta\eta-\sin\beta\eta\sinh\beta\eta)\\
&\quad-\sin\beta\cosh\beta(\cos\beta\eta\cosh\beta\eta+\sin\beta\eta\sinh\beta\eta)\\
&\quad+\sin\beta\sinh\beta(\sin\beta\eta\cosh\beta\eta+\cos\beta\eta\sinh\beta\eta)\\
&\quad-\sin\beta\sinh\beta(\cos\beta\eta\sin\beta\eta+\cosh\beta\eta\sinh\beta\eta)\\
&\quad+\cos\beta\eta\sinh\beta\eta-\sin\beta\eta\cosh\beta\eta]=0.
\end{aligned}\tag{7}$$

(ii) *Clamped–free* (16 terms):

$$\begin{aligned}
&2(1+\cos\beta\cosh\beta)+\alpha\beta[\cos\beta\sinh\beta\cos^2\beta\eta-\sin\beta\cosh\beta\cosh^2\beta\eta\\
&\quad+\cos\beta\cosh\beta(\cos\beta\eta\sinh\beta\eta-\sin\beta\eta\cosh\beta\eta)\\
&\quad+\cos\beta\sinh\beta(\sin\beta\eta\sinh\beta\eta-\cos\beta\eta\cosh\beta\eta)\\
&\quad+\sin\beta\cosh\beta(\cos\beta\eta\cosh\beta\eta+\sin\beta\eta\sinh\beta\eta)\\
&\quad-\sin\beta\sinh\beta(\cos\beta\eta\sinh\beta\eta+\sin\beta\eta\cosh\beta\eta)\\
&\quad+\sin\beta\sinh\beta(\cosh\beta\eta\sinh\beta\eta+\cos\beta\eta\sin\beta\eta)\\
&\quad+\cos\beta\eta\sinh\beta\eta-\sin\beta\eta\cosh\beta\eta]=0.
\end{aligned}\tag{8}$$

(iii) *Pinned–pinned* (5 terms):

$$\begin{aligned}
&2\sin\beta\sinh\beta+\alpha\beta[\sin\beta\sinh\beta(\cosh\beta\eta\sinh\beta\eta-\cos\beta\eta\sin\beta\eta)\\
&\quad+\cos\beta\sinh\beta\sin^2\beta\eta-\sin\beta\cosh\beta\sinh^2\beta\eta]=0,
\end{aligned}\tag{9}$$

where $\beta^4=(kl)^4=\rho A\omega^2 l^4/(EI)$ by virtue of Eq. (2).

As the mass on a beam increases to two, the frequency equation becomes much longer. For example, the characteristic equation of pinned–pinned beams carrying two concentrated masses is given by [25]:

`pp_2M_freqn` (48 terms):

$$\begin{aligned}
&4\sin\beta\sinh\beta+\alpha_1\alpha_2\beta^2[2\cos\beta\cosh\beta+\sin\beta\cosh(\beta\eta_1)\sinh(\beta\eta_1)\cosh(\beta\eta_2)\\
&\quad\times\sinh(\beta\eta_2)\sinh\beta-\cos(\beta\eta_2)\sin\beta\sin(\beta\eta_2)\cosh(\beta\eta_1)\sinh(\beta\eta_1)\sinh\beta\\
&\quad+\cos(\beta\eta_2)\cos\beta\sin(\beta\eta_2)\sinh\beta+\cos^2(\beta\eta_1)\cos\beta\cosh^2(\beta\eta_2)\cosh\beta\\
&\quad+\cos(\beta\eta_1)\sin(\beta\eta_1)\cos\beta\sinh\beta\cos^2(\beta\eta_2)+\cdots 29\text{ terms}\ldots]\\
&\quad+2\beta[(\alpha_1+\alpha_2)(\sin\beta\cosh\beta+\cos\beta\sinh\beta)\\
&\quad-\alpha_1\cos(\beta\eta_1)\sin(\beta\eta_1)\sin\beta\sinh\beta\\
&\quad+\alpha_2\cosh(\beta\eta_2)\sin\beta\sinh\beta\sinh(\beta\eta_2)+\cdots 6\text{ terms}\ldots]=0. \qquad (10)
\end{aligned}$$

For the same boundary ends, a three-mass-loaded beam as shown in Fig. 1 can now be obtained by virtue of Eq. (6):

`pp_3M_freqn` (318 terms):

$$\begin{aligned}
&8\sin\beta\sinh\beta+\alpha_1\alpha_2\alpha_3\beta^3[\sin\beta\cosh\beta\cos^2(\beta\eta_1)\cos^2(\beta\eta_3)+\cdots 193\text{ terms}\ldots]\\
&\quad+\beta^2[2\alpha_1\alpha_3\cos\beta\cosh\beta\cosh^2(\beta\eta_1)\cos^2(\beta\eta_3)+\ldots 104\text{ terms}\ldots]\\
&\quad+\beta[4\alpha_2\sin\beta\sinh\beta\cosh^2(\beta\eta_2)\sinh^2(\beta\eta_2)+\cdots 17\text{ terms}\ldots]=0. \qquad (11)
\end{aligned}$$

Many terms in the frequency equation (10) and (11) have been omitted for clarity. It is obvious that the total number of terms increases significantly from 5 to 48 and 318, if the number of masses carried by the pinned–pinned beam is from one to two and three, respectively.

The frequency equations for the clamped–clamped and clamped–free beams carrying two/three masses are not listed here owing to the lengthiness of the expressions. In fact, the frequency equation for the two-mass-loaded clamped–clamped beam contains 117 terms [26,27], while the expression for the three-mass-loaded clamped–clamped beam is more than 50 times longer than that of pinned–pinned beam, `pp_3M_freqn`, as listed in Eq. (11).

3.2. Rayleigh estimation

The fundamental frequency of a beam carrying a mass at various positions can be obtained by substituting a specified shape function into Rayleigh's quotient [22,23]:

$$\beta^4=\frac{\int[\mathrm{d}^2v(x)/\mathrm{d}x^2]^2\,\mathrm{d}x}{\int v^2(x)\,\mathrm{d}x+\alpha v^2(a)}, \qquad (12)$$

where $v(x)$ is the shape function to be specified, while $v(a)$ is the corresponding beam deflection at $x=a$, the location of concentrated mass M.

In the case of the beam carrying multiple masses, Eq. (12) can be extended to introduce the respective concentrated masses $M_j(a_j)$ located at position $x=x_j=a_j$,

$$\beta^4=\frac{\int[\mathrm{d}^2v(x)/\mathrm{d}x^2]^2\,\mathrm{d}x}{\int v^2(x)\,\mathrm{d}x+\sum_j\alpha_jv^2(a_j)}, \qquad (13)$$

where $\alpha_j = M_j/(\rho A l)$ is the mass ratio associated to M_j. For the loaded-beam system shown in Fig. 1, Eq. (13) becomes:

$$\beta^4 = \frac{\int [\mathrm{d}^2 v(x)/\mathrm{d}x^2]^2 \,\mathrm{d}x}{\int v^2(x)\,\mathrm{d}x + \alpha_1 v^2(\eta_1) + \alpha_2 v^2(\eta_2) + \alpha_3 v^2(\eta_3)}. \tag{14}$$

Different trigonometric functions have been suggested for use in Eq. (14) as a shape function $v(x)$ for the respective beam considered in this work [26,27,31]:

(i) *clamped–clamped beam*: $v(x) = A_d(1 - \cos(2\pi x/l))$,
(ii) *clamped–free beam*: $v(x) = A_d(1 - \cos(\pi x/(2l)))$,
(iii) *pinned–pinned beam*: $v(x) = A_d \sin(\pi x/l)$,

where A_d is the amplitude of beam deflection.

3.3. Error parameter defined for comparison

For the comparison of the two methods, an error parameter is defined as

$$\varepsilon(\%) = (\beta_{ray} - \beta_{eig})100/\beta_{eig}, \tag{15}$$

in which the eigenvalue β_{eig} is obtained by solving the respective eigenfrequency equation presented in Section 3.1, whereas β_{ray} is the frequency by virtue of Rayleigh's expression, Eq. (14). Eq. (15) enables us to judge the validity of the Rayleigh estimation, if compared to that obtained by solving the eigenfrequency equation. A small error implies that Rayleigh's expression is a good approximation and should be applied since the solving of the algebraic expression is much more timesaving. In fact, the eigenfrequency equation can be solved symbolically by using Maple [30], and the computation time is much longer than the solving of Rayleigh's expression (14). However, Maple with Pentium 4 is not able to symbolically solve the frequency equations of the clamped–clamped/free beams carrying three masses. The object (i.e., frequency equation) is too large to operate or simplify, according to the message given by Maple. Therefore, the root-searching scheme [32] was adopted to obtain the eigensolutions in the present work.

4. Results and discussion

4.1. Parametric comparisons

Fig. 2 shows the results of β_{eig} and β_{ray} for a clamped–free beam with the changing η_2. It indicates the changing of the fundamental frequency of the loaded beam with a mass M_2 moving from one end to another, while two other different masses fixed at $\eta_1 = 0.2$ and $\eta_3 = 0.55$, respectively. To easily compare the two results, the error ε (in percentage) by using Rayleigh's approximation with different η_3 is given in Fig. 3. It is seen that Rayleigh's estimation is always higher than the respective eigenvalue estimated by solving the characteristic equation. This is the fact has been derived by Meirovitch [28]. Also, the error reduces as the mass places towards the free end. Note that the maximum error with respect to that obtained by eigenanalysis is less than 4%, in this particular case. It is also interesting to note that the error reaches its maximum if both the two masses M_2 and M_3 were placed near the node of the beam system, $\eta = 0.55$ in Fig. 3.

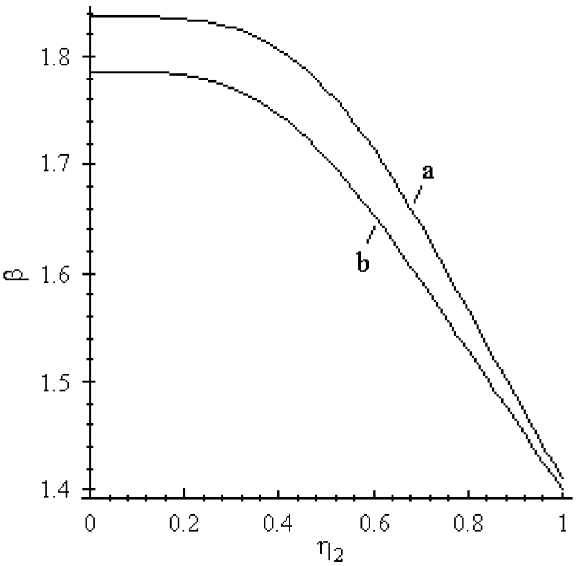

Fig. 2. Eigenvalue of the clamped–free loaded beam with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{3}$ and $\eta_1 = \frac{2}{10}$, $\eta_3 = 0.55$. (a) Rayleigh's result and (b) eigensolution.

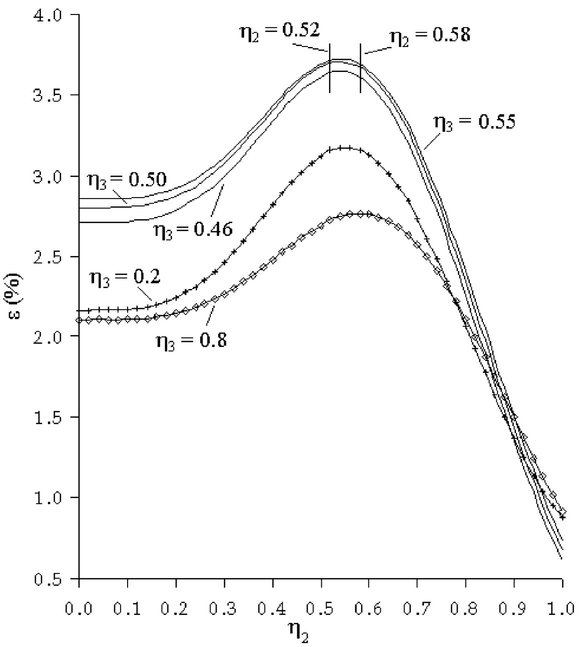

Fig. 3. Error of Rayleigh's approximation with respect to the eigensolution of the clamped–free beam with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{3}$ and $\eta_1 = \frac{2}{10}$.

Figs. 4–6 provide the errors ε (%) in changing η_2 for the three beams with various α_2. In fact, the error curve looks similar to the deflection shape of the respective beam. The error reduces, in general, for heavier M_2. For the clamped–clamped and pinned–pinned beams, two peaks occur as

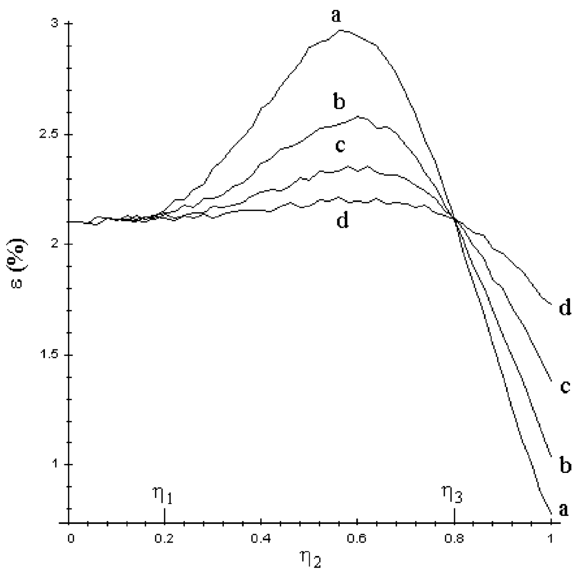

Fig. 4. Effect of α_2 on Rayleigh's error with respect to the eigensolution of the clamped–free beam with $\alpha_1 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$ and $\eta_1 = \frac{2}{10}$, $\eta_3 = \frac{8}{10}$. (a) $\alpha_2 = 1$, (b) $\alpha_2 = \frac{1}{2}$, (c) $\alpha_2 = \frac{1}{4}$ and (d) $\alpha_2 = \frac{1}{10}$.

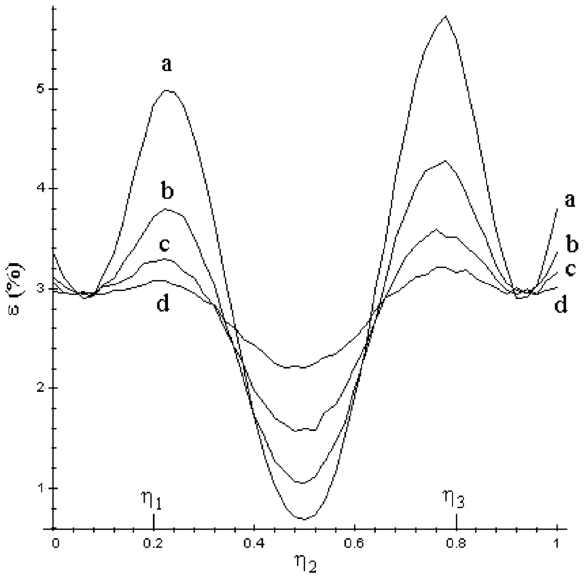

Fig. 5. Effect of α_2 on Rayleigh's error with respect to the eigensolution of the clamped–clamped beam with $\alpha_1 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$ and $\eta_1 = \frac{2}{10}$, $\eta_3 = \frac{8}{10}$. (a) $\alpha_2 = 1$, (b) $\alpha_2 = \frac{1}{2}$, (c) $\alpha_2 = \frac{1}{4}$ and (d) $\alpha_2 = \frac{1}{10}$.

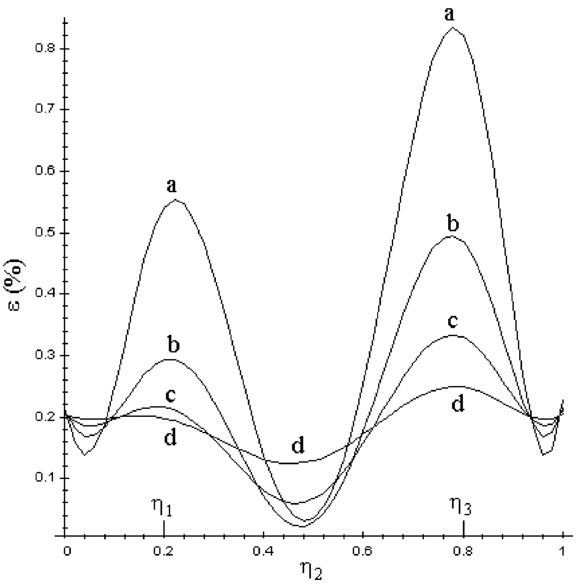

Fig. 6. Effect of α_2 on Rayleigh's error with respect to the eigensolution of the pinned–pinned beam with $\alpha_1 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$ and $\eta_1 = \frac{2}{10}$, $\eta_3 = \frac{8}{10}$. (a) $\alpha_2 = 1$, (b) $\alpha_2 = \frac{1}{2}$, (c) $\alpha_2 = \frac{1}{4}$ and (d) $\alpha_2 = \frac{1}{10}$.

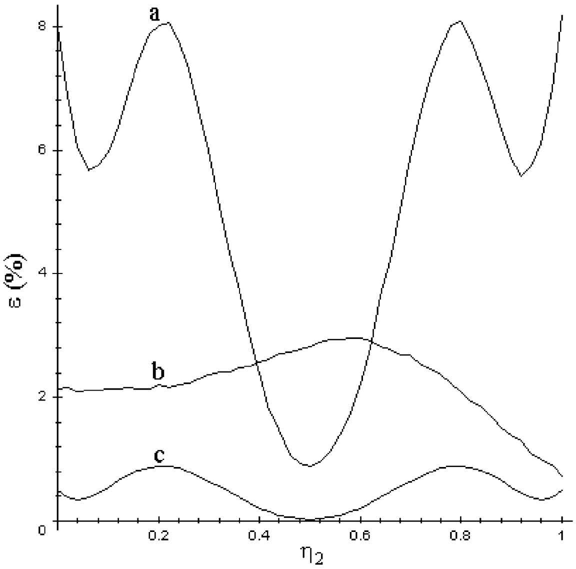

Fig. 7. Effect of boundary ends on Rayleigh's error with respect to the eigensolution of the three beams with $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and $\eta_1 = \frac{2}{10}$, $\eta_3 = \frac{8}{10}$. (a) Clamped–clamped, (b) clamped–free and (c) pinned–pinned.

the mass M_2 is placed near to any of two other masses. The peak at η_3 is higher as the total mass of $(\alpha_2 + \alpha_3)$ is higher than that of α_1.

Fig. 7 compares the errors of the three beams under the same loading. It is found that the error for the clamped–clamped/free beams is higher than that with the pinned–pinned beam. For all mass ratios of 2, the maximum error is less than 8% for the clamped–clamped beam, as illustrated in Fig. 7. This is the allowable maximum range of loaded weights for the linear model, Eq. (1), to be valid. A non-linear beam model should be adopted for cases of heavier loaded masses due to the large beam deflection.

4.2. Saving in computation

The eigensolution process for the clamped–clamped/free beams carrying three masses takes much longer time if compared to that for the pinned–pinned case. This is expected as the characteristic equation of the three-mass-loaded clamped–clamped/free beams is more than 50 times longer than that of pinned–pinned beams. Rayleigh's method by Eq. (13), with the respective trigonometry function, is highly recommended in view of its simple algebraic form and the huge time saving, if compared with that by eigenanalysis. Most importantly, the acceptable maximum errors (about 8% for the clamped–clamped beam with all α set at 2, as seen in Fig. 7). It is also believed that the computational time saved by solving Rayleigh's expression (13) could be much more significant if it is applied to the beam carrying more than three masses at various locations. Furthermore, the generation of frequency equations from Eq. (6) and the solving for β_{eig} are almost impossible for the loaded beam–mass system with higher number of masses.

5. Concluding remarks

In this paper, the validity of Rayleigh's expression (13) applied to a quick frequency estimation of uniform beams carrying multiple masses has been investigated and demonstrated. The fundamental frequencies by using the Rayleigh method have been compared to those obtained by solving the corresponding characteristic equation. It has been demonstrated that the number of terms of the characteristic equation increases significantly if the number of masses carried by the beam is increased. It is also found that Rayleigh's expression with trigonometric shape functions can generally yield good approximation if compared with the result associated to the eigenanalysis. Therefore, Rayleigh's method is highly recommended for uniform beams carrying multiple masses at various positions, in view of the significant computation time saving.

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