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Letter to the Editor

The asymptotic solution of the strongly non-linear Klein–Gordon equation

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1. Introduction

The modified LP method presented by Cheung et al. [1] is applied for extending the range to infinite dimensional systems. The strongly non-linear Klein–Gordon equation is studied. The asymptotic solution is obtained by the present method is in quite good agreement with the numerical solution. This method can be expanded into solving other strongly non-linear partial differential equations.

Because the equation and the boundary condition are limited, it is very difficult to solve the infinite dimensional system with non-linearity [2–8]. In this paper, we study the common Klein–Gordon equation

$$u_{tt} - u_{xx} + u + \varepsilon u^3 = 0, \quad u(x,0) = \cos nx, \quad u_t(x,0) = 0. \tag{1}$$

We discuss the weakly non-linear system first and then the strongly non-linear system. The result obtained by the approximate method is compared with the numerical result to examine the calculating accuracy of this method.

2. The asymptotic solution of the weakly non-linear system

In the analysis of the weakly non-linear system, that ε in Eq. (1) is a small parameter is usually assumed.

If $\varepsilon = 0$, the solution of Eq. (1) is

$$u(x,t) = \cos nx \cos k_0 t, \quad k_0^2 = n^2 + 1. \tag{2}$$

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If $\varepsilon \neq 0$, a new variable $\tau = kt$ is introduced, Eq. (1) then becomes

$$k^2 u_{\tau\tau} - u_{xx} + u + \varepsilon u^3 = 0, \quad u(x,0) = \cos nx, \quad u_\tau(\mathrm{x},0) = 0. \tag{3}$$

Let u and k be expanded in power series of ε

$$u(x,\tau) = u_0(x,\tau) + \varepsilon u_1(x,\tau) + \varepsilon^2 u_2(x,\tau) + \cdots, \tag{4}$$

$$k^2 = k_0^2 + \varepsilon k_1 + \varepsilon^2 k_2 + \cdots. \tag{5}$$

The asymptotic solution of Eq. (1) is written as follows

$$u(x,t) = \cos kt \cos nx + \varepsilon\left(\frac{3}{128k_0^2}\cos nx \cos 3kt - \frac{3}{128n^2}\cos 3nx \cos kt + \frac{1}{128}\cos 3nx \cos 3kt\right),$$

$$k = k_0\left(1 + \frac{9}{32k_0^2}\right). \tag{6}$$

3. The asymptotic solution of the strongly non-linear system

In the strongly non-linear situation ε needs not be small. If $\varepsilon = 0$, the solution of Eq. (1) is

$$u(x,t) = \cos nx \cos k_0 t, \quad k_0^2 = n^2 + 1. \tag{7}$$

If $\varepsilon \neq 0$, a new variable $\tau = kt$ is introduced, Eq. (1) became separately

$$k^2 u_{\tau\tau} - u_{xx} + u + \varepsilon u^3 = 0, \quad u(x,0) = \cos nx, \quad u_\tau(x,0) = 0. \tag{8}$$

Let k be expanded in power series of ε

$$k^2 = k_0^2 + \varepsilon k_1 + \varepsilon^2 k_2 + \cdots \tag{9}$$

A new parameter α on a similar type [1,9,10] is defined as

$$\alpha = \frac{\varepsilon k_1}{k_0^2 + \varepsilon k_1} \tag{10}$$

such that

$$\varepsilon = \frac{k_0^2}{k_1(1-\alpha)} \tag{11}$$

and

$$k^2 = \frac{k_0^2}{1-\alpha}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots) \tag{12}$$

then

$$k = k_0\left[1 + \tfrac{1}{2}\alpha + \left(\tfrac{3}{8} + \frac{\delta_2}{2}\right)\alpha^2 + \cdots)\right]. \tag{13}$$

Let u be expanded into a power series in the parameter α

$$u(x,\tau) = u_0(x,\tau) + \alpha u_1(x,\tau) + \alpha^2 u_2(x,\tau) + \cdots. \tag{14}$$

Substituting Eqs. (11), (13) and (14) into Eq. (8) yields

$$\begin{aligned}&\frac{k_0^2}{1-\alpha}(1+\delta_2\alpha^2+\delta_3\alpha^3+\cdots)(u_{0\tau\tau}+\alpha u_{1\tau\tau}+\alpha^2 u_{2\tau\tau}+\cdots)\\&\quad-(u_{0xx}+\alpha u_{1xx}+\alpha^2 u_{2xx}+\cdots)+(u_0+\alpha u_1+\alpha^2 u_2+\cdots)\\&\quad+\frac{k_0^2}{k_1(1-\alpha)}(u_0+\alpha u_1+\alpha^2 u_2+\cdots)^3=0.\end{aligned} \tag{15}$$

Equating the coefficients of like terms of α, the following set of linear partial differential equations can be obtained:

$$k_0^2 u_{0\tau\tau} - u_{0xx} + u_0 = 0, \quad u_0(x,0) = \cos nx, \quad u_{0\tau}(x,0) = 0, \tag{16}$$

$$k_0^2 u_{1\tau\tau} - u_{1xx} + u_1 = u_0 - u_{0xx} - \frac{k_0^2}{k_1}u_0^3, \quad u_1(x,0) = 0, \quad u_{1\tau}(x,0) = 0, \tag{17}$$

$$\begin{aligned}&k_0^2 u_{2\tau\tau} - u_{2xx} + u_2 = u_1 - u_{1xx} - k_0^2\delta_2 u_{0\tau\tau} - \frac{3k_0^2}{k_1}u_0^2 u_1,\\&u_2(x,0) = 0, \quad u_{2\tau}(x,0) = 0.\end{aligned} \tag{18}$$

The solution of Eq. (16) satisfying the initial conditions is given by

$$u_0 = \cos nx \cos\tau. \tag{19}$$

Substituting Eq. (19) into Eq. (17) yields

$$\begin{aligned}k_0^2 u_{1\tau\tau} - u_{1xx} + u_1 &= \left(1 + n^2 - \frac{9k_0^2}{16k_1}\right)\cos nx\cos\tau\\&\quad-\frac{k_0^2}{16k_1}(3\cos nx\cos 3\tau + \cos 3nx\cos 3\tau + 3\cos 3nx\cos\tau).\end{aligned} \tag{20}$$

To eliminate the secular term, k_1 should be obtained, $k_1 = \frac{9}{16}$. The solution u_1 satisfying the initial conditions is

$$u_1 = \frac{1}{24}\cos nx\cos 3\tau - \frac{k_0^2}{24n^2}\cos 3nx\cos\tau + \frac{k_0^2}{72}\cos 3nx\cos 3\tau. \tag{21}$$

Substituting Eqs. (19) and (21) into Eq. (18) yields

$$\begin{aligned}k_0^2 u_{2\tau\tau} - u_{2xx} + u_2 &= \left[k_0^2\delta_2 - \frac{3k_0^2}{k_1}\left(\frac{1}{48} - \frac{k_0^2}{48n^2} + \frac{k_0^2}{288}\right)\right]\cos nx\cos\tau\\&\quad+A_1\cos nx\cos 3\tau + A_2\cos nx\cos 5\tau + A_3\cos 3nx\cos\tau\\&\quad+A_4\cos 3nx\cos 3\tau + A_5\cos 3nx\cos 5\tau + A_6\cos 5nx\cos\tau\\&\quad+A_7\cos 5nx\cos 3\tau + A_8\cos 5nx\cos 5\tau,\end{aligned} \tag{22}$$

where

$$A_1 = \frac{k_0^2}{24} - \frac{3k_0^2(1/16 + k_0^2/144 - k_0^2/96n^2)}{4k_1}, \quad A_2 = -\frac{3k_0^2(1/32 + k_0^2/288)}{4k_1},$$

$$A_3 = -\frac{3k_0^2}{8} - \frac{k_0^2}{24n^2} - \frac{3k_0^2(1/96 + k_0^2/144 - k_0^2/16n^2)}{4k_1},$$

$$A_4 = \frac{k_0^2}{72} + \frac{k_0^2}{8n^2} + \frac{3k_0^2(1 + k_0^2 - k_0^2/n^2)}{192k_1}, \quad A_5 = -\frac{k_0^2}{128k_1},$$

$$A_6 = -\frac{3k_0^2(k_0^2/288 - k_0^2/32n^2)}{4k_1}, \quad A_7 = -\frac{3k_0^2(k_0^2/144 - k_0^2/96n^2)}{4k_1}, \quad A_8 = -\frac{k_0^2}{384k_1}. \tag{23}$$

To eliminate the secular term, we obtain

$$\delta_2 = \frac{1}{4k_1}\left(\frac{1}{48} - \frac{k_0^2}{48n^2} + \frac{k_0^2}{288}\right).$$

Finally the asymptotic solution of the strongly non-linear Klein–Gordon Equation is given by

$$u(x,t) = \cos nx \cos kt + \alpha\left(\tfrac{1}{24}\cos nx \cos 3kt - \frac{k_0^2}{24n^2}\cos 3nx \cos kt + \frac{k_0^2}{72}\cos 3nx \cos 3kt\right),$$

$$k = k_0\left[1 + \tfrac{1}{2}\alpha + \left(\tfrac{3}{8} + \frac{\delta_2}{2}\right)\alpha^2\right]. \tag{24}$$

4. Comparison between the asymptotic solution and the numerical solution

This section gives the results calculated by the asymptotic method and the numerical method of central difference interpolation. We take the bounded domain [0,50], time step $\Delta t = 0.05$, space step $\Delta x = 0.05$, $\varepsilon = 2$ and $n = 1$.

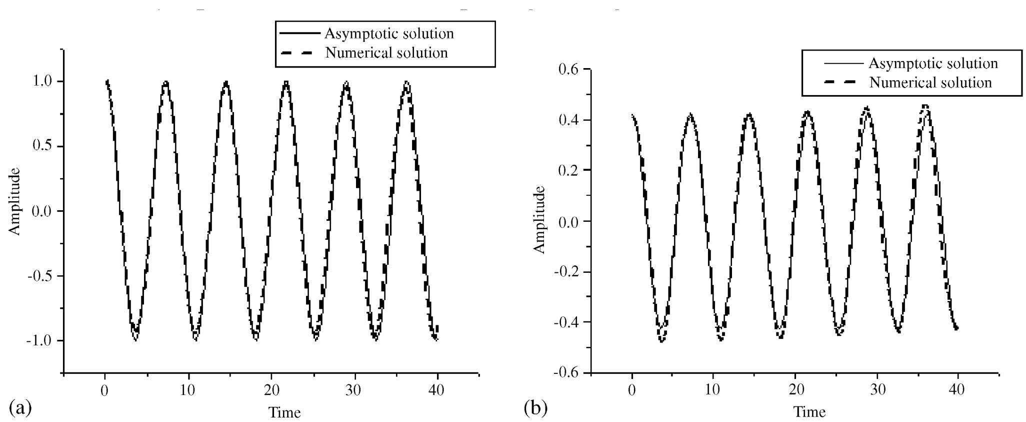

Fig. 1. Solution of equation with $x = 25$ (a), $x = 20$ (b).

The approximate analytical results obtained by using Eq. (24) are also plotted in the Fig. 1. The asymptotic solution is in quite good agreement with the numerical solution.

5. Conclusion

The modified LP method can be applied to infinite dimensional systems. It is an effective method for studying the strongly non-linear Klein–Gordon equation. The approximate results obtained by the present method have very high accuracy comparing with the numerical method. This method has enormous potentialities to be expanded into solving other strongly non-linear partial differential equations.

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