ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 268 (2003) 1041–1053

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Letter to the Editor

The complete-similitude scale models for predicting the vibration characteristics of the elastically restrained flat plates subjected to dynamic loads

Jia-Jang Wu*

Department of Marine Engineering, National Kaohsiung Institute of Marine Technology, No. 142, Hai-Chuan Road, Nan-Tzu, Kaohsiung 811, Taiwan, Republic of China

Received 27 February 2003; accepted 17 March 2003

1. Introduction

In order to predict the physical characteristics of the full-size system from those of its scale models, the similitude conditions (or the scaling laws) between the full-size system and its scale models must be satisfied. Several researchers have studied the relating problems. For example, Qian et al. [1] have studied the scaling laws for impact damage in fibre composites. Vassalos [2] has investigated the modelling and similitude of marine structures. Some valuable information concerning the appropriate use of models in the design of marine structures is presented. Safoniuk et al. [3] have proposed an approach to scale-up the three-phase fluidized beds with the aid of the Buckingham π theorem [4]. Stimitses and Rezaeepazhand et al. [5–7] have established a scale model for predicting the free vibration and buckling of laminated shell. In their reports, the similitude theory [8] is used to establish the similarity between the chosen structural systems. Wu et al. [9] have presented the scaling laws for the prediction of the vibration characteristics of a full-size crane structure from those of a scale model by means of the similitude theory [8] and the dimensional analysis [10].

From the foregoing literature, one sees that the researchers usually established the scaling laws between the full-size system and scale model based on the theory of similitude. Then, the experiments or numerical analyses were performed to validate the presented scaling laws. Since the information concerning the scaling issues for vibration characteristics of the plate-typed structures subjected to moving loads is not found yet, the title problem is studied here and it is hoped to provide a systematic method for establishing the complete-similitude scale models. It has been found that the vibration characteristics of the full-size system, such as natural frequencies, mode shapes and the transverse deflections, can be accurately predicted by means of the corresponding ones of its scale models and the associated scaling factors.

*Tel.: +886-7-8100888; fax: +886-6-2808458.
E-mail address: jjangwu@mail.nkimt.edu.tw (J.-J. Wu).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00303-1

2. Derivation of scaling laws

In general, the physical characteristics of a full-size system will be proportional to those of its scale model if the latter is properly scaled. Based on the similitude theory, one may assure the similarity between the physical characteristics of a full-size system and the corresponding ones of the scale model. For the static analysis, the geometric similitude between the full-size system and its scale model is enough. For the dynamic analysis, both the kinematic and the dynamic similitudes between the two systems are also required, in addition to the geometric similitude. In this section, the similarity conditions (i.e., *scaling laws*) for the dynamic similitude (or similarity) between the full-size plate-typed system and its scale model are presented.

For an undamped uniform rectangular plate (see Fig. 1), the equation of motion is given by [11]

$$D\left[\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right] + \mu \frac{\partial^2 w(x,y,t)}{\partial t^2} = F(x,y,t), \tag{1}$$

where

$$D = Eh^3/[12(1-v^2)] \tag{2}$$

is the bending rigidity of the plate, E is the Young's modulus, h is thickness of the plate, v is the Poisson ratio, μ is mass per unit area of the plate, while $F(x,y,t)$ and $w(x,y,t)$, respectively, represent the external load per unit area of the plate and the vertical (transverse) deflection of the plate at position (x,y) and time t.

If a concentrated force with magnitude $f(t) = F(x,y,t)\cdot b\ell$ applies at (x_f, y_f) of the plate (see Fig. 1), then Eq. (1) can be re-written as

$$\left[\frac{Eh^3}{12(1-v^2)}\right] b\ell \left[\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right] + \rho b h \ell \frac{\partial^2 w(x,y,t)}{\partial t^2} = f(t)\cdot \delta(x-x_f)\cdot \delta(y-y_f), \tag{3}$$

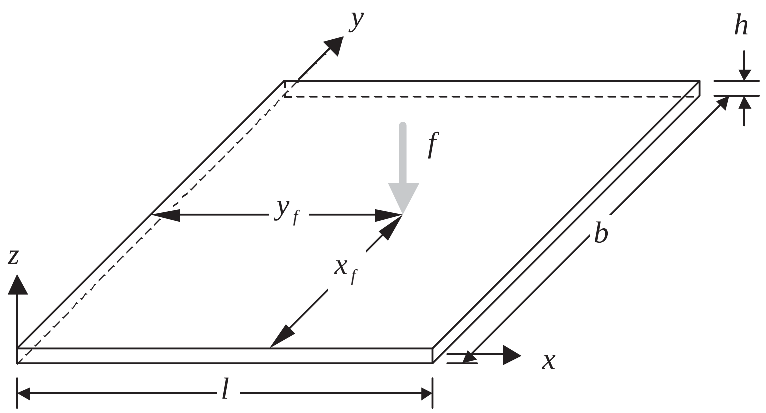

Fig. 1. A flat plate subjected to a concentrated force f located at (x_f, y_f).

where b and ℓ, respectively, represent the width and length of the rectangular plate, ρ is the mass density of the plate material and $\delta(\cdot)$ denotes the Dirac delta function. It is noted that the two important scalable parameters, plate width b and length ℓ disappearing in Eq. (1) is now to appear in Eq. (3).

From Eq. (3), one obtains a set of scalable parameters capable of representing the physical characteristics of the system

$$\{\bar{\phi}\} = [b \quad \ell \quad h \quad w \quad x \quad y \quad t \quad f]^{\mathrm{T}}, \tag{4}$$

where $\{\bar{\phi}\}$ denotes a vector composed of the following parameters: plate width (b), plate length (ℓ), plate thickness (h), transverse deflection (w), planar (horizontal) co-ordinates (x and y), time (t) and point force (f).

The requirement for the complete similitude is that the ratios between all the scalable parameters for the full-size system and those for the scale model must be equal to some constant ratios. These constant ratios are generally called the *scaling factors* and are necessarily dimensionless. Based on the parameters appearing in Eq. (4), the scaling factors are defined by

$$\lambda_\phi = \phi_s/\phi_F \quad \text{with } \phi = b, \ell, h, w, x, y, t, f, \tag{5}$$

where λ_ϕ represents the *scaling factor* for the parameters ϕ $(= b, \ell, h, w, x, y, t, f)$, and the subscripts F and s, respectively, represent the "full-size" system and the "scale" model.

For the full-size system, Eq. (3) can be written as

$$\left[\frac{Eh_F^3}{12(1-\nu^2)}\right] b_F \ell_F \left[\frac{\partial^4 w_F}{\partial x_F^4} + 2\frac{\partial^4 w_F}{\partial x_F^2 \partial y_F^2} + \frac{\partial^4 w_F}{\partial y_F^4}\right] + \rho b_F h_F \ell_F \frac{\partial^2 w_F}{\partial t_F^2} = f_F(t_F)\cdot\delta(x_F - x_{f_F})\cdot\delta(y_F - y_{f_F}). \tag{6}$$

It is evident that w_F denotes the transverse deflection of the full-size plate at the position (x_F, y_F) and time t_F, i.e., $w_F \equiv w_F(x_F, y_F, t_F)$.

Similarly, for the scale model, Eq. (3) can be written as

$$\left[\frac{Eh_s^3}{12(1-\nu^2)}\right] b_s \ell_s \left[\frac{\partial^4 w_s}{\partial x_s^4} + 2\frac{\partial^4 w_s}{\partial x_s^2 \partial y_s^2} + \frac{\partial^4 w_s}{\partial y_s^4}\right] + \rho b_s h_s \ell_s \frac{\partial^2 w_s}{\partial t_s^2} = f_s(t_s)\cdot\delta(x_s - x_{f_s})\cdot\delta(y_s - y_{f_s}). \tag{7}$$

It is noted that $w_s \equiv w_s(x_s, y_s, t_s)$, besides, the mass density for the full-size system and that for the scale model are assumed to be the same, i.e., $\rho_F = \rho_s = \rho$.

Substituting Eq. (5) into Eq. (7) leads to

$$(\lambda_h^3 \lambda_b \lambda_\ell)\left[\frac{Eh_F^3}{12(1-\nu^2)}\right] b_F \ell_F \left[\left(\frac{\lambda_w}{\lambda_x^4}\right)\frac{\partial^4 w_F}{\partial x_F^4} + 2\left(\frac{\lambda_w}{\lambda_x^2 \lambda_y^2}\right)\frac{\partial^4 w_F}{\partial x_F^2 y_F^2} + \left(\frac{\lambda_w}{\lambda_y^4}\right)\frac{\partial^4 w_F}{\partial y_F^4}\right] + \left(\frac{\lambda_b \lambda_h \lambda_\ell \lambda_w}{\lambda_{\mathrm{t}}^2}\right)\rho b_F h_F \ell_F \frac{\partial^2 w_F}{\partial t_F^2} = \lambda_f f_F(t_F)\cdot\delta(x_F - \lambda_x x_{f_F})\cdot\delta(y_F - \lambda_y y_{f_F}). \tag{8}$$

From Eqs. (6) and (8), one sees that the solution of Eq. (8) will be proportional to that of Eq. (6) if the ratios between the coefficients of $\partial^4 w_F/\partial x_F^4$, $\partial^4 w_F/\partial x_F^2 \partial y_F^2$, $\partial^4 w_F/\partial y_F^4$, $\partial^2 w_F/\partial t_F^2$ and $f_F(t_F)$ in Eq. (8) and the corresponding ones in Eq. (6) are equal to a constant, i.e.,

$$\lambda_h^3 \lambda_b \lambda_\ell \frac{\lambda_w}{\lambda_x^4} = \lambda_h^3 \lambda_b \lambda_\ell \frac{\lambda_w}{\lambda_x^2 \lambda_y^2} = \lambda_h^3 \lambda_b \lambda_\ell \frac{\lambda_w}{\lambda_y^4} = \frac{\lambda_b \lambda_h \lambda_\ell \lambda_w}{\lambda_t^2} = \lambda_f \quad \text{(a constant)} \tag{9}$$

Eqs. (9) constitute the scaling laws, since all the scaling factors, λ_ϕ ($\phi = b, \ell, h, w, x, y, t, f$), must satisfy Eqs. (9) then the scale model may be completely similar to the full-size system. For convenience, all the scaling factors appearing in the scaling laws defined by Eq. (9) are called the *explicit* scaling factors hereafter.

It is noted that the scaling laws derived in this paper are based on the theory of "thin" plates. For other types of plates, the scaling laws must be re-derived according to their equations of motion. However, the scaling laws will not change with the dimensions of the plates if their equations of motion are in the same form.

3. Solution for the explicit scaling factors

Eqs. (9) consist of four equations and eight unknowns (λ_ϕ, $\phi = b, \ell, h, w, x, y, t, f$). Because the total number of equations is less than that of the unknowns, solving Eqs. (9) for the unknowns λ_ϕ one may obtain many sets of scaling factors. In this paper, one solution for the scaling factors was determined by selecting the scaling factor for the transverse deflection defined by the equation of motion, (6) or (8), λ_w, as the fundamental scaling parameter and introducing the following relationships obtained from the dimensional analysis theory [10]

$$\lambda_\kappa = \lambda_w \quad (\kappa = b, \ell, h, w, x, y, t), \tag{10a}$$

$$\lambda_f = \lambda_w^2. \tag{10b}$$

It is evident that the scaling factors given by Eq. (10) satisfy the scaling laws defined by Eqs. (9). For this reason, Eqs. (10) were selected as the solution for the scaling factors in this paper. It is noted that Eq. (10b) is derived from the expression $f(t) = F(x, y, t) \cdot b\ell$.

Eqs. (10) implies that if the width b_s, the height h_s and the length ℓ_s of the scale model are equal to λ_w times the corresponding ones of the full-size system, and the external load f_s of the scale model is equal to λ_w^2 times that of the full-size system, then the dynamic responses of the scale model, w_s, due to the external load f_s applied at position (x_s, y_s) and at time t_s will be equal to λ_w times those of the full-size system, w_F, due to the external load $f_F = f_s/\lambda_w^2$ applied at position (x_F, y_F) and at time t_F. Where $t_F = t_s/\lambda_t$, $x_F = x_s/\lambda_x$ and $y_F = y_s/\lambda_y$.

4. Determination of the implicit scaling factors

The object of this paper is to predict the physical characteristics of a full-size elastically restrained flat plat subjected to the moving loads by means of its scale model, hence, all the physical parameters affecting the dynamic behaviour of the full-size system must be properly scaled. It is evident that, in addition to the *explicit* scaling factors appearing in the scaling laws derived from the governing equations for the full-size system and the scale model, the *implicit* scaling factors disappearing in the scaling laws but relating to the supporting (boundary) conditions and the excitation mechanisms must also be suitably defined, then the complete-similitude conditions between the full-size system and the scale model may be satisfied. The *explicit* scaling factors include those for the length, width, thickness and displacement of the plate, those for the positions and magnitude of the external force, and that for time. They have been determined by Eqs. (10). However, the *implicit* scaling factors, such as those for the translational springs and rotational springs being used to support the flat plate, those for the excitation frequency and moving-load speed, and those for the damping ratios and natural frequencies of the plate, are not defined yet. These *implicit* scaling factors will be determined in the following by means of the dimensional analysis theory. It is noted that the scaling factor for the transverse deflection of the plate, λ_w, is used as the fundamental scaling parameter in this paper.

Since the dimension for the natural frequency (ω) of the structure or that for the excitation frequency (Ω) of the external load is the reciprocal of time (t), i.e.,

$$D_\omega = D_\Omega = 1/D_t, \tag{11}$$

where the symbol D_v represents "the dimension of v" ($v = \omega$, Ω or t), the scaling factor for the natural frequency of the structure, λ_ω, and that for the excitation frequency of the external load, λ_Ω, are given by

$$\lambda_\omega = \lambda_\Omega = 1/\lambda_t = 1/\lambda_w. \tag{12}$$

It is noted that the last dimensional analysis result given by Eq. (12) agrees with the result of Ref. [9] derived from the physical formula: $\omega \propto \sqrt{EI/(\rho A)}$.

By means of the similar dimensional analysis technique, one may obtain the scaling factors for the translational spring (k_ℓ), the rotational spring (k_r), the moving-load speed (V) and the damping ratio (ξ). The results were listed in Table 1.

In deriving the scaling factor for the rotational springs (k_r), it is noted that the rotation angle (θ) is a non-dimensional quantity, thus, $D_\theta = 1$. Besides, the damping ratio (ξ) is non-dimensional, thus, its scaling factor is $\lambda_\xi = 1$.

From Table 1 one sees that both the values of λ_V and λ_ξ are equal to one. This means that based on the *explicit* scaling factors given by Eqs. (10), both the moving-load speed (V) and the damping ratio (ξ) for the scale model must be equal to those for the full-size system, then the scale model may be complete similar to the full-size model. In other words, it is not necessary that each parameter of the full-size system must be scaled down (or up), then the scale model may be complete similar to the full-size system.

Table 1
The scaling factors for the natural frequency (ω), excitation frequency (Ω), translational spring (k_ℓ), rotational spring (k_r), moving-load speed (V) and damping ratio (ξ)

Parameters	Dimensional analyses	Scaling factors
Natural frequency (ω) or excitation frequency (Ω)	$D_\omega = D_\Omega = 1/D_t$	$\lambda_\omega = \lambda_\Omega = 1/\lambda_t = 1/\lambda_w$
Translational spring (k_ℓ)	$D_{k_\ell} = D_f/D_w$	$\lambda_{k_\ell} = \lambda_f/\lambda_w = \lambda_w^2/\lambda_w = \lambda_w$
Rotational spring (k_r)	$D_{k_r} = D_f D_w/D_\theta$	$\lambda_{k_r} = \lambda_f \lambda_w/1 = \lambda_w^2 \lambda_w/1 = \lambda_w^3$
Moving-load speed (V)	$D_V = D_w/D_t$	$\lambda_V = \lambda_w/\lambda_t = \lambda_w/\lambda_w = 1$
Damping ratio (ξ)	—	$\lambda_\xi = 1$

5. Dynamic responses of the rectangular plate subjected to dynamic loads

In this paper, the natural frequencies and the associated mode shapes of the elastically restrained flat plate (see Figs. 2, 4 and 6) were determined by the finite element method (FEM). Either the full-size system or the scale model was subdivided into 10×10 identical rectangular plate elements bounded by 121 nodes (see Figs. 2, 4 and 6). The Jacobi method [12] is used to calculate the lowest 10 natural frequencies and mode shapes of the full-size system and the corresponding ones of the scale model. Since the computing time required by the Jacobi method will dramatically increase with increasing the total degree of freedom for the entire structural system, the other appropriate technique [12] should be used when the complex structure with many degrees of freedom is investigated. The forced vibration responses, $\{w(x, y, t)\}$, of the full-size system and the scale model due to the dynamic loads, $\{\bar{f}(t)\}$, were determined by the mode-superposition method incorporated with the Duhamel integration [12].

The fundamental scaling factor for the transverse deflection of the plate was assumed to be $\lambda_w = 1/10$. The lowest ten modes, each with damping ratio $\xi_i = 0.005$ ($i = 1$ to 10), were used for the superposition method. The time step size is $\Delta t_F = 0.005$ s for the full-size system and $\Delta t_s = \Delta t_F \lambda_t = \Delta t_F \lambda_w = 0.0005$ s for the scale model and all the initial conditions were assumed to be "at rest".

6. Numerical results and discussions

In this section, the scaling factors given by Eqs. (10) and Table 1 were validated with the free and forced vibration characteristics of a flat plate elastically restrained by the translational and rotational springs along the left side AB and the right side CD of the plate (see Figs. 2, 4 and 6), and subjected to a stationary harmonic load, $\tilde{f}(t) = \tilde{f}_0 \sin \Omega t$, and a point load with magnitude f_0 moving from left side to right side along the centreline of the plate in the $\bar{x}$ direction with constant

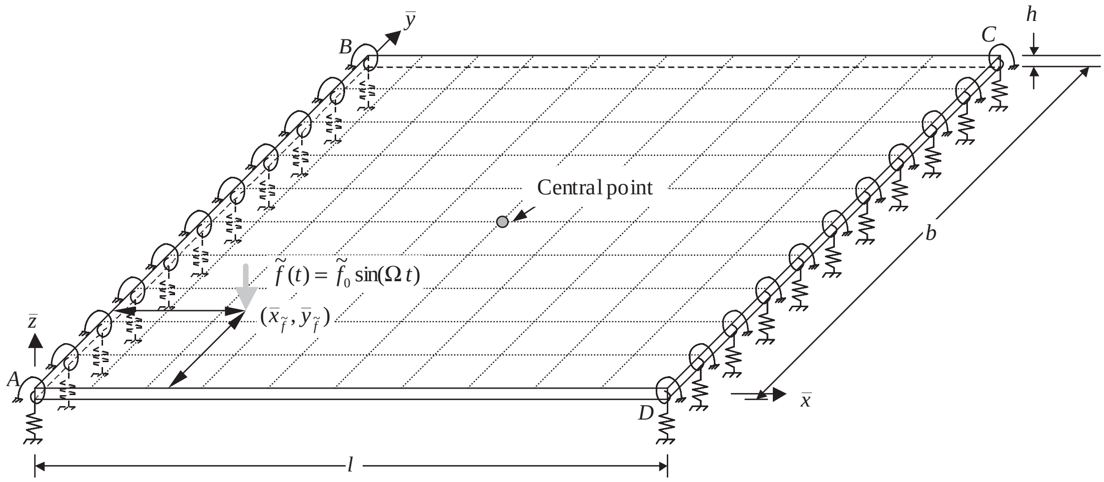

Fig. 2. An elastically restrained flat plate subjected to a stationary harmonic load, $\tilde{f}(t) = \tilde{f}_0 \sin \Omega t$, located at $(\bar{x}_{\tilde{f}}, \bar{y}_{\tilde{f}})$.

speed V. The full-size plate and its scale model were made of the same material with Young's modulus $E = 206.8\,\text{GN/m}^2$, mass density $\rho = 7820\,\text{kg/m}^3$ and the Poisson ratio $\nu = 0.29$. Based on the assumed scaling factor, $\lambda_w = 1/10$, all the pertinent parameters for the full-size system and the scale model were calculated and listed in Table 2. The requirement for the selection of a fundamental scaling factor is that the relationships between all the scaling factors defined by Eqs. (10a) and (10b) must be satisfied. Because the scaling factors for length (λ_ℓ), width (λ_b), thickness (λ_h) are equal to that for transverse ($\bar{z}$) deflection (λ_w), i.e., $\lambda_\ell = \lambda_b = \lambda_h = \lambda_w$, as one may see from Eq. (10), the transverse ($\bar{z}$) responses of the plate will remain unchanged no matter whether λ_ℓ, λ_b, λ_h or λ_w is selected as the fundamental scaling factor.

6.1. Validation of scaling laws by free vibration characteristics

In order to perform the forced vibration analyses using the mode superposition method, the lowest ten natural frequencies and mode shapes of the vibrating systems were determined. Table 3 shows the lowest five natural frequencies for the full-size system and the 1/10 scale model. In which, the natural frequencies listed in the second and the third columns are, respectively, obtained from the Jacobi method and the Lanczos method [13]. Since the associated natural frequencies obtained from the two different methods are very close to each other, the computer program developed for this research based on the Jacobi method should be reliable. From the final column of the table, one sees that the natural frequency ratio, $\lambda_{\omega,i} = \omega_{s,i}/\omega_{F,i}$, for each mode

Table 2
Parameters for elastically restrained flat plates with scaling factor $\lambda_w = 1/10$

Parameters	Full-size system	Scale model
Length ℓ (m)	$\ell_F = 10.0$	$\ell_s = \ell_F\lambda_\ell = \ell_F\lambda_w = 1.0$
Width b (m)	$b_F = 10.0$	$b_s = b_F\lambda_b = b_F\lambda_w = 1.0$
Thickness h(m)	$h_F = 0.1$	$h_s = h_F\lambda_h = h_F\lambda_w = 0.01$
Damping ratio ξ	$\xi_F = 0.005$	$\xi_s = \xi_F\lambda_\xi = 0.005$ ($\lambda_\xi = 1$)
Translational spring constant, k_ℓ (MN/m)	$k_{\ell F} = 290$	$k_{\ell s} = k_{\ell F}\lambda_{k_\ell} = k_{\ell F}\lambda_w = 29$
Rotational spring constant, k_r (MN m/rad)	$k_{rF} = 340$	$k_{rs} = k_{rF}\lambda_{k_r} = k_{rF}\lambda_w^3 = 0.34$
Size of each plate element, $\Delta\ell \times \Delta b \times \Delta h$ (m)	1.0×1.0×0.1	0.1 × 0.1 × 0.01

Table 3
The lowest five natural frequencies of elastically restrained flat plates

Mode no. i	Natural frequencies, ω_i (Hz)			Frequency ratios, $\lambda_{\omega,i} = \omega_{s,i}/\omega_{F,i}$
	Full-size system, $\omega_{F,i}$		Scale model, $\omega_{s,i}$	
	Jacobi	Lanczos [13]		
1	5.3759	5.3786	53.7589	9.9999
2	6.3033	6.3065	63.0332	10.0000
3	10.0779	10.0831	100.7795	10.0000
4	14.8467	14.8542	148.4671	10.0000
5	16.0156	16.0237	160.1560	10.0000

is very close to 10.0, i.e., $\lambda_{\omega,i} \approx 10.0$ for $i = 1 \sim 5$. This result agrees with the scaling factor for frequency, $\lambda_\omega = 1/\lambda_w = 1/\frac{1}{10} = 10$, as one may see from Table 1. It is noted that each of the natural frequencies for the full-size system shown in the 2nd and 3rd columns of Table 3 and the corresponding ones for the scale model shown in the 4th column of Table 3, have the same corresponding mode shapes.

6.2. Forced vibration characteristics due to a stationary harmonic load

In this section, the forced vibration characteristics of the full-size system and the scale model respectively subjected to a stationary harmonic force, $\tilde{f}_F(t_F) = \tilde{f}_{F0}\sin(\Omega_F t_F)$ and $\tilde{f}_s(t_s) = \tilde{f}_{s0}\sin(\Omega_s t_s)$, were investigated and then the reliability of the presented scaling laws were validated. The location of the force $\tilde{f}_F(t_F)$ is at $\bar{x}_{\tilde{f}_F} = 2.5$ and $\bar{y}_{\tilde{f}_F} = 2.5$ m on the full-size system, while that of $\tilde{f}_s(t_s)$ is at $\bar{x}_{\tilde{f}_s} = 0.25$ and $\bar{y}_{\tilde{f}_s} = 0.25$ m on the scale model. The magnitudes for the pertinent parameters associated with the two stationary harmonic forces, $\tilde{f}_F(t_F)$ and $\tilde{f}_s(t_s)$, were calculated and shown in Table 4 based on the assumed scaling factor $\lambda_w = 1/10$.

Fig. 3 shows three time–history curves for the vertical ($\bar{z}$) central displacements of the full-size plate. Among which, the thick dashed curve (**- - -**) and the thin solid curve with circles (—○—), respectively, represent the time histories for the vertical ($\bar{z}$) central displacements of the full-size plate, $\bar{z}_F(t)$, by using the mode superposition method (MSM) and the Newmark method [14], while the thin solid curve with stars (—★—) represents those obtained from the scale model and the scaling laws, i.e., $\bar{z}_F(t) = \bar{z}_s(t)/\lambda_w = 10\bar{z}_s(t)$ with $\lambda_w = 1/10$. Because the thick dashed curve (- - -) is in good agreement with the thin solid curve with circles (—○—), it is believed that the computer program developed for this paper based on the mode superposition method (MSM) should be reliable. For convenience, the values of $\bar{z}_F(t)$ and $\bar{z}_s(t)/\lambda_w$ are called the *theoretical* value and *predicted* value of the vertical ($\bar{z}$) central displacements of the full-size plate, respectively, and the associated curves, $\bar{z}_F(t_F)$ versus t_F and $\bar{z}_s(t_s)/\lambda_w$ versus t_s/λ_t, are called the *theoretical* and *predicted* time histories, respectively. From the Fig. 3, it can be seen that the *theoretical* and *predicted* time histories are in good agreement.

6.3. Forced vibration characteristics due to a moving point load

This subsection validates the scaling laws using the forced vibration characteristics due to a moving point load. All the conditions for this example are exactly the same as those for the last example except that the stationary harmonic load, $\tilde{f}(t)$, is replaced by a point load with magnitude

Table 4
Parameters for the stationary harmonic loads applied on the full-size system and the scale model with scaling factor $\lambda_w = 1/10$

Parameters	Full-size system	Scale model
Force amplitude, f_0 (N)	$f_{F0} = 7480.0$	$f_{s0} = f_{F0}\lambda_f = f_{F0}\lambda_w^2 = 74.8$
Excitation frequency, Ω (Hz)	$\Omega_F = 5.3759$	$\Omega_s = \Omega_F\lambda_\Omega = \Omega_F/\lambda_w = 53.759$
Time duration, $\bar{t}(s)$	$\bar{t}_F = 10.0$	$\bar{t}_s = t_F\lambda_t = t_F\lambda_w = 1.0$
$\bar{x}$ co-ordinate of load	$\bar{x}_{\tilde{f}_F} = 2.5$ m	$\bar{x}_{\tilde{f}_s} = \bar{x}_{\tilde{f}_F}\lambda_x = \bar{x}_{\tilde{f}_F}\lambda_w = 0.25$ m
$\bar{y}$-co-ordinate of load	$\bar{y}_{\tilde{f}_F} = 2.5$ m	$\bar{y}_{\tilde{f}_s} = \bar{y}_{\tilde{f}_F}\lambda_y = \bar{y}_{\tilde{f}_F}\lambda_w = 0.25$ m

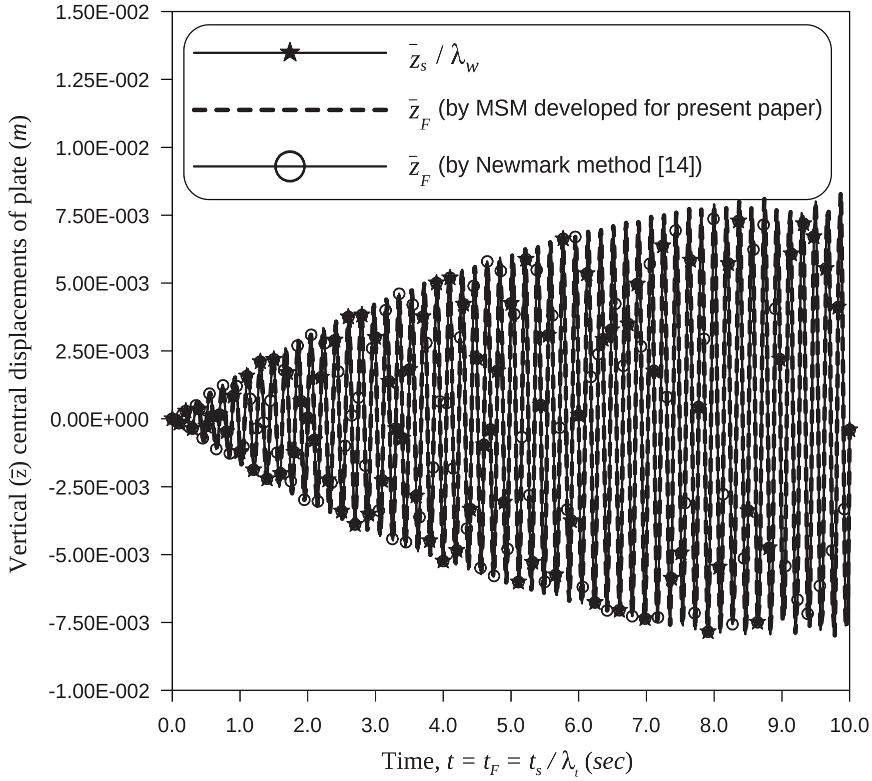

Fig. 3. Theoretical and predicted time histories for the vertical ($\bar{z}$) central displacements of the elastically restrained flat plate due to a stationary harmonic load $\tilde{f}_F(t) = 7480\sin(5.3759t_F)\ N$ with scaling factor $\lambda_w = 1/10$. (The corresponding load for the scale model is $\tilde{f}_s(t) = 74.80\sin(53.759t_s)N$).

f_0 moving from left side to right side along the centreline of the plate in the $\bar{x}$ direction with constant speed V, as shown in Fig. 4. The magnitude and moving speed of the point load, and the time duration that numerical analyses were performed were listed in Table 5. The initial and final positions for f_F are ($\bar{x}_{f_F} = 0\,\mathrm{m}$, $\bar{y}_{f_F} = 5\,\mathrm{m}$) and ($\bar{x}_{f_F} = 10\,\mathrm{m}$, $\bar{y}_{f_F} = 5\,\mathrm{m}$) on the full-size system, respectively, while those for f_s are ($\bar{x}_{fs} = 0\,\mathrm{m}$, $\bar{y}_{fs} = 0.5\,\mathrm{m}$) and ($\bar{x}_{fs} = 11\,\mathrm{m}$, $\bar{y}_{fs} = 0.5\,\mathrm{m}$) on the scale model, respectively.

The time histories for the vertical ($\bar{z}$) central displacements of the full-size plate were shown in Fig. 5. The denotations for the curves in the figure are exactly the same as those in Fig. 3. It is apparent that the *theoretical* and *predicted* time histories are also in good agreement. From all the foregoing investigations, one sees that both the free and forced vibration characteristics of a full-size system can be precisely predicted with the scale model and the associated scaling laws.

6.4. Dynamic responses of an elastically restrained flat plate supported by an elastic foundation and subjected to a stationary harmonic load and a moving point load

After the reliability of the theory the computer programs for this paper have been confirmed, this subsection tries to predict the dynamic responses of the foregoing elastically restrained

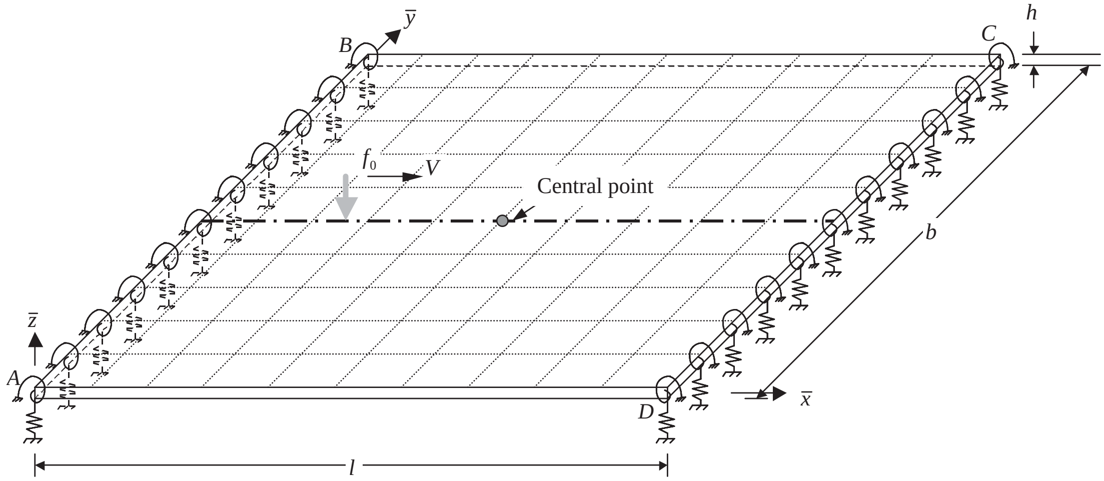

Fig. 4. An elastically restrained flat plate subjected to a point load f_0 moving from left side to right side along the centreline of the plate in the $\bar{x}$ direction with constant speed V.

Table 5
Parameters for the moving loads on the full-size system and the scale model with scaling factor $\lambda_w = 1/10$

Parameters	Full-size system	Scale model
Force amplitude, f_0 (N)	$f_{F0} = 845.0$	$f_{s0} = f_F\lambda_f = f_F\lambda_w^2 = 8.45$
Moving-load speed, V (m/s)	$V_F = 1.0$	$V_s = V_F\lambda_V = 1.0$ ($\lambda_V = 1$)
Time duration, $\bar{t}$ (s)	$\bar{t}_F = 10.0$	$\bar{t}_s = \bar{t}_F\lambda_t = \bar{t}_F\lambda_w = 1.0$
Initial position of moving load	$\bar{x}_{f_F} = 0\,\text{m}$	$\bar{x}_{fs} = \bar{x}_{f_F}\lambda_x = \bar{x}_{f_F}\lambda_w = 0\,\text{m}$
	$\bar{y}_{f_F} = 5\,\text{m}$	$\bar{y}_{fs} = \bar{y}_{f_F}\lambda_y = \bar{y}_{f_F}\lambda_w = 0.5\,\text{m}$
Final position of moving load	$\bar{x}_{f_F} = 10\,\text{m}$	$\bar{x}_{fs} = \bar{x}_{f_F}\lambda_x = \bar{x}_{f_F}\lambda_w = 1\,\text{m}$
	$\bar{y}_{f_F} = 5\,\text{m}$	$\bar{y}_{fs} = y_{f_F}\lambda_y = \bar{y}_{f_F}\lambda_w = 0.5\,\text{m}$

full-size flat plate supported by an elastic foundation and subjected to the combined excitations of a stationary harmonic load and a moving point load by means of the scale model and the scaling laws. In this example, all the conditions for the stationary harmonic load and the moving point load are exactly the same as those of the last two examples, respectively. However, the elastically restrained flat plate (see Fig. 4) is further supported by an elastic foundation, as shown in Fig. 6. The spring constant for each translational spring of the elastic foundation is $\bar{k}_{\ell F} = 228\,\text{kN/m}$, for the full-size system, and $\bar{k}_{\ell s} = \bar{k}_{\ell F}\lambda_{k_\ell} = 22.8\,\text{kN/m}$, for the scale model. The total number of translational springs for the elastic foundation is $N_{spring} = 99$, which is equal to the total number of nodes excluding those on the left side AB and on the right side CD of the flat plate.

Table 6 shows the lowest five natural frequencies for the full-size system, $\omega_{F,i}$, and those for the scale model, $\omega_{s,i}$, while Fig. 7 shows the theoretical and predicted time histories for the vertical ($\bar{z}$) central displacements of the full-size plate. From the frequency ratios shown in the final column of Table 6, $\lambda_{\omega,i} = \omega_{s,i}/\omega_{F,i}$ ($i = 1-5$), and the coincidence between the time histories shown in Fig. 7,

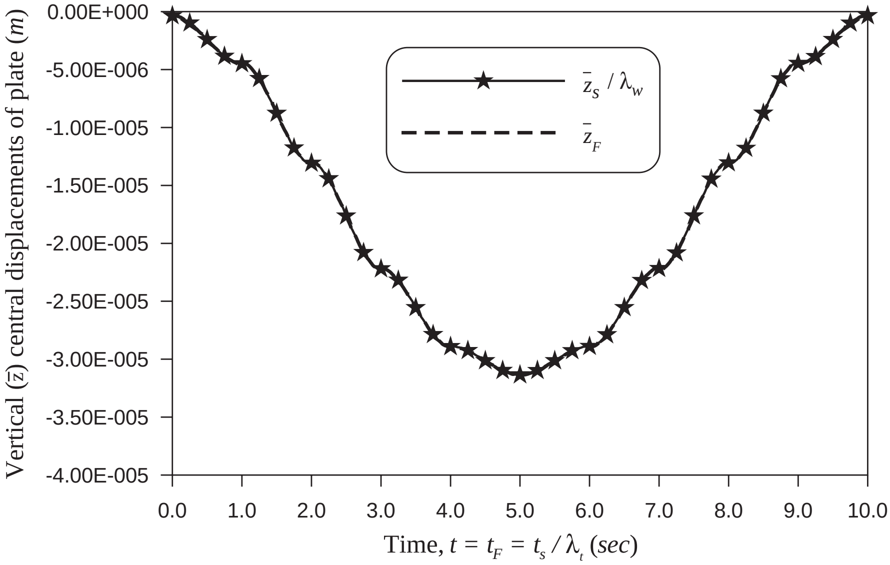

Fig. 5. Theoretical and predicted time histories for the vertical ($\bar{z}$) central displacements of the elastically restrained flat plate due to a moving point load with magnitude $f_F(t) = 845.0\text{N}$ and speed $V_F = 1.0\,\text{m/s}$ with a scaling factor $\lambda_w = 1/10$ and damping ratio $\xi_F = \xi_s = 0.005$ (the corresponding load for the scale model is $f_s(t) = 8.45\,\text{N}$ and $V_s = V_F = 1.0\,\text{m/s}$.)

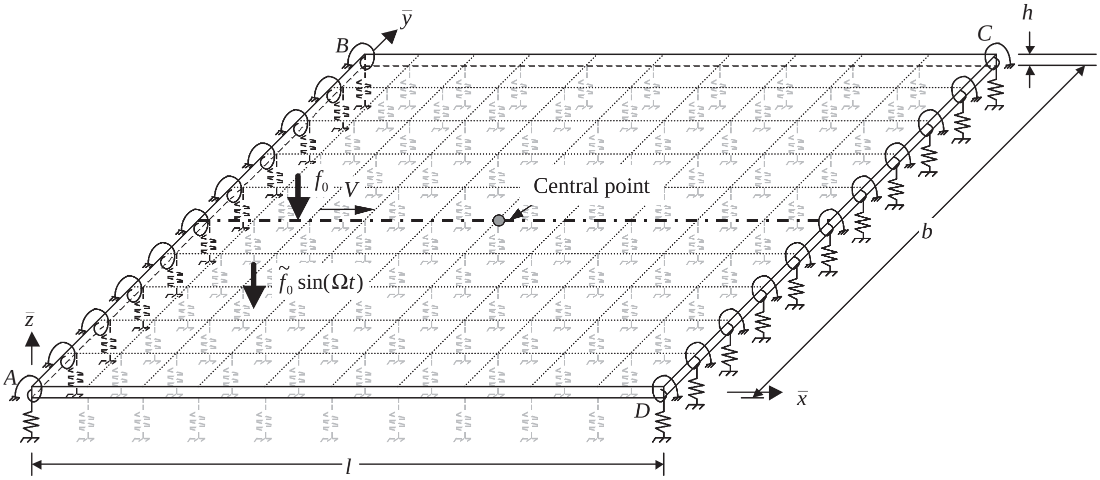

Fig. 6. An elastically restrained flat plate supported by an elastic foundation and subjected to a stationary harmonic load, $\tilde{f}(t) = \tilde{f}_0 \sin \Omega t$, and a point load f_0 moving from left side to right side along the centreline of the plate in the $\bar{x}$ direction with constant speed V.

it is apparent that the dynamic characteristics of a general vibrating system subjected to the general excitations (such as the case shown in Fig. 6) can also be accurately predicted by means of the scale model and the scaling laws presented in this paper.

Table 6
The lowest five natural frequencies of an elastically restrained flat plate supported by an elastic foundation with scaling factor $\lambda_w = 1/10$

Mode no. i	Natural frequencies, ω_i (Hz)		Frequency ratios, $\lambda_{\omega,i} = \omega_{s,i}/\omega_{F,i}$
	Full-size system, $\omega_{F,i}$	Scale model, $\omega_{s,i}$	
1	6.0955	60.9552	10.0000
2	7.0169	70.1692	10.0000
3	10.5611	105.6108	9.9999
4	15.1259	151.2591	10.0000
5	16.3093	163.0930	10.0000

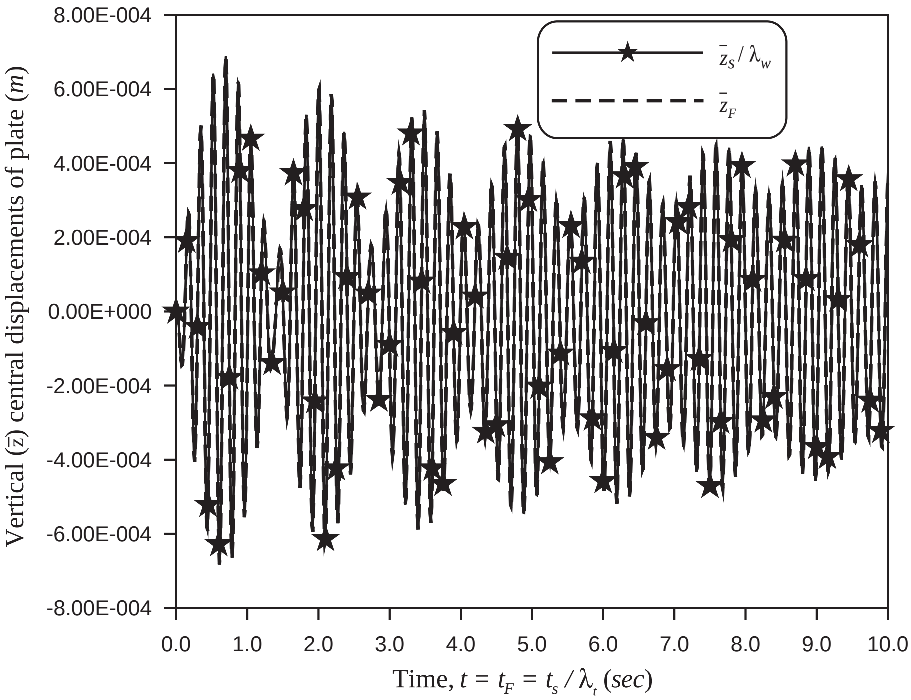

Fig. 7. Theoretical and predicted time histories for the vertical ($\bar{z}$) central displacements of the elastically restrained flat plate supported by an elastic foundation and subjected to a stationary harmonic load $\tilde{f}_F(t_F) = 7480\sin(5.3759t_F)N$ and a moving point load with magnitude $f_F(t_F)845.0N$ and speed $V_F = 1.0\,\text{m/s}$, with scaling factor $\lambda_w = 1/10$ and damping ratio $\xi_F = \xi_s = 0.005$ (the corresponding loads for the scale model are $\tilde{f}_s(t_s) = 74.80\sin(53.759t_s)N$ and $f_s(t_s) = 8.45N$, and $V_s = V_F = 1.0\,\text{m/s}$).

7. Conclusions

1. In order to establish a complete-similitude scale model, one must assure the geometric, kinematic and dynamic similarities between the scale model and its full-size structural system.

To this end, both the *explicit* and *implicit* scaling factors should be determined. The *explicit* scaling factors may be obtained from the scaling laws derived from the equations of motion, and the *implicit* scaling factors relating to the supporting conditions and the excitation mechanisms may be obtained by means of the theory of dimensional analysis.

2. To achieve the complete similitude, the ratios between the scalable parameters of the full-size system and the corresponding ones of the scale model are usually different from each other. For example, the scaling factors for length, time, force, natural frequency, forcing frequency, translational spring, rotational spring, moving-load speed and damping ratio are respectively given by $\lambda_w, \lambda_w, \lambda_w^2, 1/\lambda_w, 1/\lambda_w, \lambda_w, \lambda_w^3, 1$ and 1, in this paper. Therefore, any set of selected scaling factors must be validated before they are applied to designing a scale model for practical experiments.

Acknowledgements

The author wishes to acknowledge the financial support of the National Science Council, the Republic of China, under contract No. NSC-91-2611-E-022-002.

References

[1] Y. Qian, S.R. Swanson, R.J. Nuismer, R.B. Bucinell, An experimental study of scaling rules for impact damage in fiber composites, Journal of Composite Material 24 (1990) 559–570.
[2] D. Vassalos, Physical modelling and similitude of marine structures, Ocean Engineering 26 (1999) 111–123.
[3] M. Safoniuk, J.R. Grace, L. Hackman, C.A. Mcknight, Use of dimensional similitude for scale-up of hydrodynamics in three-phase fluidised beds, Chemical Engineering Science 54 (1999) 4961–4966.
[4] E. Buckingham, Modal experiments and the forms of empirical equations, Transactions of the American Society of Mechanical Engineers 37 (1915) 263–296.
[5] J. Rezaeepazhand, G.J. Simitses, J.H. Starnes, Use of scaled-down models for predicting vibration response of laminated plates, Composite Structures 30 (1995) 419–426.
[6] J. Rezaeepazhand, G.J. Simitses, J.H. Starnes, Design of scaled down models for predicting shell vibration response, Journal of Sound and Vibration 195 (2) (1996) 301–311.
[7] G.J. Simitses, J. Rezaeepazhand, Structural similitude for laminated structures, Composites Engineering 3 (1993) 751–765.
[8] W.E. Baker, P.S. Westine, F.T. Dodge, Similarity Methods in Engineering Dynamics: Theory and Practice of Scale Modelling, Hayden Book Company, 1973.
[9] J.J. Wu, M.P. Cartmell, A.R. Whittaker, Prediction of the vibration characteristics of a full-size structure from those of a scale model, Computers & Structures 80 (2002) 1461–1472.
[10] P.W. Bridgman, Dimensional Analysis, Yale University Press, New Haven, CT, 1992.
[11] L. Frýba, Vibration of Solids and Structures Under Moving Loads, Noordhoff International Publishing, The Netherlands, 1971.
[12] K.J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1982.
[13] G.H. Golub, R. Underwood, J.H. Wilkinson, The Lanczos algorithm for the symmetric Ax = Lambda*Bx problem, Stanford University, Computer Science Department, 1972.
[14] J.S. Wu, M.L. Lee, T.S. Lai, The dynamic analysis of a flat plate under a moving load by the finite element method, International Journal for Numerical Methods in Engineering 24 (1987) 743–762.