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Letter to the Editor

A classical perturbation technique which is valid for large parameters

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There are various perturbation techniques for constructing analytical approximations to the oscillatory solutions of second order, non-linear differential equations [1,2]. But many of them apply to weakly non-linear cases only. To overcome the limitations, many novel techniques have been proposed in recent years. For example, Cheung et al. [3] proposed a modified Lindstedt–Poincare method, and Lim et al. [4] presented a modified Mickens procedure for certain non-linear oscillators. Recently, He [5] proposed a perturbation technique, which is valid for large parameters. But unfortunately, ω_1 and B in Ref. [5] are the functions of η which is unknown.

The main purpose of this paper is to point out that there already exists a classical perturbation technique, which is valid for large parameters. In this paper, the following well-known Duffing equation is considered:

$$x'' + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{1}$$

where ε is a parameter.

The solution of Eq. (1) is assumed in the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots. \tag{2}$$

The fundamental frequency ω^2 is given by [6–9]

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{3}$$

where, at this point, the ω_i are unknown constants. This expansion has been also employed by Cheung et al. [3].

Substituting Eqs. (2) and (3) into Eq. (1) gives

$$\begin{aligned}(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + (\omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)\\ + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 = 0.\end{aligned} \tag{4}$$

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This equation is satisfied by setting the coefficients of the powers of ε equal to zero, resulting in

$$x_0'' + \omega^2 x_0 = 0, \tag{5}$$

$$x_1'' + \omega^2 x_1 = \omega_1 x_0 - x_0^3, \tag{6}$$

$$x_2'' + \omega^2 x_2 = \omega_2 x_0 + \omega_1 x_1 - 3x_0^2 x_1. \tag{7}$$

The initial conditions are replaced by

$$x_0(0) = A, \quad x_0'(0) = 0, \quad x_i(0) = 0, \quad x_i'(0) = 0, \quad i \geqslant 1. \tag{8}$$

The solution to Eq. (5) is

$$x_0 = A\cos\omega t. \tag{9}$$

Using the "normal" classical perturbation technique [1,2] and taking into account the initial conditions given by Eq. (8), we can easily obtain

$$\omega_1 = \tfrac{3}{4}A^2, \quad x_1 = \frac{A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t), \tag{10}$$

$$\omega_2 = -\frac{3A^4}{128\omega^2}, \quad x_2 = \frac{A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t). \tag{11}$$

Therefore, the second approximate solution to Eq. (1) becomes

$$x = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t) \tag{12}$$

with

$$\omega^2 = \omega_0^2 + \tfrac{3}{4}\varepsilon A^2 - \frac{3\varepsilon^2 A^4}{128\omega^2}. \tag{13}$$

This equation can be rewritten as

$$\omega^4 - \left(\omega_0^2 + \tfrac{3}{4}\varepsilon A^2\right)\omega^2 + \tfrac{3}{128}\varepsilon^2 A^4 = 0. \tag{14}$$

Solving for ω^2 gives

$$\omega^2 = \tfrac{1}{2}\left(\omega_0^2 + \tfrac{3}{4}\varepsilon A^2\right) \pm \tfrac{1}{2}\sqrt{\left(\omega_0^2 + \tfrac{3}{4}\varepsilon A^2\right)^2 - \tfrac{3}{32}\varepsilon^2 A^4}. \tag{15}$$

Obviously, if $\varepsilon = 0$, we should have $\omega^2 = \omega_0^2$. Therefore, we get

$$\omega = \tfrac{1}{4}\sqrt{8\omega_0^2 + 6\varepsilon A^2 + \sqrt{64\omega_0^2 + 96\omega_0^2\varepsilon A^2 + 30\varepsilon^2 A^4}}. \tag{16}$$

To compare the present results with the results in the literature, we take $\omega_0^2 = 1$. Then, Eq. (16) becomes

$$\omega = \omega_p = \tfrac{1}{4}\sqrt{8 + 6\varepsilon A^2 + \sqrt{64 + 96\varepsilon A^2 + 30\varepsilon^2 A^4}}. \tag{17}$$

The exact frequency of the periodic motion of the Duffing equation is given by [10]

$$\omega_e = \frac{\pi\sqrt{1+\varepsilon A^2}}{2}\left(\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-m^2\sin^2\theta}}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)} \tag{18}$$

and the corresponding approximate frequencies obtained by Lim et al. [4] and He [5] are

$$\omega_L = \tfrac{1}{8}\sqrt{32+25\varepsilon A^2+\sqrt{1024+1472\varepsilon A^2+433\varepsilon^2A^4}} \tag{19}$$

and

$$\omega_H = \tfrac{1}{4}\sqrt{8+6\varepsilon A^2+\sqrt{64+96\varepsilon A^2+42\varepsilon^2A^4}}, \tag{20}$$

respectively.

For comparison, the exact frequency obtained by integrating Eq. (18) and the approximate frequencies ω_p, ω_L and ω_H computed by Eqs. (17), (19) and (20), respectively, are listed in Table 1. Table 1 indicates that formula (17) is more accurate than formulas (19) and (20), and can give excellent approximate frequencies for both small and large parameters and oscillation amplitude. We also have

$$\begin{aligned}\lim_{\varepsilon A^2\to\infty}\frac{\omega_p}{\omega_e} &= \frac{\sqrt{6+\sqrt{30}}}{2\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-0.5\sin^2\theta}}\\ &= \frac{\sqrt{6+\sqrt{30}}}{2\pi}1.8541 = 0.9997.\end{aligned} \tag{21}$$

In summary, a classical perturbation technique has been used to solve the Duffing equation. We find that the parameter ε does not need to be small. For any value of ε, the maximal relative error of the second approximate frequency with respect to the exact solution is less than 0.03%.

Table 1
Comparison of approximate frequencies with the corresponding exact frequency for the Duffing equation

εA^2	ω_e Eq. (18) [10]	ω_L Eq. (19) [4]	ω_H Eq. (20) [5]	ω_p Eq. (17) present
0.2	1.07200	1.07200	1.07276	1.07200
0.4	1.13891	1.13889	1.14144	1.13891
0.6	1.20173	1.20170	1.20656	1.20173
0.8	1.26118	1.26112	1.26859	1.26118
1	1.31778	1.31767	1.32789	1.31776
2	1.56911	1.56873	1.59278	1.56905
5	2.15042	2.14912	2.20688	2.15018
10	2.86664	2.86408	2.96097	2.86613
100	8.53359	8.52220	8.88639	8.53110
1000	26.8107	26.7734	27.9465	26.8025
10000	84.7245	84.6088	88.3257	84.7013

Why does this classical technique work for large parameters? It is an interesting question and needs further study. The normal classical perturbation method [1,2,10] uses the expansion

$$\omega = \omega_0 + \varepsilon\omega_1' + \varepsilon^2\omega_2' + \cdots \tag{22}$$

instead of expansion (3), which is the important difference of the normal classical perturbation method with respect to this "innovative" classical perturbation method. For example, we have the first approximate frequency (for $\omega_0 = 1$)

$$\omega = \sqrt{1 + \tfrac{3}{8}\varepsilon A^2}. \tag{23}$$

But the normal classical perturbation method [1,10] gives

$$\omega = 1 + \tfrac{3}{8}\varepsilon A^2. \tag{24}$$

Formula (23) can give good approximate frequencies for both small and large parameters [4,5]. But for large parameters, formula (24) is not valid. $\sqrt{1 + \frac{3}{4}\varepsilon A^2} \approx 1 + \frac{3}{8}\varepsilon A^2$ only when ε is small. Maybe using expansion (22) is the reason why the normal classical perturbation method is not able to provide accurate result when the parameter is large.

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