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Letter to the Editor

Application of a new differential quadrature element method to free vibrational analysis of beams and frame structures

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1. Introduction

The differential quadrature (DQ) method, introduced by Bellman and Casti in 1971 [1], has drawn caused much attention in recent years owing to its favorable features, such as the method is simple and can yield highly accurate results with only a few grid points for certain problems. In 1988, Bert and his co-workers first applied the DQ method to solve structural mechanics problems [2,3]. Refs. [4–6] give the details on the development and applications of the DQ method.

Since the DQ method has only the function values at grid points as the independent variables, difficulty arises for applying the boundary conditions, if the number of boundary conditions is greater than one. Bert and his co-workers [2,3] introduced a δ-point apart from the boundary point by a small distance as an additional boundary point and applied the other boundary condition at that point. It is found [2,3,7] that, however, the solution accuracy may not be assured since δ is problem-dependent. Alternatively, some DQ equations at inner grid points can be replaced by the additional boundary conditions. It is found that, however, the solution accuracy may vary depending on which DQ equations at inner grids are replaced by the boundary conditions [6].

A new method to apply the boundary conditions to increase the solution accuracy has been introduced by the first author [8]. The essential idea is to build the boundary conditions during formulation of the weighting coefficients for higher order derivatives. But the method cannot be used for all boundary conditions. Later Malik and Bert [9] tried to extend this idea to apply all boundary conditions, however, the δ-point has to be introduced for some combinations of boundary conditions. Detailed discussions on the way to implement boundary conditions in DQ method can be found in the excellent introduction of Ref. [10].

As has been noticed, that the conventional DQ method lacks some flexibility to solve real structural problems. Striz et al. [11] first introduced the quadrature element method (QEM) to

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analyze beam structures with various loads, including discontinuous loads. However, the method is not quite convenient and accurate since the δ-point is introduced to establish the element equations. Later Wang et al. [12] proposed a method to remove the δ-point in QEM by assigning two degrees of freedom to each end point for a fourth order differential equation. The method is called the differential QEM or simply DQEM. All boundary conditions can be easily applied by the DQEM and accurate solutions can be obtained. The same idea is independently proposed by Chen et al. [13]. Bert and his co-workers [14,15] also introduced another version of DQ element by using the weak form similar to the high order finite element. Han and Liew [16] first introduced the DQEM for thick plates. A variety of structure problems have been solved with the various proposed differential QEMs [12–20] since then. Recently, Wu and his co-workers [10,21–23] proposed a generalized differential quadrature rule (GDQR). The GDQR constructs the DQ in a general situation and can easily deal with any problems involving more than one condition at any discrete point. The explicit weighting coefficients of the GDQR are also derived and a variety of problems involving higher order (third, fourth, sixth and eighth order) differential equations have been successfully solved. For plate problems, only three degrees of freedom are used for corner points in the GDQR to deal with the three independent boundary conditions while four degrees of freedom have to be used in the DQEM [14,15,19].

In this paper, a new version of differential QEM is introduced. The new method is similar to the DQEM [12,13] or GDQR [10,21–23] in the way that the number of degrees of freedom is the same as the number of the independent boundary conditions at the boundary points, but the essential difference from the DQEM or GDQR is that the weighting coefficient of the first order derivative is exactly the same as that of the conventional DQ method. More specifically, the weighting coefficients are determined by the Hermite interpolation shape function in the GDQR or DQEM, but are determined by the Lagrange interpolation function for the present method and the explicit formulae are provided by Quan and Chang [24] and Shu and Richards [25]. The multiple degrees of freedoms are introduced merely for the application of boundary conditions, therefore, the additional degree of freedom for the beam case is not necessary for the rotation. It could also be the curvature. The present method is also different from the conventional DQ method in the way of formulating the weighting coefficients of higher order derivatives. In what follows, the new idea to determine the weighting coefficients of higher order derivatives is described first, the new version of DQEM is then formulated; followed by the applications to analysis of free vibration problems of beams and beam structures; finally, conclusions are drawn based on the results presented herein and some future research works are pointed out.

2. The new version of DQ method and DQ element method

For simplicity and purpose of illustrations, one-dimensional problems are considered herein. In the ordinary DQ method, the solution function $w(x)$ can be assumed as

$$w(x) = \sum_{j=1}^{N} L_j(x) w_j, \tag{1}$$

where N, $L_j(x)$ and w_j are the total number of grid points in the entire domain including the end points, the Lagrange interpolation function and the solution values at grid point j, respectively.

Then the kth order derivative of the solution function at grid point i can be computed by

$$w_i^{(k)} = \sum_{j=1}^{N} L_j^{(k)}(x_i) w_j = \sum_{j=1}^{N} E_{ij} w_j \quad (i = 1, 2, \ldots, N), \tag{2}$$

where E_{ij} are called the weighting coefficients of the kth order derivative.

Thus, the ordinary differential governing equations can be approximated by a set of algebraic equations with unknowns of w_j in terms of DQ method. When appropriate boundary conditions are applied, a unique solution can be obtained.

Let A_{ij} be the weighting coefficients of the first order derivative, which can be computed explicitly by [6,24,25]

$$A_{ij} = \frac{\omega_N'(x_i)}{(x_i - x_j)\omega_N'(x_j)} \quad (i \neq j), \quad A_{ii} = \sum_{j=1, i\neq j}^{N} \frac{1}{(x_i - x_j)}, \tag{3}$$

where

$$\omega_N(x) = (x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_i)(x - x_{i+1})\cdots(x - x_N),$$

$$\omega_N'(x) = (x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_N).$$

The weighting coefficients of second, third, and fourth order derivatives, B_{ij}, C_{ij}, D_{ij}, can be computed by

$$B_{ij} = \sum_{k=1}^{N} A_{ik} A_{kj}, \quad C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}, \quad D_{ij} = \sum_{k=1}^{N} B_{ik} B_{kj}. \tag{4}$$

The essence of the new way to apply the boundary conditions is that two degrees of freedom at each end point, similar to the DQEM in references [12,13], are introduced for a fourth order differential equation. It should be mentioned that the present method is readily extended to higher order differential equations, but three or four degrees of freedom at each end point should be introduced for a sixth or eight order differential equation. It is also noticed that the present method is, however, different from the QEM [11] since the δ method is not used, and from the DQEM [12,13] since the weighting coefficients of second, third, and fourth order derivatives at inner points are computed differently by

$$\begin{aligned} w_i'' &= \sum_{j=1}^{N} B_{ij} w_j = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{ik} A_{kj} w_j, \\ w_i''' &= A_{i1} w_1'' + A_{iN} w_N'' + \sum_{j=1}^{N} \sum_{k=2}^{N-1} A_{ik} B_{kj} w_j, \\ w_i^{\mathrm{IV}} &= B_{i1} w_1'' + B_{iN} w_N'' + \sum_{j=1}^{N} \sum_{k=2}^{N-1} B_{ik} B_{kj} w_j \quad (i = 2, 3, \ldots, N-1). \end{aligned} \tag{5}$$

The weighting coefficients of second, and third order derivatives at end points are computed by

$$w_1'' = A_{11}w_1' + A_{1N}w_N' + \sum_{j=1}^{N}\sum_{k=2}^{N-1} A_{1k}A_{kj}w_j = \sum_{j=1}^{N+2} \bar{B}_{1j}\delta_j,$$

$$w_N'' = A_{N1}w_1' + A_{NN}w_N' + \sum_{j=1}^{N}\sum_{k=2}^{N-1} A_{Nk}B_{kj}w_j = \sum_{j=1}^{N+2} \bar{B}_{Nj}\delta_j, \tag{6a}$$

where $\{\delta\}^{\mathrm{T}} = \{w_1, w_1', w_N, w_N', w_2, \ldots, w_{N-1}\}$:

$$w_1''' = A_{11}w_1'' + A_{1N}w_N'' + \sum_{j=1}^{N}\sum_{k=2}^{N-1} A_{1k}B_{kj}w_j,$$

$$w_N''' = A_{N1}w_1'' + A_{NN}w_N'' + \sum_{j=1}^{N}\sum_{k=2}^{N-1} A_{Nk}B_{kj}w_j. \tag{6b}$$

It is seen that the weighting coefficients of the second order derivatives are computed differently at the inner grid by Eq. (5) and at end points by Eq. (6a). This is the key step for success by extending the method in Refs. [8,9] without the δ method. Similar procedures can be used for determining the weighting coefficients used for the six or eight order differential equations. Since the degrees of freedom at the end points are w_1, w_1', w_N, w_N', substituting Eq. (6a) into Eq. (5) and Eq. (6b) yields

$$w_i'' = \sum_{j=1}^{N+2} \bar{B}_{ij}\delta_j, \quad w_i''' = \sum_{j=1}^{N+2} \bar{C}_{ij}\delta_j, \quad w_i^{\mathrm{IV}} = \sum_{j=1}^{N+2} \bar{D}_{ij}\delta_j \quad (i = 2, 3, \ldots, N-1), \tag{7a}$$

$$w_1''' = \sum_{j=1}^{N+2} \bar{C}_{1j}\delta_j, \quad w_N''' = \sum_{j=1}^{N+2} \bar{C}_{Nj}\delta_j. \tag{7b}$$

For the case of a Bernoulli–Euler beam under loadings, the governing differential equation can be expressed as

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - q(x) + \rho A\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \quad (x \in [0, L]), \tag{8}$$

where E, I, L, $q(x)$, P, ρ, and A are Young's modulus, the principal moment of inertia about the y-axis, the beam length, the distributed load, the axial load, mass density and the cross-sectional area, respectively. The shear force $Q(x)$ and bending moment $M(x)$ are

$$EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = Q(x), \quad EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = M(x) \quad (x \in [0, L]). \tag{9}$$

Eqs. (8) and (9) can be expressed in terms of the weighting coefficients by utilizing the DQ method. Various boundary conditions can be easily applied and unique solution can be obtained for the beam subjected to static and dynamic loadings.

The formulation of the new version of DQ beam element is exactly the same as the one presented in Refs. [12,17], once the weighting coefficients are determined by using Eqs. (3), (6a), (7a) and (7b). The DQ element equation for a Bernoulli–Euler beam can be symbolically written as

$$[K]\{\delta\} = \{F\}, \tag{10}$$

where $[K], \{\delta\}, \{F\}$ are called the DQ weighting coefficient matrix, generalized displacement vector, and generalized load vector, respectively. Details may be found in Ref. [17] since the only difference between the new version DQEM and the DQEM in Ref. [17] is the way the weighting coefficients are determined.

3. Applications and discussions

As the first example, consider the flexural vibration of prismatic beams with six combinations of boundary conditions, i.e., SS–SS, SS–C, SS–F, C–C, C–F, and F–F, where SS, C and F denote the simply supported, clamped and free boundaries, respectively. Use one DQ element method to solve this problem. The new version of DQEM equations is the same for all boundary conditions and is given by

$$\begin{bmatrix} EI\sum_{j=1}^{N+2}\bar{C}_{1j}\delta_j \\ -EI\sum_{j=1}^{N+2}\bar{B}_{1j}\delta_j \\ -EI\sum_{j=1}^{N+2}\bar{C}_{Nj}\delta_j \\ EI\sum_{j=1}^{N+2}\bar{B}_{Nj}\delta_j \\ EI\sum_{j=1}^{N+2}\bar{D}_{2j}\delta_j \\ EI\sum_{j=1}^{N+2}\bar{D}_{3j}\delta_j \\ \cdots \\ EI\sum_{j=1}^{N+2}\bar{D}_{N-1,j}\delta_j \end{bmatrix} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_N \\ M_N \\ \rho A\omega^2 w_2 \\ \rho A\omega^2 w_3 \\ \cdots \\ \cdots \\ \rho A\omega^2 w_{N-1} \end{bmatrix}. \tag{11}$$

The negative sign introduced in the second and third equations in the above equation is merely for convenience of assembling the global weighting coefficients for a beam structure. In other words, generalized forces of a beam element at both ends are in the same direction, similar to the finite element formulations. Eq. (11) can be rewritten in the following partition form for all boundary conditions, namely,

$$\begin{bmatrix} K_{ee}K_{ei} \\ K_{ie}K_{ii} \end{bmatrix}\begin{Bmatrix} \delta_e \\ \delta_i \end{Bmatrix} = \begin{Bmatrix} F_e \\ \rho A\omega^2\{\delta_i\} \end{Bmatrix}, \tag{12}$$

where subscripts e and i denote the two end points and all internal points, respectively. Multiplying out Eq. (12) yields

$$\begin{aligned} [K_{ee}]\{\delta_e\} + [K_{ei}]\{\delta_i\} &= \{F_e\}, \\ [K_{ie}]\{\delta_e\} + [K_{ii}]\{\delta_i\} &= \rho A\omega^2\{\delta_i\}. \end{aligned} \tag{13}$$

Thus,

$$\{\delta_e\} = [K_{ee}]^{-1}(\{F_e\} - [K_{ei}]\{\delta_i\}). \tag{14}$$

Table 1
Fundamental frequency ($\bar{\omega}$) of beams with various boundary conditions ($\bar{\omega} = L^2\omega\sqrt{\rho A/EI}, N = 9$)

BCs	SS–SS	C–SS or F–SS[a]	C–F	C–C or F–F[a]
Present	9.8696	15.418	3.5160	22.373 or 22.374[b]
Leissa [26]	9.8696	15.418	3.5160	22.373

[a] The smallest non-zero frequency.
[b] The BC is F–F.

Substituting Eq. (14) into the second equation of Eq. (13) yields

$$([K_{ii}] - [K_{ie}][K_{ee}]^{-1}[K_{ei}])\{\delta_i\} = \rho A\omega^2\{\delta_i\} - [K_{ie}][K_{ee}]^{-1}\{F_e\}. \tag{15}$$

For free vibration analysis of beams, $[K_{ie}][K_{ee}]^{-1}\{F_e\}$ will not appear in the final equation. It is either zero due to zero generalized forces or eliminated due to zero generalized displacements. For example, $\delta_1 = w_1 = 0,\ \ \delta_2 = w_1' = 0$ if the left end of the beam ($x = 0$) is clamped, thus the first two equations in Eq. (11) can be dropped; $\delta_1 = w_1 = 0,\ \ M_1 = 0$ if the left end of the beam ($x = 0$) is simply supported, thus the first equation in Eq. (11) can be dropped; $\delta_2 = w_1' = 0,\ \ Q_1 = 0$ if the left end of the beam ($x = 0$) is a sliding type, thus the second equation in Eq. (11) can be dropped; and if the left end of the beam ($x = 0$) is free, $Q_1 = 0,\ \ M_1 = 0$. Therefore, the equation to obtain the frequencies of beams with various boundary conditions can be symbolically written as

$$[\bar{K}]\{\delta_i\} = \rho A\omega^2 L^4/EI\{\delta_i\} = \bar{\omega}^2\{\delta_i\}. \tag{16}$$

Solving Eq. (16) yields frequencies. The fundamental frequencies are listed in Table 1 for various boundary conditions and compared with the analytical solutions by Leissa [26]. It can be seen that good accuracy is achieved by the DQ element method with $N = 9$. To expedite the convergence rate, the following non-uniform grid spacing is used in the analysis, namely,

$$x_i = \frac{L}{2}\left(1 - \cos\frac{k\pi}{N-1}\right) \quad (k = 0, 1, \ldots, N-1). \tag{17}$$

This example demonstrates a new way to apply the boundary conditions. The new approach is not only as convenient as the original method proposed by Bert et al. [2,3], but also as accurate as the method proposed by Wang and Bert [8] and Wang et al. [12]. Therefore, the proposed method is also recommended for use in practice. It should be pointed out, however, that the formulae to compute weighting coefficients are entirely different from those given in Refs. [10,12–14,17–19,21–23].

Next consider a fixed-end portal frame shown in Fig. 1. For simplicity, E, ρ, A, I, and L for all three members are assumed the same. When such frame vibrates, the motion is predominantly flexural. It is customary to assume that the frame is inextensible ($u_9 = u_{17}$). The procedures to obtain the weighting coefficient matrix for a structure are similar to the finite element method but in a strong form, namely, w, w' are assumed the same at the common grid points and the equilibrium equations are set at all common grid points. After imposing the appropriate boundary conditions, unique solutions can be obtained for the structure under static and dynamic loadings. Details may be found, for example, in Refs. [17,27]. It should be mentioned that concentrated inertial forces ($\rho AL\omega^2 u_9/2$) are added at points 9 and 17 in Fig. 1 along the x direction to account for the movement of the inextensible beam element 2 in the x direction. The zero displacements of

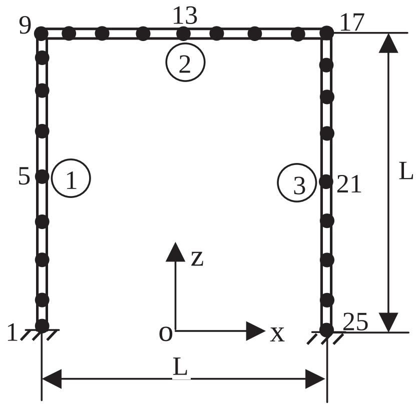

Fig. 1. Portal frame structure discretized by three DQ beam elements.

Table 2
Frequencies of the portal frame ($\bar{\omega} = \omega L^2\sqrt{\rho A/EI}$)

$\bar{\omega}$	Analytical [27,28]	DQ7N [17]	Present
Asymmetric	3.2046	3.2046	3.2046
Symmetric	12.648	12.648	12.648

rotations for the portal frame shown in Fig. 1 are u_1, w_1, w_1', u_{25}, w_{25}, w_{25}', w_9 and w_{17}. When the vibration modes are symmetric, $u_9 = u_{17} = 0$, which will be automatically satisfied in the analysis. It is known that the first two lowest eigenvalues are corresponding to the antisymmetric and symmetric modes [27,28]. The natural frequencies for the lowest antisymmetric and symmetric modes are determined by using the new version of DQ element method. Nine grid points are used for each beam element. The results are shown in Table 2. It can be seen that the results agree very well with the analytical solutions and the previous DQEM results when the same order polynomial is used to determine the weighting coefficients.

4. Summary remarks

A new formulation to compute the weighting coefficients has been presented in detail based on the extension of the way to apply the boundary conditions in Ref. [8] without the usage of the δ method. Then a new differential QEM is introduced following the same procedures as in Refs. [12,17]. The essential difference between the present method and the DQEM in Ref. [12] and GDQR in Refs. [10,21–23] is the way to compute the weighting coefficients. The present version of DQEM has also provided a new approach to apply the boundary conditions to the conventional differential quadrature method. The present formulations of the weighting coefficients can be extended to sixth or eight order differential equations without any difficulties. Based on the numerical results, it is found that the present DQ element method combines the attractive features of rapid convergence and the high accuracy of the DQM with the generality of the FEM formulations for application to structural analyses. Although there is not much difference between the present DQ element method and the existing DQEM in the literature for one-dimensional problems, the method is, however, readily extended to two-dimensional problems. Preliminary results show that numerical instability problems encountered by the previous DQEM in solving two-dimensional problems is overcome by the present method since only three degrees of freedom (the same as the GDQR) is needed for the corner point of a rectangular plate element.

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