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An optimization of frame structures with exact dynamic constraints based on Timoshenko beam theory

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Abstract

Sensitivity analyses of eigensolutions and eigenfunctions of 3-D frame structures using the exact frequency equation from the transfer dynamic stiffness matrix that was derived on Timoshenko beam theory were developed in this paper. Based on the sensitivity data of frame structures, the minimum weight design with an exact frequency constraint can be carried out efficiently. Three examples that demonstrated the results obtained by the proposed method, are in good agreement with those computed by ANSYS.

1. Introduction

Sensitivity analyses concern the relationship between design variables to the structural response such as displacement, natural frequency, etc. Once the sensitivity data is available, most structural optimization problem can be solved using efficient gradient based method. Huag et al. [1] presented several design sensitivity formulas and numerical methods. Design optimization is a problem associated with all fields of engineering. Optimization deals with problems of minimizing or maximizing a function with several variables usually subject to equality and inequality constraints. A lot of different approaches have been developed to find optimum designs. When gradient information is available, the approaches generally show better convergence. Also, many methods and algorithms have been developed for optimum design of structural system. Most of the methods deal with continuous design variables and use mathematical programming techniques based on discrete models such as finite element method. Tong and Liu [2] presented

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an optimization procedure based on discrete model for the minimum weight optimization. Also, natural frequency responses can reflect the relationship between dynamic loads and structural responses and they are used to represent the dynamic limit in some structural dynamic optimization problems based on the discrete system. Sergeyev and Mroz [3] developed an efficient method of sensitivity and optimization of space frame with natural frequency constraints, which based on the discrete model. Negm and Maalawi [4] described an exact frequency optimization analysis for a typical wind turbine tower structure without gradient constraints, which is only based on Euler–Bernoulli beam theory in bending vibration.

Hence, we want to extend the sensitivity to 3-D frame vibration by using exact formulation based on Timoshenko beam theory [5,6]. In structural optimization, if the structure is analyzed using continuous methods such as the transfer matrix method (TMM) [7–9]. Dynamic properties such as natural frequencies and frequency responses can be computed more accurately. Thus, we suggest performing minimum weight design with consideration of the exact eigenvalue constraint and gradient.

A method presented in this paper is to investigate the use of the transfer dynamic stiffness matrix based on Timoshenko beam theory in the exact sensitivity analyses of eigenvalues and eigenfunctions and to perform the optimal design of beam and frame structures with frequency constraints using exact frequency sensitivity data.

2. Formulations of eigenvalue and eigenfunction problem

Given a 3-D frame consists of N prismatic members, the vibration analysis of each member involves the solution of four differential equations: i.e., axial vibration, torsional vibration and flexural vibration equations in two planes. So the vibration analysis of this frame involves the solution of $4N$ differential equations. If the solution is exact, it satisfies the governing differential equation and all the boundaries and interconnection condition. The direction of positive end forces and displacements defined in the TMM are shown as Fig. 1 and the states for vibration mode are defined in Table 1.

2.1. Eigenvalue problem formulation

In the Timoshenko beam theory, the equation of motions for torsional, axial and flexural vibrations in two planes can be expressed as the following.

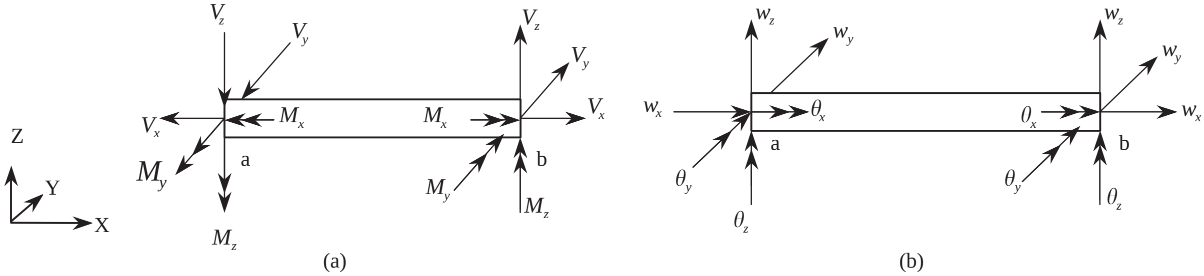

Fig. 1. End forces (a) and displacements (b).

Table 1
Definition of states for vibration mode

Vibration mode	State	
	Displacement	Force
Axial	w_x	V_x
Torsional	θ_x	M_x
Bending in $x-y$ plane	w_y, θ_z	V_y, M_z
Bending in $x-z$ plane	w_z, θ_y	V_z, M_y

For flexural vibration in $x-y$ plane:

$$\frac{\partial^4 w_y}{\partial x^4} + \left(\frac{\sigma + \tau_{xy}}{l^2}\right)\frac{\partial^2 w_y}{\partial x^2} - \left(\frac{\beta_{xy}^4 - \sigma\tau_{xy}}{l^4}\right) w_y = 0. \tag{1}$$

For flexural vibration in $x-z$ plane:

$$\frac{\partial^4 w_z}{\partial x^4} + \left(\frac{\sigma + \tau_{xz}}{l^2}\right)\frac{\partial^2 w_z}{\partial x^2} - \left(\frac{\beta_{xz}^4 - \sigma\tau_{xz}}{l^4}\right) w_z = 0. \tag{2}$$

For axial vibration in x direction:

$$\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} + \beta_e^2 w_x = 0. \tag{3}$$

For torsional vibration within x direction:

$$\frac{\mathrm{d}^2 \theta_x}{\mathrm{d}x^2} + \beta_t^2 \theta_x = 0, \tag{4}$$

where $\mu = \rho A$, $A_s = kA$, $\sigma = (\mu\omega^2/GA_s)l^2$, $r_z = \sqrt{I_z/A}$, $\tau_{xy} = (\mu r_z^2\omega^2/EI_z)l^2$, $\beta_{xy}^4 = (\mu\omega^2/EI_z)l^4$, $r_y = \sqrt{I_y/A}$, $\tau_{xz} = (\mu r_y^2\omega^2/EI_z)l^2$, $\beta_{xz}^4 = (\mu\omega^2/EI_z)l^4$, $\beta_e^2 = \omega^2/(E/\rho)$, $\beta_t^2 = \omega^2/(GJ_t/\rho I_p)$, and ρ, E, A, I_z, I_y, I_p, G, k, and $J_t(x)$ are mass density, Young's modulus, cross-sectional area, moment of inertia about z- and y-axis, mass polar moment of inertia of the shaft per unit length, shear modulus, Timoshenko's shear coefficient and the torsional constant depended on the shape of its cross-section, respectively.

The general solutions of deformation $w_y(x)$, slope $\theta_z(x)$, shear force $V_y(x)$ and moment $M_z(x)$ for transverse free vibration in $x-y$ plane are shown below:

$$\begin{aligned} w_y(x) = -\frac{l^4}{\beta^4 EI_z}\Bigg[& A_1 \frac{\lambda_1}{l}\sinh\left(\lambda_1 \frac{x}{l}\right) + A_2 \frac{\lambda_1}{l}\cosh\left(\lambda_1 \frac{x}{l}\right) \\ & - A_3 \frac{\lambda_2}{l}\sin\left(\lambda_2 \frac{x}{l}\right) + A_4 \frac{\lambda_2}{l}\cos\left(\lambda_2 \frac{x}{l}\right)\Bigg], \end{aligned} \tag{5}$$

$$\begin{aligned} \theta_z(x) = \frac{-l^2}{\beta^4 EI_z}\Bigg\{ & (\sigma + \lambda_1^2)\left[A_1 \cosh\left(\lambda_1 \frac{x}{l}\right) + A_2 \sinh\left(\lambda_1 \frac{x}{l}\right)\right]. \\ & + (\sigma - \lambda_2^2)\left[A_3 \cos\left(\lambda_2 \frac{x}{l}\right) + A_4 \frac{\lambda_2}{l}\sin\left(\lambda_2 \frac{x}{l}\right)\right]\Bigg\}, \end{aligned} \tag{6}$$

$$V_y(x) = A_1 \cosh\left(\lambda_1 \frac{x}{l},\right) + A_2 \sinh\left(\lambda_1 \frac{x}{l}\right) + A_3 \cos\left(\lambda_2 \frac{x}{l}\right) + A_4 \sin\left(\lambda_2 \frac{x}{l}\right), \tag{7}$$

$$M_z(x) = \frac{-l^2}{\beta^4}\left\{(\sigma + \lambda_1^2)\frac{\lambda_1}{l}\left[A_1 \sinh\left(\lambda_1 \frac{x}{l}\right) + A_2 \cosh\left(\lambda_1 \frac{x}{l}\right)\right] - (\sigma - \lambda_2^2)\frac{\lambda_2}{l}\left[A_3 \sin\left(\lambda_2 \frac{x}{l}\right) - A_4 \cos\left(\lambda_2 \frac{x}{l}\right)\right]\right\}. \tag{8}$$

The general solutions of deformation $w_z(x)$, slope $\theta_y(x)$, shear force $V_z(x)$ and moment $M_y(x)$ for transverse free vibration in $x-z$ plane are shown below:

$$w_z(x) = -\frac{l^4}{\beta^4 EI_y}\left[B_1 \frac{\lambda_1}{l} \sinh\left(\lambda_1 \frac{x}{l}\right) + B_2 \frac{\lambda_1}{l} \cosh\left(\lambda_1 \frac{x}{l}\right) - B_3 \frac{\lambda_2}{l} \sin\left(\lambda_2 \frac{x}{l}\right) + B_4 \frac{\lambda_2}{l} \cos\left(\lambda_2 \frac{x}{l}\right)\right], \tag{9}$$

$$\theta_y(x) = \frac{l^2}{\beta^4 EI_y}\left\{(\sigma + \lambda_1^2)\left[B_1 \cosh\left(\lambda_1 \frac{x}{l}\right) + B_2 \sinh\left(\lambda_1 \frac{x}{l}\right)\right]. + (\sigma - \lambda_2^2)\left[B_3 \cos\left(\lambda_2 \frac{x}{l}\right) + B_4 \frac{\lambda_2}{l} \sin\left(\lambda_2 \frac{x}{l}\right)\right]\right\}, \tag{10}$$

$$V_z(x) = B_1 \cosh\left(\lambda_1 \frac{x}{l}\right) + B_2 \sinh\left(\lambda_1 \frac{x}{l}\right) + B_3 \cos\left(\lambda_2 \frac{x}{l}\right) + B_4 \sin\left(\lambda_2 \frac{x}{l}\right), \tag{11}$$

$$M_y(x) = \frac{l^2}{\beta^4}\left\{(\sigma + \lambda_1^2)\frac{\lambda_1}{l}\left[B_1 \sinh\left(\lambda_1 \frac{x}{l}\right) + B_2 \cosh\left(\lambda_1 \frac{x}{l}\right)\right] - (\sigma - \lambda_2^2)\frac{\lambda_2}{l}\left[B_3 \sin\left(\lambda_2 \frac{x}{l}\right) - B_4 \cos\left(\lambda_2 \frac{x}{l}\right)\right]\right\}. \tag{12}$$

The general solutions of deformation $w_x(x)$ and shear force $V_x(x)$ for axial free vibration are shown below:

$$w_x(x) = C_1 \cos \beta_e x + C_2 \sin \beta_e x, \tag{13}$$

$$V_x = -C_1\, EA\, \beta_e \sin \beta_e x + C_2\, EA\, \beta_e \cos \beta_e x. \tag{14}$$

The general solutions of slope $\theta_x(x)$ and moment $M_x(x)$ for torsional free vibration are shown below:

$$\theta_x(x) = D_1 \cos \beta_t x + D_2 \sin \beta_t x, \tag{15}$$

$$M_x = -D_1\, GJ_t\, \beta_t \sin \beta_t x + D_2\, GJ_t\, \beta_t \cos \beta_t x. \tag{16}$$

In general, the force and displacement can be expressed in a matrix form as

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = BC, \tag{17}$$

where

For axial vibration:

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_x \\ V_x \end{Bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\beta_e x & \sin\beta_e x \\ -EA\,\beta_e \sin\beta_e x & EA\,\beta_e \cos\beta_e x \end{bmatrix}.$$

For torsional vibration:

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} \theta_x \\ M_x \end{Bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\beta_t x & \sin\beta_t x \\ -GJ_t\,\beta_t \sin\beta_t x & GJ_t\,\beta_t \cos\beta_t x \end{bmatrix}.$$

For flexural vibration in $x-y$ plane:

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_y \\ \theta_z \\ V_y \\ M_z \end{Bmatrix},$$

and

$$B = \begin{bmatrix} \frac{-l^3\lambda_1}{\beta^4\,EI_z}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^3\lambda_1}{\beta^4\,EI_z}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^3\lambda_2}{\beta^4\,EI_z}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l^3\lambda_2}{\beta^4\,EI_z}\cos\left(\lambda_2\frac{x}{l}\right) \\ \frac{-l^2(\sigma+\lambda_1^2)}{\beta^4\,EI_z}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^2(\sigma+\lambda_1^2)}{\beta^4\,EI_z}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^2(\sigma-\lambda_2^2)}{\beta^4\,EI_z}\cos\left(\lambda_2\frac{x}{l}\right) & \frac{-l^2(\sigma-\lambda_2^2)}{\beta^4\,EI_z}\sin\left(\lambda_2\frac{x}{l}\right) \\ \cosh\left(\lambda_1\frac{x}{l}\right) & \sinh\left(\lambda_1\frac{x}{l}\right) & \cos\left(\lambda_2\frac{x}{l}\right) & \sin\left(\lambda_2\frac{x}{l}\right) \\ \frac{-l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\cos\left(\lambda_2\frac{x}{l}\right) \end{bmatrix}.$$

For flexural vibration in $x-z$ plane:

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_z \\ \theta_y \\ V_z \\ M_y \end{Bmatrix},$$

and

$$B = \begin{bmatrix} \frac{-l^3\lambda_1}{\beta^4\,EI_y}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^3\lambda_1}{\beta^4\,EI_y}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^3\lambda_2}{\beta^4\,EI_y}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l^3\lambda_2}{\beta^4\,EI_y}\cos\left(\lambda_2\frac{x}{l}\right) \\ \frac{l^2(\sigma+\lambda_1^2)}{\beta^4\,EI_y}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^2(\sigma+\lambda_1^2)}{\beta^4\,EI_y}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{l^2(\sigma-\lambda_2^2)}{\beta^4\,EI_y}\cos\left(\lambda_2\frac{x}{l}\right) & \frac{l^2(\sigma-\lambda_2^2)}{\beta^4\,EI_y}\sin\left(\lambda_2\frac{x}{l}\right) \\ \cosh\left(\lambda_1\frac{x}{l}\right) & \sinh\left(\lambda_1\frac{x}{l}\right) & \cos\left(\lambda_2\frac{x}{l}\right) & \sin\left(\lambda_2\frac{x}{l}\right) \\ \frac{l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\cosh\left(\lambda_1\frac{x}{l}\right) & -\frac{l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\cos\left(\lambda_2\frac{x}{l}\right) \end{bmatrix}.$$

Note that C is unknown coefficient vector and depends on the boundary conditions.

2.1.1. Transfer matrix

The unknown constants in terms of states can be expressed as

$$S_x = \begin{Bmatrix} d_x \\ f_x \end{Bmatrix} = B(x)C. \tag{18}$$

At point $x = 0$, Eq. (12) can be obtained as

$$S_0 = \begin{Bmatrix} d_0 \\ f_0 \end{Bmatrix} = B(0)C. \tag{19}$$

Thus, the coefficient C can be obtained by

$$C = B(0)^{-1} \cdot S_0. \tag{20}$$

Substitute Eq. (20) into Eq. (19) to yield

$$S_x = B(x)B(0)^{-1} \cdot S_0 = T \cdot S_0, \tag{21}$$

where T is the transfer matrix and expressed as

$$T = B(x)B(0)^{-1} \tag{22}$$

2.1.2. Derivation of transfer dynamic stiffness matrix

For rearranging TMM into a dynamic stiffness matrix form, the transfer matrix solution in mixed form appears as Eq. (21)

$$S_x = T(x) \cdot S_0. \tag{23}$$

Eq. (23) can be rewritten as

$$S_x = \begin{Bmatrix} d_x \\ f_x \end{Bmatrix} = \begin{bmatrix} T_{dd} & T_{df} \\ T_{fd} & T_{ff} \end{bmatrix} \begin{Bmatrix} d_0 \\ f_0 \end{Bmatrix} = T(x) \cdot S_0. \tag{24}$$

Rearrange Eq. (24) as

$$\begin{Bmatrix} f_0 \\ f_x \end{Bmatrix} = \begin{bmatrix} -T_{df}^{-1} T_{dd} & T_{df}^{-1} \\ T_{fd} - T_{ff} T_{df}^{-1} T_{dd} & T_{ff} T_{df}^{-1} \end{bmatrix} \begin{Bmatrix} d_0 \\ d_x \end{Bmatrix} = K \cdot \begin{Bmatrix} d_0 \\ d_x \end{Bmatrix}, \tag{25}$$

where K is the 12×12 transfer dynamic stiffness matrix for a 3-D beam member.

The global transfer dynamic stiffness matrix K_G is obtained by assembling the TDSM for each member in the structure. After applying the boundary condition, the eigenvalue problem for free vibration can be written as

$$K_R(\alpha) U_R = 0, \tag{26}$$

where K_R, U_R are the reduced global transfer dynamic stiffness matrix and eigenvector, respectively.

Assume that all entries in K_R are a continuously differentiable with respect to design variables.

2.2. Eigenfunction problem formulation

Once we know the natural frequencies and the eigenvectors, which are the responses at the nodes, we may compute eigenfunction for the continuous structure use the dynamic shape function approach. Thus, we will derive the dynamic shape function first.

Recall Eqs. (5) and (6), the general solutions of deformation and slope for transverse free vibration in $x-y$ plane are

$$w_y(x) = -\frac{l^4}{\beta^4 EI_z}\left[C_1\frac{\lambda_1}{l}\sinh\left(\lambda_1\frac{x}{l}\right) + C_2\frac{\lambda_1}{l}\cosh\left(\lambda_1\frac{x}{l}\right) - C_3\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right) + C_4\frac{\lambda_2}{l}\cos\left(\lambda_2\frac{x}{l}\right)\right], \tag{5'}$$

$$\theta_z(x) = \frac{-l^2}{\beta^4 EI_z}\left\{(\sigma+\lambda_1^2)\left[C_1\cosh\left(\lambda_1\frac{x}{l}\right) + C_2\sinh\left(\lambda_1\frac{x}{l}\right)\right]. + (\sigma-\lambda_2^2)\left[C_3\cos\left(\lambda_2\frac{x}{l}\right) + C_4\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right)\right]\right\}. \tag{6'}$$

Recall Eqs. (9) and (10), the general solutions of deformation and slope for transverse free vibration in $x-z$ plane are

$$w_z(x) = -\frac{l^4}{\beta^4 EI_y}\left[B_1\frac{\lambda_1}{l}\sinh\left(\lambda_1\frac{x}{l}\right) + B_2\frac{\lambda_1}{l}\cosh\left(\lambda_1\frac{x}{l}\right) - B_3\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right) + B_4\frac{\lambda_2}{l}\cos\left(\lambda_2\frac{x}{l}\right)\right], \tag{9'}$$

$$\theta_y(x) = \frac{l^2}{\beta^4 EI_y}\left\{(\sigma+\lambda_1^2)\left[B_1\cosh\left(\lambda_1\frac{x}{l}\right) + B_2\sinh\left(\lambda_1\frac{x}{l}\right)\right]. + (\sigma-\lambda_2^2)\left[B_3\cos\left(\lambda_2\frac{x}{l}\right) + B_4\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right)\right]\right\}. \tag{10'}$$

Recall Eq. (13), the general solution of the deflection for axial free vibration must be taken to be

$$w_x(x) = C_1\cos\beta x + C_2\sin\beta x. \tag{13'}$$

Recall Eq. (15), the general solution of the slope for torsional free vibration must be taken to be

$$\theta_x(x) = C_1\cos\beta x + C_2\sin\beta x. \tag{15'}$$

In general, the displacement function for beam element can be expressed as

$$[U] = [D(x)]\cdot C. \tag{27}$$

For a typical beam element, the displacement at both ends can be taken as

$$[U^e] = \begin{bmatrix} U_i \\ U_j \end{bmatrix} = \begin{bmatrix} D(x_i=0) \\ D(x_j=L) \end{bmatrix}\cdot[C] = [A]\cdot C. \tag{28}$$

Solve Eq. (28) for the coefficient, we get

$$[C] = [A]^{-1} \cdot [U^e]. \tag{29}$$

Substituting Eq. (29) into Eq. (27), we have

$$[U] = [D(x)] \cdot [A]^{-1} \cdot [U^e] = [N] \cdot [U^e], \tag{30}$$

where N is dynamic shape function and expressed as

$$[N] = [D(x)] \cdot [A]^{-1}. \tag{31}$$

Thus, the eigenfunction can be determined by the eigenvector and the dynamic shape function. For each beam member, the eigenfuction corresponding to a particular frequency can be computed from Eq. (30). Note that U is the eigenfunction and U^e is the end displacement for each beam member.

3. Sensitivity analysis

The computation for derivatives of eigensolution and eigenfunction with respect to design variables α will be performed in the following:

3.1. Eigensolution sensitivity

Let ω_n be a natural frequency and let the corresponding eigenvector $\{\phi\}$ be normalized to unit length. Thus, Eq. (26) can be rewritten as

$$[K_R(\omega_n)]\{\phi\} = \{0\}, \tag{32}$$

and

$$\{\phi\}^{\mathrm{T}}\{\phi\} = 1. \tag{33}$$

Differentiate the above equation with respect to α, we have

$$[K_R']\{\phi\} + [K_R]\{\phi'\} = 0, \tag{34}$$

and

$$\{\phi\}^{\mathrm{T}}\{\phi\} = 0. \tag{35}$$

Now, using the chain rule of differentiation as given below:

$$K_R' = \frac{\mathrm{d}K_R}{\mathrm{d}\alpha} = \frac{\partial K_R}{\partial \alpha} + \frac{\partial K_R}{\partial \omega_n}\frac{\partial \omega_n}{\partial \alpha}. \tag{36}$$

Substituting Eq. (36) into Eq. (34) yields

$$\frac{\partial K_R}{\partial \alpha}\phi + \frac{\partial K_R}{\partial \omega_n}\frac{\partial \omega_n}{\partial \alpha}\phi + K_R\phi' = 0. \tag{37}$$

Rewriting the above equation, we have

$$\frac{\partial K_R}{\partial \omega_n}\phi\omega_n' + K_R\phi' = -\frac{\partial K_R}{\partial \alpha}\phi. \tag{38}$$

Putting Eqs. (38) and (35) together, we have

$$[Z]\{y\} = \{Q\}, \tag{39}$$

where

$$\{y\} = \begin{Bmatrix} \phi' \\ \omega_n' \end{Bmatrix},$$

$$[z] = \begin{bmatrix} K_R & \dfrac{\partial K_R}{\partial \omega_n}\phi \\ \phi^{\mathrm{T}} & 0 \end{bmatrix},$$

$$[Q] = \begin{bmatrix} -\dfrac{\partial K_R}{\partial \alpha}\phi \\ 0 \end{bmatrix}.$$

Note that both the eigenvalue and eigenvector sensitivity, ω_n' and ϕ', are computed from Eq. (39). $\partial K_R/\partial\alpha$ and $\partial K_R/\partial\omega_n$ are assembled from the corresponding derivative of the dynamic stiffness matrix of the system.

3.2. Eigenfunction sensitivity

Once eigenvector sensitivity ϕ' for the whole structure is available, the eigenfunction for each member in the structure can be obtained by

$$U_G^{q'} = N_G^q \cdot \phi_e^{q'} + N_G^{q'} \phi_e^q, \tag{40}$$

where $U_G^{q'}$ is the eigenfunction sensitivity for the member q in global co-ordinate, N_G^q the dynamic shape function of the member q in global co-ordinate, $\phi_e^{q'}$ is the eigenvector sensitivity for the member q in global co-ordinate, $N_G^{q'}$ is the derivative with respect to a design variable of the dynamic shape function of the member q in global co-ordinate.

4. Optimization design

In this section, exact natural frequency solutions are employed in dynamic response constraints. Therefore, the minimum weight of frame structures with continuous design variables can be achieved by solving the following constrained minimization problem:

$$\text{Find} \quad x_i \tag{41}$$

$$\text{Minimize} \quad W_e(x) = \textstyle\sum_{i=1}^{m} A(x_i)\rho_i L_i\,, \tag{42}$$

$$\begin{aligned} \text{Subject to} \quad & \lambda_i \geqslant \lambda_{ir} \quad \text{or} \quad g(x) = \frac{-\lambda_i}{\lambda_{ir}} + 1 \leqslant 0, \\ & x_i \in S_i, \end{aligned} \tag{43}$$

where $W_e(x)$ is the weight of the frame structure. x_i is the ith design variable, which is the dimension for the cross-section of the ith member of a frame with the same sectional area. A_i, ρ_i and L_i are the cross-sectional area corresponding to x_i, density and length of the ith member, respectively. λ_i is the ith eigenvalue of the frame structure. λ_{ir} is the frequency limit for the ith eigenvalue. S_i is a given continuous dimensions for the ith design variable, which implicitly include the lower bound $LB_i$and upper bound UB_i of the ith design variable.

Once the sensitivity data are available, the objective and constraint sensitivity can be computed by using the following equations to be useful in reducing the set of the optimization:

$$\nabla g_d(x_i) = \frac{-\nabla \lambda_i}{\lambda_{ir}}, \tag{44}$$

$$\nabla W_e(x) = \sum_{i=1}^{m} \nabla A(x_i)\rho_i L_i, \tag{45}$$

where $\nabla g_d(x_i)$ is the constraint gradient for ith design variable, $\nabla W_e(x)$the objective gradient, $\nabla \lambda_i = \partial \lambda(x_i)/\partial x_i$, $\nabla A(x_i) = \partial A(x_i)/\partial x_i$.

5. Numerical examples

Three numerical examples, a 3-D portal frame, a two-level portal frame and a three-story space structure demonstrated the principle and algorithm described in the above sections. Fig. 2 shows a 3-D portal frame with four nodes and three members. The structure is completely fixed at nodes 1 and 4. Fig. 3 shows a two-level portal frame with six nodes and six members. The structure is

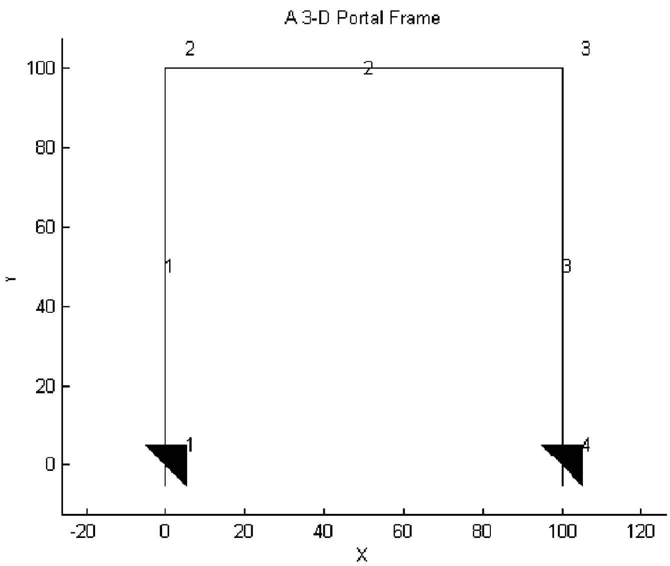

Fig. 2. 3-D portal frame.

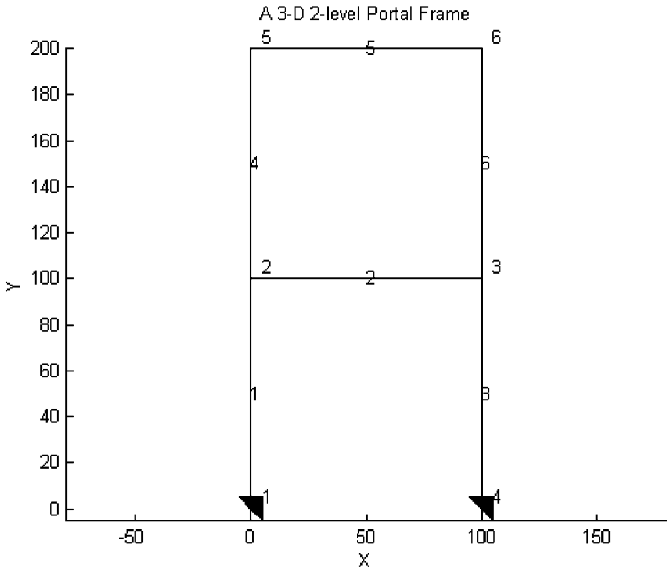

Fig. 3. Two-level portal frame.

completely fixed at nodes 1 and 4. Fig. 4 shows a three-story space structure with 16 nodes and 24 members. The structure is completely fixed at nodes 1–4. The length of each member for those three examples is as shown in Table 2, including Young's modulus, shear modulus, density and cross-sectional area.

5.1. Sensitivity of natural frequency

For 3-D portal frame, let X_1, X_2, and X_3 be the diameters of members 1, 2 and 3, respectively. The resulting first natural frequency of this example is 3.6151 Hz by Timoshenko beam theory. The sensitivity of this first natural frequency to the three design variables is shown in Table 3. These results are validated by the central difference calculation (using $\partial\omega_1/\partial x_i = \omega_1(x_i + \Delta x_i) - \omega_1(x_i - \Delta x_i)/2\Delta x_i$, $i = 1$–3, $\Delta x_i = 0.001$). These sensitivity data indicated that increasing X_1 and X_3 would increase ω_1 while increase X_2 will cause ω_1 to decrease.

For two-level portal frame, Let X_1, X_2, X_3,X_4, X_5, and X_6 designate the diameters of members 1, 2, 3, 4, 5, and 6, respectively. The resulting first natural frequency of this example is 1.0885 Hz by Timoshenko beam theory. The sensitivity of this first natural frequency to the six design variables is shown in Table 4. These results are validated by the central difference calculation. These sensitivity data indicated that increasing X_1 and X_3 would increase ω_1 while increase X_2, X_4, X_5, and X_6 will cause ω_1 to decrease.

For three-story space structure, let X_1, X_2, X_3, X_4, X_5, and X_6 be the diameters of members 1–4, 5–9, 10–12, 13–16, 17–20, and 21–24, respectively. The first natural frequency of this example is 1.4399 Hz by Timoshenko beam theory. The sensitivity data shown in Table 5 are validated by the central difference calculation. These sensitivity data indicated that increasing X_1, X_2, X_3, and X_4 would increase ω_1 while increase, X_5 and X_6 will cause ω_1 to decrease.

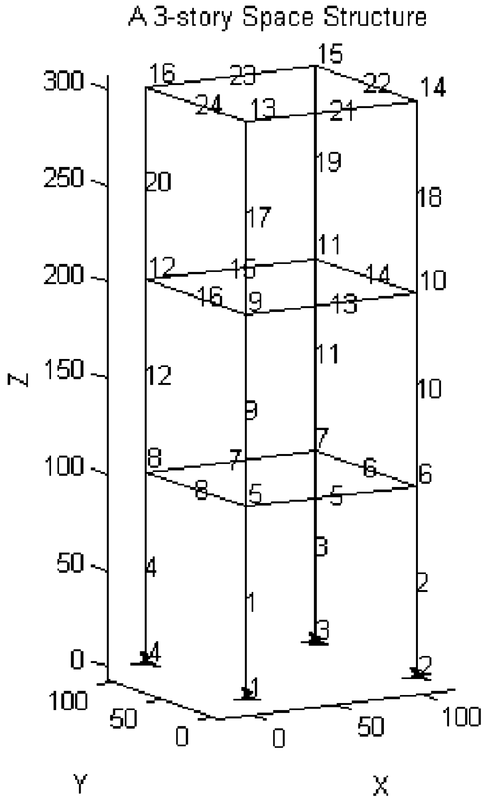

Fig. 4. Three-story space structure.

Table 2
Material properties for three examples

Length (in)	100
Young's modulus E (psi)	1×10^7
Shear modulus G (psi)	0.8×10^7
Mass density ρ (lb/in^3)	0.1/386.4
Cross-sectional area (in^2)	4

Table 3
Sensitivity of the first natural frequency for 3-D portal frame

	Timoshenko beam theory
$\partial\omega_1/\partial X_1$	8.3795
$\partial\omega_1/\partial X_2$	−6.6970
$\partial\omega_1/\partial X_3$	8.3795

Table 4
Sensitivity of the first natural frequency for two-level portal frame

	Timoshenko beam theory
$\partial\omega_1/\partial X_1$	2.7633
$\partial\omega_1/\partial X_2$	−0.1572
$\partial\omega_1/\partial X_3$	2.7633
$\partial\omega_1/\partial X_4$	−0.4441
$\partial\omega_1/\partial X_5$	−1.4510
$\partial\omega_1/\partial X_6$	−0.4441

Table 5
Sensitivity of the first natural frequency for three-story space structure

	Timoshenko beam theory
$\partial\omega_1/\partial X_1$	2.3351
$\partial\omega_1/\partial X_2$	2.1302
$\partial\omega_1/\partial X_3$	0.8794
$\partial\omega_1/\partial X_4$	0.5504
$\partial\omega_1/\partial X_5$	−0.7193
$\partial\omega_1/\partial X_6$	−1.1742

Table 6
Comparison of optimal design using proposed method for 3-D portal frame

	Without gradient	With gradient
Minimum weight	0.42342	0.42369
Constraint value	−2.1602E−9	−2.1602E−9
$X1$	3.149	3.15
$X2$	1	1
$X3$	3.1489	3.15

5.2. *Optimal design example*

To illustrate the optimal design problem formulated in Section 4, the following three examples were solved using constrained minimization by MATLAB.

For 3-D portal frame, we want to find the diameter of each member to minimize the weight of structure which maintaining the first natural frequency larger than 50 rad/s or 7.9577 Hz. The design variables X_1, X_2, and X_3 are the diameters of members 1, 2 and 3, respectively, for 3-D portal frame. The lower and upper bounds for each design variables are 1 and 10 in, respectively.

The results using the proposed method are given in Table 6. In addition, the optimal design problem was solved with or without gradient data. The design history without using constraint gradient is shown in Fig. 5. Fig. 6 shows design history when gradient data are used. In Fig. 5, the

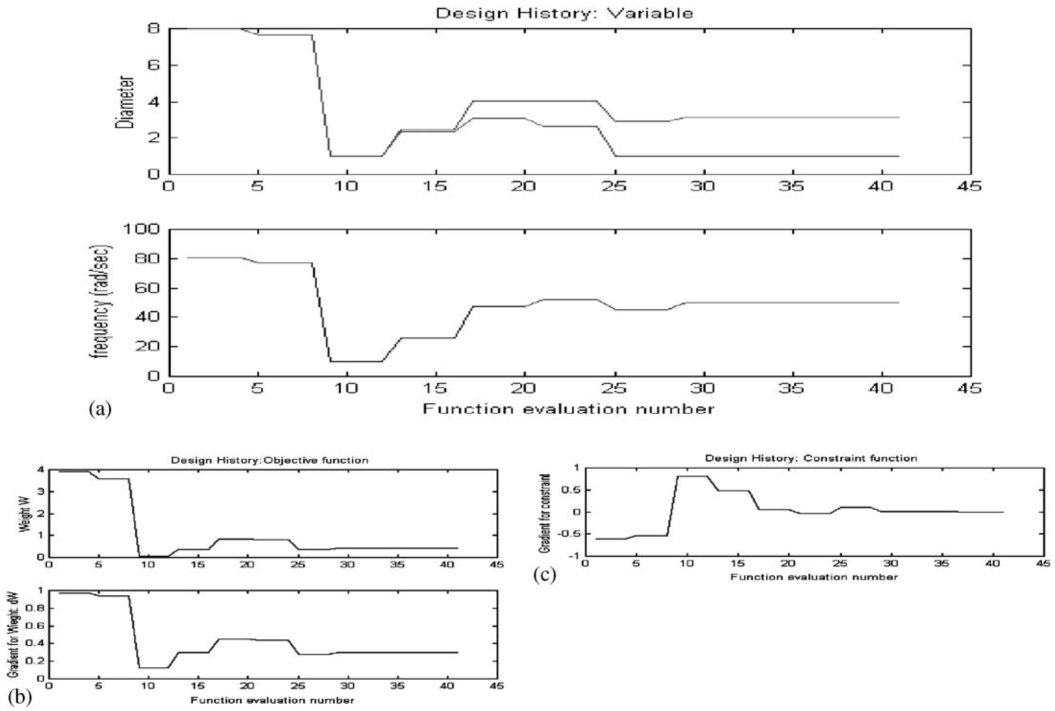

Fig. 5. Design history without constraint and objective gradients for 3-D portal frame using proposed method: (a) history of variables and frequency, (b) history of objective function, and (c) history of constraint function.

number of solution iterations without gradient is 41. But, in Fig. 6, the number of solution iterations using gradient is 13. So note that when gradient data are used, the solution converges in fewer functional evaluations.

The optimal design shown in Table 7 was obtained by ANSYS based on subproblem approximation method and first order optimization method. The evaluation for solution converges 13 without gradient and 6 with gradient, respectively.

For two-level portal frame, we want to find the diameter of each member to minimize the weight of structure which maintaining the first natural frequency larger than 20 rad/s or 3.183 Hz. The design variables X_1, X_2, X_3,X_4, X_5, and X_6 are the diameters of members 1, 2, 3, 4, 5, and 6, respectively. The lower and upper bounds for each design variables are 1 and 10 in, respectively.

The results using the proposed method are given in Table 8. In addition, this optimal design problem is solved with or without gradient data. The design history without using constraint gradient is shown in Fig. 7. Fig. 8 shows design history when gradient data are used. In Fig. 7, the number of solution iterations without gradient is 342. But, in Fig. 8, the number of solution iterations using gradient is 23. So note that when gradient data are used, the solution can converge in fewer functional evaluations.

The optimal design shown in Table 9 is obtained by ANSYS based on subproblem approximation method and first order optimization method. The evaluation for solution converges 21 without gradient and 7 with gradient, respectively.

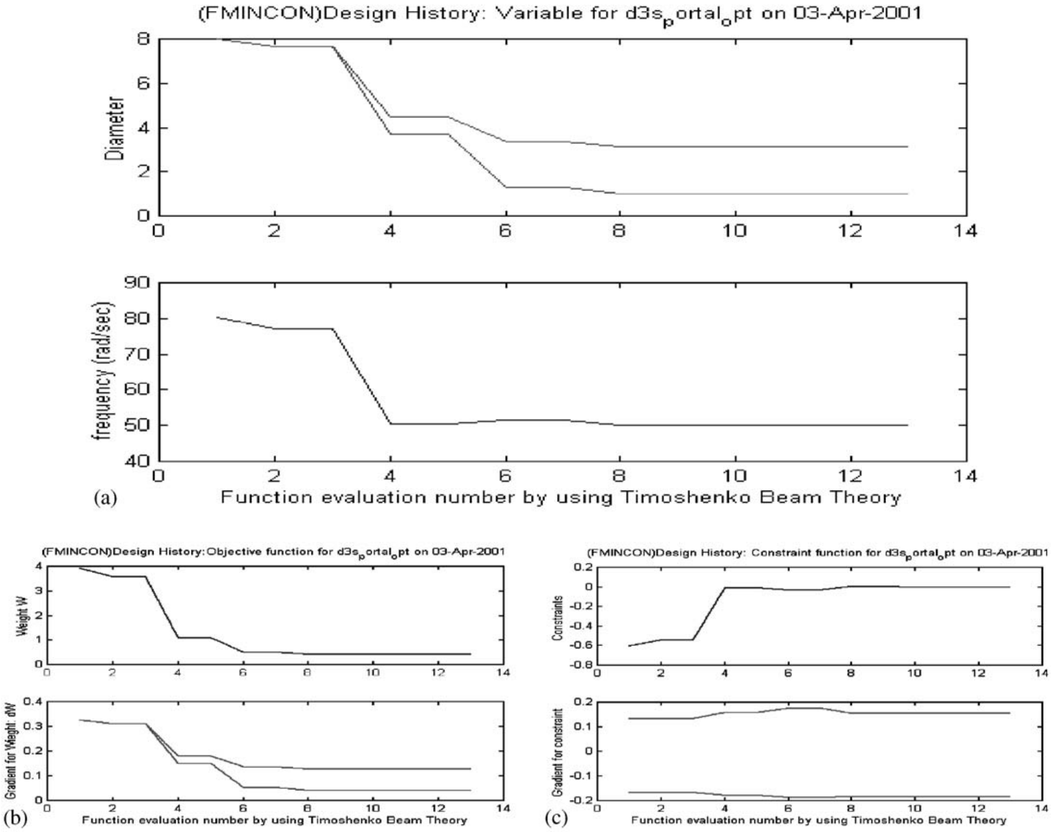

Fig. 6. Design history with constraint and objective gradients for 3-D portal frame using proposed method: (a) history of variables and frequency, (b) history of objective function and objective gradient, and (c) history of constraint function and constraint gradient.

Table 7
Comparison of optimal design using ANSYS for 3-D portal frame

	Without gradient	With gradient
Minimum weight	0.50618	0.43467
Constraint value	−0.44824	0.12574
$X1$	3.6	3.1926
$X2$	1.1583	1
$X3$	3.256	3.1926

For three-story space structure, we want to find the diameter of each member to minimize the weight of structure which maintaining the first natural frequency larger than 20 rad/s or 3.183 Hz. The design variables X_1, X_2, X_3,X_4, X_5, and X_6 are the diameters of members 1–4, 5–9, 10–12, 13–16, 17–20, and 21–24, respectively. The lower and upper bounds for each design variables are 1 and 10 in, respectively.

Table 8
Comparison of optimal design using proposed method for two-level portal frame

	Without gradient	With gradient
Minimum weight	0.83704	0.75406
Constraint value	−1.2122E−6	−1.1183E−8
$X1$	4.1414	3.7062
$X2$	1	1
$X3$	4.0163	3.7062
$X4$	1.7171	1.9528
$X5$	1.0001	1
$X6$	1.7177	1.9528

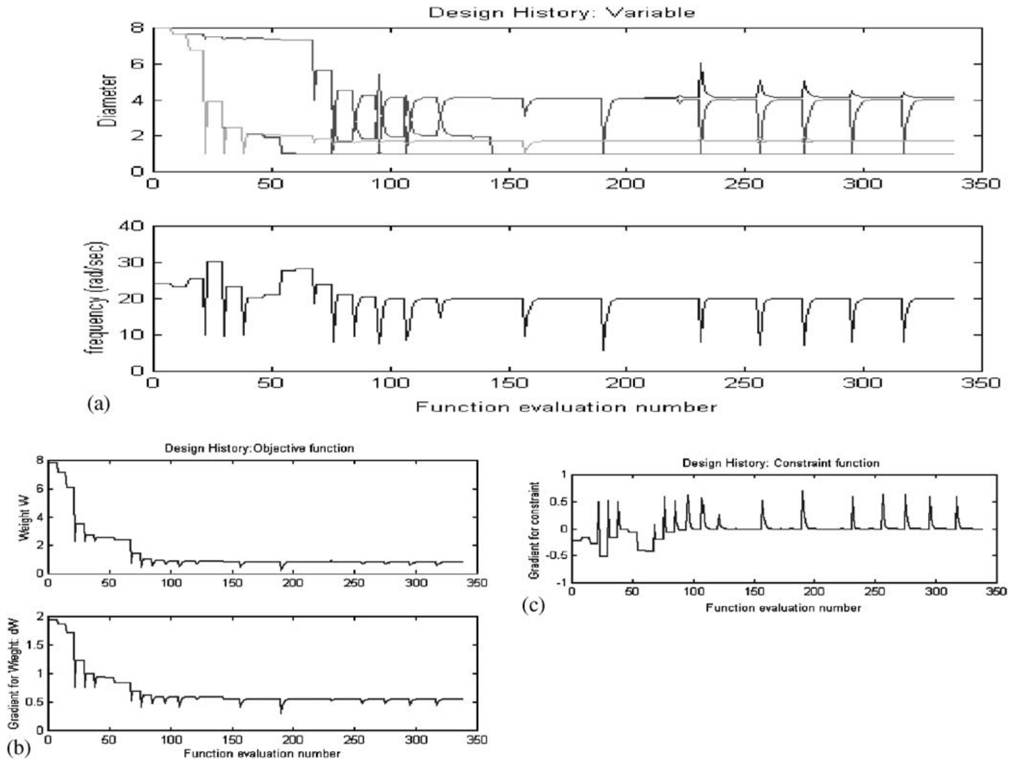

Fig. 7. Design history without constraint and objective gradients for two-level portal frame using proposed method: (a) history of variables and frequency, (b) history of objective function, and (c) history of constraint function.

The results using the proposed method and ANSYS are given in Table 10. In addition, in each theory, this optimal design problem is solved with gradient data directly after its solutions are good agreement within the first two examples. Similar minimum weights are obtained from both methods.

The design history with using gradients based on Timoshenko beam theory is shown in Fig. 9. The number of its iterations using gradient shown in Fig. 9 is 19.

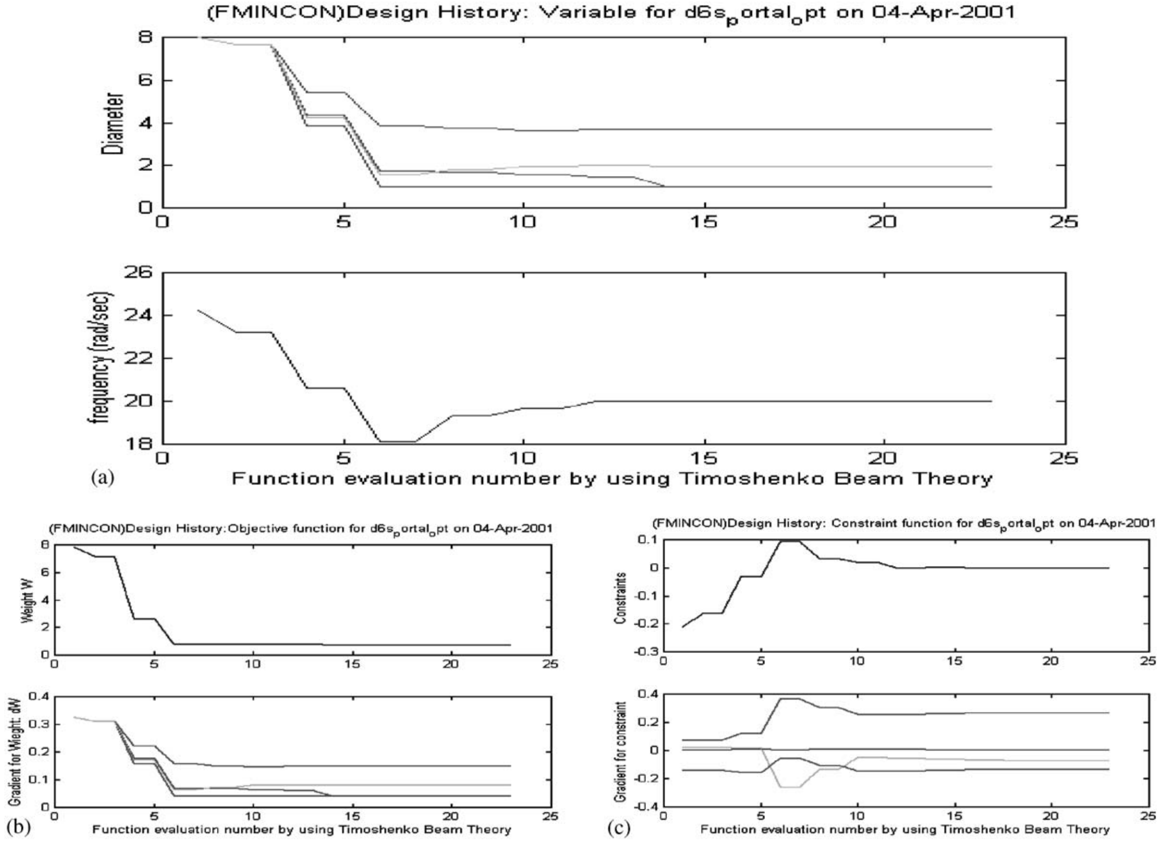

Fig. 8. Design history with constraint and objective gradients for two-level portal frame using proposed method: (a) history of variables and frequency. (b) history of objective function and objective gradient, and (c) history of constraint function and constraint gradient.

Table 9
Comparison of optimal design using ANSYS for two-level portal frame

	Without gradient	With gradient
Minimum weight	1.5726	0.80218
Constraint value	0.14852	0.13498E−2
$X1$	5.4261	3.8338
$X2$	2.0865	1.5402
$X3$	4.6181	3.8338
$X4$	3.5718	1.83
$X5$	2.644	1
$X6$	1.5806	1.83

6. Conclusions

The transfer dynamic stiffness matrix formulation is used in the sensitivity analysis of space frame structures. The exact sensitivity of eigenvalues with respect to design variables has

Table 10
Comparison of optimal design with gradients using proposed method and ANSYS for three-story space structure

	Proposed method	ANSYS
Minimum weight	4.6772	4.683
Constraint value	−7.588E−9	−0.41294E−1
$X1$	4.0248	4.2662
$X2$	3.9694	4.0789
$X3$	3.3107	3.1194
$X4$	3.0763	2.8519
$X5$	2.0366	1.9738
$X6$	1	1.0

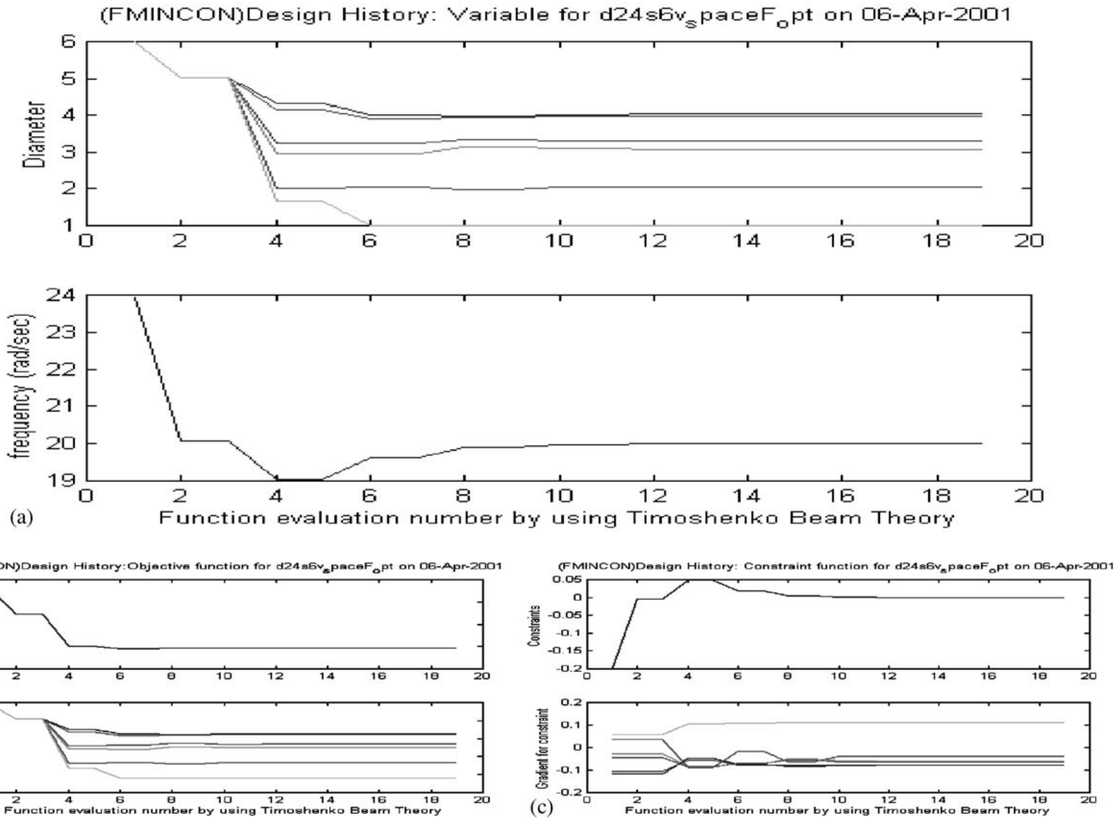

Fig. 9. Design history with constraint and objective gradients for three-story structure using proposed method: (a) history of variables and frequency, (b) history of objective function and objective gradient, and (c) history of constraint function and constraint gradient.

computed and also has been used for optimum design. The influence of design variables to the natural frequency was illustrated in examples. This derivation provides engineers valuable information for structural design and modification. Numerical examples demonstrated the accuracy of natural frequency sensitivity computed by exact formulation.

In optimal design, the proposed method can solve the problem of minimum weight design with constraints of eigenvalue using gradient. Three design examples demonstrate that. When gradient data are used, the optimal design process converges faster than the one with no gradient data used and the optimal design result presented is also agreed with that by ANSYS.

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