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Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method

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Abstract

A study of the free vibration of Timoshenko beams and axisymmetric Mindlin plates is presented. The analysis is based on the Chebyshev pseudospectral method, which has been widely used in the solution of fluid mechanics problems. Clamped, simply supported, free and sliding boundary conditions of Timoshenko beams are treated, and numerical results are presented for different thickness-to-length ratios. Eigenvalues of the axisymmetric vibration of Mindlin plates with clamped, simply supported and free boundary conditions are presented for various thickness-to-radius ratios.

1. Introduction

The vibration of beams and plates is important in many applications pertaining to mechanical, civil and aerospace engineering. Beams and plates used in real practice may have appreciable thickness where the transverse shear and the rotary inertia are not negligible as assumed in the classical theories. As a result, the thick beam model based on the Timoshenko theory and the thick plate model based on the Mindlin theory have gained more popularity.

Research on the vibration of Timoshenko beams and Mindlin plates can be divided into three categories. Firstly, there exist exact solutions only for a very restricted number of simple cases. Secondly, studies of semi-analytic solutions, including the differential quadrature method [1–3] and the boundary characteristic orthogonal polynomials [4,5], are available. Finally, there are the most widely used discretization methods such as the finite element method and the finite difference method. As it is more useful to have analytical results than to resort to numerical methods, most

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efforts focus on developing efficient semi-analytic solutions. Although the Rayleigh–Ritz method with the differential quadrature method and the boundary characteristic orthogonal polynomials have been successful in the analysis of beams and plates, there are some drawbacks inherent in these methods. For example, they require a process of constructing either weighting coefficients or characteristic polynomials since there are no readily available formulas.

The pseudospectral method can be considered to be a spectral method that performs a collocation process. As the formulation is straightforward and powerful enough to produce approximate solutions close to exact solutions, this method has been highly successful in many areas such as turbulence modelling, weather prediction and non-linear waves [6]. Even though this method can be used for the solution of structural mechanics problems, it has been largely unnoticed by the structural mechanics community, and few articles are available where the pseudospectral method has been applied.

Soni and Amba-Rao [7] and Gupta and Lal [8] are among those who have applied the pseudospectral method to the axisymmetric vibration analysis of circular and annular Mindlin plates. Recently, the usefulness of the pseudospectral method in the solution of structural mechanics problems has been demonstrated in a static analysis of the L-shaped Reissner–Mindlin plate [9]. In the present work, the pseudospectral method is applied to the eigenvalue analysis of Timoshenko beams and axisymmetric circular Mindlin plates.

2. Timoshenko beams

The equations of motion of a homogeneous beam of rectangular cross-section based on the Timoshenko theory are derived as

$$\begin{aligned} \frac{\partial M}{\partial x} + V &= \rho I \frac{\partial^2 \Theta}{\partial t^2}, \\ \frac{\partial V}{\partial x} &= \rho h \frac{\partial^2 W}{\partial t^2}, \end{aligned} \tag{1}$$

where $W(x,t)$ and $\Theta(x,t)$ are the lateral deflection and the rotation of the normal line, ρ is the mass density, h is the thickness of the beam, and $I = h^3/12$ is the second moment of area per unit width. The stress resultants M and V are defined by

$$\begin{aligned} M &= EI \frac{\partial \Theta}{\partial x}, \\ V &= \alpha G h \left(\frac{\partial W}{\partial x} - \Theta \right), \end{aligned} \tag{2}$$

where E is the modulus of elasticity, G is the shear modulus, and α is the shear coefficient. Assuming the sinusoidal motion in time

$$\begin{aligned} \Theta(x,t) &= \theta(x) \cos \omega t, \\ W(x,t) &= w(x) \cos \omega t, \end{aligned} \tag{3}$$

the substitution of Eq. (2) into Eq. (1) yields

$$
\begin{aligned}
&EI\frac{d^2\theta}{dx^2}+\alpha hG\left(\frac{dw}{dx}-\theta\right)=-\omega^2\rho I\theta,\\
&-\alpha hG\frac{d\theta}{dx}+\alpha hG\frac{d^2w}{dx^2}=-\omega^2\rho hw.
\end{aligned}
\tag{4}
$$

When the range of the independent variable is given by $x\in[-L/2, L/2]$, where L is the length of the beam, it is convenient to use the normalized variable

$$
z=\frac{2x}{L}\in[-1,1],
\tag{5}
$$

and (4) is rewritten as

$$
\begin{aligned}
&EI\left(\frac{2}{L}\right)^2\theta''-\alpha hG\theta+\alpha hG\frac{2}{L}w'=-\omega^2\rho I\theta,\\
&-\alpha hG\frac{2}{L}\theta'+\alpha hG\left(\frac{2}{L}\right)^2w''=-\omega^2\rho hw,
\end{aligned}
\tag{6}
$$

where $'$ stands for the differentiation with respect to z.

The boundary conditions are represented as follows:

clamped (C):

$$
\theta=0,\quad w=0,
\tag{7}
$$

pinned (P):

$$
M=0,\quad w=0,
\tag{8}
$$

free (F):

$$
M=0,\quad V=0,
\tag{9}
$$

and *sliding* (S)

$$
\theta=0,\quad V=0,
\tag{10}
$$

at the extremity $z=\pm1$.

In their attempts to compute the natural frequencies of axisymmetric circular and annular Mindlin plates, Soni and Amba-Rao [7] and Gupta and Lal [8] formed fourth order linear differential equations with variable coefficients in terms of the bending rotation by eliminating the lateral deflection, and applied the pseudospectral method. The boundary conditions that did not contain the eigenvalue were combined with the governing equations to form the characteristic equations from which the eigenvalues were calculated. In the present study, the conceptual simplicity of the pseudospectral method is used, and the solution of Eq. (6) is pursued without eliminating w.

The series expansions of the exact solutions $\theta(z)$ and $w(z)$ have an infinite number of terms. In this study, however, the eigenfunctions $\theta(z)$ and $w(z)$ are approximated by the $(N+2)$th partial sums as follows:

$$\theta(z) \approx \tilde{\theta}(z) = \sum_{n=1}^{N+2} a_n T_{n-1}(z),$$
$$w(z) \approx \tilde{w}(z) = \sum_{n=1}^{N+2} b_n T_{n-1}(z), \tag{11}$$

where a_n and b_n are the expansion coefficients, and $T_{n-1}(z)$ is the Chebyshev polynomial of the first kind of degree of $n-1$. Mikami and Yoshimura suggested an efficient way to handle the boundary conditions by adopting two less collocation points than the number of expansion terms [10]. The pseudospectral algebraic system of equations is formed by setting the residuals of Eq. (6) equal to zero at the Chebyshev interpolation grid points

$$z_i = -\cos\frac{\pi(2i-1)}{2N}, \quad i = 1, \ldots, N. \tag{12}$$

Expansions (11) are substituted into Eq. (6) and are collocated at z_i to yield

$$\sum_{n=1}^{N+2}\left[a_n\left\{\frac{4EI}{L^2}T''_{n-1}(z_i) - \alpha hGT_{n-1}(z_i)\right\} + \frac{2\alpha hG}{L}b_n T'_{n-1}(z_i)\right]$$
$$= -\omega^2\rho I\sum_{n=1}^{N+2} a_n T_{n-1}(z_i), \tag{13}$$

$$\sum_{n=1}^{N+2}\left\{-\frac{2\alpha hG}{L}a_n T'_{n-1}(z_i) + \frac{4\alpha hG}{L^2}b_n T''_{n-1}(z_i)\right\} = -\omega^2\rho h\sum_{n=1}^{N+2} b_n T_{n-1}(z_i), \quad i = 1, \ldots, N.$$

This can be rearranged in the matrix form

$$[\mathbf{H}]\{\mathbf{d}\} + [\mathbf{H}^\dagger]\{\mathbf{d}^\dagger\} = \omega^2([\mathbf{S}]\{\mathbf{d}\} + [\mathbf{S}^\dagger]\{\mathbf{d}^\dagger\}), \tag{14}$$

where the vectors $\{\mathbf{d}\}$ and $\{\mathbf{d}^\dagger\}$ are defined by

$$\{\mathbf{d}\} = \{a_1 a_2 \cdots a_N b_1 b_2 \cdots b_N\}^{\mathrm{T}},$$
$$\{\mathbf{d}^\dagger\} = \{a_{N+1} a_{N+2} b_{N+1} b_{N+2}\}^{\mathrm{T}}. \tag{15}$$

The size of matrices $[\mathbf{H}]$ and $[\mathbf{S}]$ is $2N \times 2N$, and that of $[\mathbf{H}^\dagger]$ and $[\mathbf{S}^\dagger]$ is $2N \times 4$. The total number of unknowns in $\{\mathbf{d}\}$ and $\{\mathbf{d}^\dagger\}$ is $2N + 4$ whereas the number of equations in Eq. (13) is $2N$. The remaining four equations are obtained from the boundary conditions. When Eq. (11) is substituted into Eqs. (7)–(10), the boundary conditions at $z_b = \pm 1$ are expressed as follows:

clamped (C):

$$\sum_{n=1}^{N+2} a_n T_{n-1}(z_b) = 0, \quad \sum_{n=1}^{N+2} b_n T_{n-1}(z_b) = 0, \tag{16}$$

pinned (P):

$$\sum_{n=1}^{N+2} a_n T'_{n-1}(z_b) = 0, \quad \sum_{n=1}^{N+2} b_n T_{n-1}(z_b) = 0, \tag{17}$$

free (F):

$$\sum_{n=1}^{N+2} a_n T'_{n-1}(z_b) = 0, \quad \sum_{n=1}^{N+2} \left\{ a_n T_{n-1}(z_b) - \frac{2b_n}{L} T'_{n-1}(z_b) \right\} = 0, \tag{18}$$

and *sliding* (S):

$$\sum_{n=1}^{N+2} a_n T_{n-1}(z_b) = 0, \quad \sum_{n=1}^{N+2} \left\{ a_n T_{n-1}(z_b) - \frac{2b_n}{L} T'_{n-1}(z_b) \right\} = 0. \tag{19}$$

The physical boundary condition of the Timoshenko beam is accomplished by picking one set of conditions up from Eqs. (16)–(19) at $z_b = -1$ and another at $z_b = 1$. The clamped–clamped (C–C) boundary condition, for example, consists of the following four equations:

$$\sum_{n=1}^{N+2} a_n T_{n-1}(-1) = 0, \quad \sum_{n=1}^{N+2} b_n T_{n-1}(-1) = 0,$$
$$\sum_{n=1}^{N+2} a_n T_{n-1}(1) = 0, \quad \sum_{n=1}^{N+2} b_n T_{n-1}(1) = 0. \tag{20}$$

Eq. (20) can be rearranged in the matrix form

$$[\mathbf{U}]\{\mathbf{d}\} + [\mathbf{V}]\{\mathbf{d}^\dagger\} = \{\mathbf{0}\}, \tag{21}$$

where $\{\mathbf{0}\}$ is a zero vector. Since $\{\mathbf{d}^\dagger\}$ can be expressed as

$$\{\mathbf{d}^\dagger\} = -[\mathbf{V}]^{-1}[\mathbf{U}]\{\mathbf{d}\}, \tag{22}$$

Eq. (14) can be reformulated as

$$([\mathbf{H}] - [\mathbf{H}^\dagger][\mathbf{V}]^{-1}[\mathbf{U}])\{\mathbf{d}\} = \omega^2([\mathbf{S}] - [\mathbf{S}^\dagger][\mathbf{V}]^{-1}[\mathbf{U}])\{\mathbf{d}\}. \tag{23}$$

The solution of Eq. (23) yields the estimate for the natural frequencies and the corresponding eigenmodes. This procedure can be applied to any boundary condition pair of C–C, C–P, C–F, C–S, P–P, P–F, P–S, F–F, F–S or S–S.

The algebraic problem is solved for the eigenvalues using the Eispack GRR subroutine. A preliminary run for the convergence check of the eigenvalues of the Timoshenko beam with C–C boundary condition is carried out for $h/L = 0.01$, and the result is given in Table 1. The number of collocation points which determines the size of the problem changes from 10 to 40. This clearly shows the rapid convergence of the pseudospectral method that requires less than $N = 20$ for the first 6 eigenvalues to converge to 6 significant digits, and less than $N = 35$ for the convergence of the lowest 15 modes to 6 significant digits. The numbers given in Tables 1–5 are non-dimensionalized frequency parameters λ_i defined as

$$\lambda_i^2 = \omega_i L^2 \sqrt{\frac{m}{EI}}, \tag{24}$$

where m is mass per unit length of the beam. Throughout the paper, Poisson's ratio and the shear coefficient of the beam are $\nu = 0.3$ and $\alpha = 5/6$, respectively. Computational results with $N = 35$ for C–C, P–P, F–F and P–S boundary conditions are given in Tables 2–5, where the eigenvalues based on the classical theory [11] are added for comparison. The natural frequencies are calculated

Table 1
Convergence test of the non-dimensionalized frequency parameter λ_i of the Timoshenko beam as the number of the collocation points N increases (clamped–clamped boundary condition, $\nu = 0.3$, $\alpha = 5/6$, $h/L = 0.01$)

Mode	$N = 10$	$N = 15$	$N = 20$	$N = 25$	$N = 30$	$N = 35$	$N = 40$
1	4.72840	4.72840	4.72840	4.72840	4.72840	4.72840	4.72840
2	7.84691	7.84690	7.84690	7.84690	7.84690	7.84690	7.84690
3	10.9827	10.9800	10.9800	10.9800	10.9800	10.9800	10.9800
4	14.1132	14.1062	14.1062	14.1062	14.1062	14.1062	14.1062
5	18.1397	17.2246	17.2246	17.2246	17.2246	17.2246	17.2246
6	21.5723	20.3422	20.3338	20.3338	20.3338	20.3338	20.3338
7	38.0443	23.4481	23.4325	23.4325	23.4325	23.4325	23.4325
8	42.2513	27.1739	26.5192	26.5192	26.5192	26.5192	26.5192
9	—	30.4163	29.6033	29.5926	29.5926	29.5926	29.5926
10	—	40.1211	32.6684	32.6515	32.6514	32.6514	32.6514
11	—	43.8761	36.2185	35.6947	35.6946	35.6946	35.6946
12	—	79.1294	39.3662	38.7331	38.7209	38.7209	38.7209
13	—	83.0320	46.4220	41.7474	41.7294	41.7293	41.7293
14	—	—	49.8430	45.1366	44.7191	44.7189	44.7189
15	—	—	68.0861	48.1952	47.7008	47.6888	47.6888

Table 2
Non-dimensionalized frequency parameter λ_i of the Timoshenko beam (clamped–clamped boundary condition, $\nu = 0.3$, $\alpha = 5/6$, $N = 35$)

Mode	Classical theory	h/L						
		0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	4.73004	4.72998	4.72963	4.72840	4.72350	4.68991	4.57955	4.24201
2	7.85320	7.85295	7.85163	7.84690	7.82817	7.70352	7.33122	6.41794
3	10.9956	10.9950	10.9917	10.9800	10.9341	10.6401	9.85611	8.28532
4	14.1372	14.1359	14.1294	14.1062	14.0154	13.4611	12.1454	9.90372
5	17.2788	17.2766	17.2651	17.2246	17.0679	16.1590	14.2324	11.3487
6	20.4204	20.4168	20.3985	20.3338	20.0868	18.7318	16.1487	12.6402
7	23.5619	23.5567	23.5292	23.4325	23.0682	21.1825	17.9215	13.4567
8	26.7035	26.6960	26.6567	26.5192	26.0086	23.5168	19.5723	13.8101
9	29.8451	29.8348	29.7808	29.5926	28.9052	25.7421	21.1185	14.4806
10	32.9867	32.9729	32.9009	32.6514	31.7558	27.8662	22.5735	14.9383
11	36.1283	36.1103	36.0168	35.6946	34.5587	29.8969	23.9479	15.6996
12	39.2699	39.2470	39.1281	38.7209	37.3126	31.8418	25.2479	16.0040
13	42.4115	42.3829	42.2345	41.7293	40.0169	33.7078	26.2831	16.9621
14	45.5531	45.5178	45.3355	44.7189	42.6712	35.5011	26.4595	16.9999
15	48.6947	48.6519	48.4308	47.6888	45.2754	37.2275	26.9237	17.9357

for different thickness-to-length ratios from $h/L = 0.002$ to 0.2. These results show that the Timoshenko beam results are very close to the Bernoulli–Euler results when h/L is less than 0.01. As h/L grows larger, however, the computed natural frequencies tend to show some quantitative differences from the Bernoulli–Euler results.

Table 3
Non-dimensionalized frequency parameter λ_i of the Timoshenko beam (pinned–pinned boundary condition, $\nu = 0.3$, $\alpha = 5/6$, $N = 35$)

Mode	Classical theory	h/L						
		0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	3.14159	3.14158	3.14153	3.14133	3.14053	3.13498	3.11568	3.04533
2	6.28319	6.28310	6.28265	6.28106	6.27471	6.23136	6.09066	5.67155
3	9.42478	9.42449	9.42298	9.41761	9.39632	9.25537	8.84052	7.83952
4	12.5664	12.5657	12.5621	12.5494	12.4994	12.1813	11.3431	9.65709
5	15.7080	15.7066	15.6997	15.6749	15.5784	14.9926	13.6132	11.2220
6	18.8496	18.8473	18.8352	18.7926	18.6282	17.6810	15.6790	12.6022
7	21.9911	21.9875	21.9684	21.9011	21.6443	20.2447	17.5705	13.0323
8	25.1327	25.1273	25.0988	24.9988	24.6227	22.6862	19.3142	13.4443
9	28.2743	28.2666	28.2261	28.0845	27.5599	25.0111	20.9325	13.8433
10	31.4159	31.4053	31.3498	31.1568	30.4533	27.2263	22.4441	14.4378
11	34.5575	34.5434	34.4697	34.2145	33.3006	29.3394	23.8639	14.9766
12	37.6991	37.6807	37.5853	37.2565	36.1001	31.3581	25.2044	15.6676
13	40.8407	40.8174	40.6962	40.2815	38.8507	33.2896	26.0647	16.0241
14	43.9823	43.9531	43.8021	43.2886	41.5517	35.1410	26.2814	16.9584
15	47.1239	47.0880	46.9027	46.2769	44.2026	36.9186	26.4758	17.0019

Table 4
Non-dimensionalized frequency parameter λ_i of the Timoshenko beam (free–free boundary condition, $\nu = 0.3$, $\alpha = 5/6$, $N = 35$)

Mode	Classical theory	h/L						
		0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	4.73004	4.73000	4.72982	4.72918	4.72659	4.70873	4.64849	4.44958
2	7.85320	7.85304	7.85217	7.84908	7.83679	7.75404	7.49719	6.80257
3	10.9956	10.9952	10.9928	10.9843	10.9508	10.7332	10.1255	8.77287
4	14.1372	14.1362	14.1311	14.1131	14.0426	13.6040	12.5076	10.4094
5	17.2788	17.2770	17.2678	17.2350	17.1078	16.3550	14.6682	11.7942
6	20.4204	20.4174	20.4022	20.3483	20.1415	18.9813	16.6358	12.8163
7	23.5619	23.5575	23.5341	23.4516	23.1394	21.4834	18.4375	13.5584
8	26.7035	26.6970	26.6630	26.5436	26.0979	23.8654	20.0959	13.6520
9	29.8451	29.8360	29.7885	29.6228	29.0138	26.1335	21.6283	14.6971
10	32.9867	32.9744	32.9104	32.6881	31.8846	28.2949	23.0452	14.7384
11	36.1283	36.1122	36.0282	35.7382	34.7084	30.3571	24.3472	15.8190
12	39.2699	39.2492	39.1415	38.7719	37.4839	32.3275	25.5006	15.9135
13	42.4115	42.3854	42.2500	41.7882	40.2099	34.2132	26.2976	16.9742
14	45.5531	45.5207	45.3534	44.7861	42.8861	36.0205	26.3874	16.9918
15	48.6947	48.6552	48.4512	47.7647	45.5121	37.7554	27.1340	17.9829

Table 5
Non-dimensionalized frequency parameter λ_i of the Timoshenko beam (pinned–sliding boundary condition, $\nu = 0.3$, $\alpha = 5/6$, $N = 35$)

Mode	Classical theory	h/L						
		0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	1.57080	1.57080	1.57080	1.57076	1.57066	1.56997	1.56749	1.55784
2	4.71239	4.71235	4.71216	4.71149	4.70880	4.69027	4.62769	4.42026
3	7.85398	7.85382	7.85294	7.84983	7.83746	7.75423	7.49632	6.80658
4	10.9956	10.9951	10.9927	10.9842	10.9505	10.7319	10.1223	8.78525
5	14.1372	14.1362	14.1311	14.1130	14.0423	13.6020	12.5056	10.4663
6	17.2788	17.2770	17.2677	17.2348	17.1073	16.3524	14.6697	11.9320
7	20.4204	20.4174	20.4021	20.3481	20.1408	18.9784	16.6448	13.1407
8	23.5619	23.5575	23.5340	23.4514	23.1384	21.4804	18.4593	13.2379
9	26.7035	26.6970	26.6629	26.5432	26.0966	23.8628	20.1378	13.8936
10	29.8451	29.8360	29.7884	29.6224	29.0123	26.1320	21.7007	14.4219
11	32.9867	32.9744	32.9103	32.6876	31.8828	28.2952	23.1646	15.0377
12	36.1283	36.1121	36.0280	35.7375	34.7064	30.3601	24.5434	15.5100
13	39.2699	39.2492	39.1413	38.7712	37.4816	32.3343	25.8482	16.3112
14	42.4115	42.3854	42.2498	41.7874	40.2074	34.2249	26.1196	16.5209
15	45.5531	45.5207	45.3531	44.7852	42.8834	36.0386	26.5417	17.4685

3. Axisymmetric Mindlin plates

The eigenvalue problem of the axisymmetric vibration of circular Mindlin plates can be solved through the same procedure as the Timoshenko beam problem. The equations of motion of a homogeneous, isotropic axisymmetric circular plate based on the Mindlin theory are

$$\begin{aligned} &\frac{\partial M_r}{\partial r} + \frac{1}{r}(M_r - M_\theta) - Q = \rho I \frac{\partial^2 \Psi}{\partial t^2}, \\ &\frac{\partial Q}{\partial r} + \frac{1}{r} Q = \rho h \frac{\partial^2 W}{\partial t^2}, \end{aligned} \tag{25}$$

where the transverse deflection $W(r,t)$ and the bending rotation normal to the midplane in the radial direction $\Psi(r,t)$ are the dependent variables. The stress resultants M_r, M_θ and Q are defined by

$$\begin{aligned} M_r &= D\left(\frac{\partial \Psi}{\partial r} + \frac{\nu}{r}\Psi\right), \\ M_\theta &= D\left(\frac{\Psi}{r} + \nu\frac{\partial \Psi}{\partial r}\right), \\ Q &= \kappa^2 G h\left(\Psi + \frac{\partial W}{\partial r}\right), \end{aligned} \tag{26}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, and $\kappa^2 = \pi^2/12$ is the shear correction factor.

Assuming the sinusoidal motion in time

$$\Psi(r,t) = \psi(r)\cos\omega t,\\ W(r,t) = w(r)\cos\omega t \tag{27}$$

the substitution of Eq. (26) into Eq. (25) yields

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \left(\frac{1}{r^2} + \frac{\kappa^2 Gh}{D}\right)\psi - \frac{\kappa^2 Gh}{D}\frac{dw}{dr} = -\omega^2\frac{\rho I}{D}\psi,\\ \frac{d\psi}{dr} + \frac{\psi}{r} + \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = -\omega^2\frac{\rho}{\kappa^2 G}w. \tag{28}$$

We consider the following boundary conditions:

$$\textit{clamped}:\ w = 0,\quad \psi = 0,\\ \textit{simply supported}:\ w = 0,\quad M_r = 0,\\ \textit{free}:\ M_r = 0,\quad Q = 0. \tag{29}$$

The distance from the origin in a circular plate, r, can be normalized as

$$z = \frac{r}{R} \in [0, 1], \tag{30}$$

where R is the radius of the circular plate, and Eq. (28) becomes

$$\frac{1}{R^2}\psi'' + \frac{1}{zR^2}\psi' - \left(\frac{1}{z^2R^2} + \frac{\kappa^2 Gh}{D}\right)\psi - \frac{\kappa^2 Gh}{RD}w' = -\omega^2\frac{\rho I}{D}\psi,\\ \frac{1}{R}\psi' + \frac{1}{zR}\psi + \frac{1}{R^2}w'' + \frac{1}{zR^2}w' = -\omega^2\frac{\rho}{\kappa^2 G}w. \tag{31}$$

Fornberg [12] recommended that in the axisymmetric analysis using the pseudospectral method it is advantageous to extend the range of the independent variable over $[-1, 1]$ and then to use the symmetry and antisymmetry to reduce the actual calculations to within $[0, 1]$. The conditions at the center of the plate for the axisymmetric vibration are $\psi = 0$ and $Q = 0$. The boundary conditions at $z = 0$ are satisfied when the approximations of $\psi(z)$ and $w(z)$ are given by

$$\psi(z) \approx \tilde{\psi}(z) = \sum_{n=1}^{N+1} a_n T_{2n-1}(z),\\ w(z) \approx \tilde{w}(z) = \sum_{n=1}^{N+1} b_n T_{2n-2}(z). \tag{32}$$

The number of collocation points is less than that of expansion terms in Eq. (32) by one because the conditions at $z = 0$ have already been considered, and the collocation points are defined by

$$z_i = \cos\frac{\pi(2i-1)}{4N},\quad i = 1, \ldots, N. \tag{33}$$

The eigenfunction expansions (32) are substituted into Eq. (31) and are collocated at z_i to yield

$$\sum_{n=1}^{N+1} a_n\left\{\frac{1}{R^2}T''_{2n-1}(z_i)+\frac{1}{z_iR^2}T'_{2n-1}(z_i)-\left(\frac{1}{z_i^2R^2}+\frac{\kappa^2Gh}{D}\right)T_{2n-1}(z_i)\right\} - \frac{\kappa^2Gh}{RD}\sum_{n=1}^{N+1} b_nT'_{2n-2}(z_i) = -\omega^2\frac{\rho I}{D}\sum_{n=1}^{N+1} a_nT_{2n-1}(z_i),$$

$$\sum_{n=1}^{N+1} a_n\left\{\frac{1}{R}T'_{2n-1}(z_i)+\frac{1}{z_iR}T_{2n-1}(z_i)\right\} + \sum_{n=1}^{N+1} b_n\left\{\frac{1}{R^2}T''_{2n-2}(z_i)+\frac{1}{z_iR^2}T'_{2n-2}(z_i)\right\} = -\omega^2\frac{\rho}{\kappa^2G}\sum_{n=1}^{N+1} b_nT_{2n-2}(z_i), \quad i=1,\ldots,N. \tag{34}$$

The eigenvalue problem (34) can be written in the matrix form Eq. (14), where the vector $\{\mathbf{d}^\dagger\}$ is redefined by

$$\{\mathbf{d}^\dagger\} = \{a_{N+1}b_{N+1}\}^{\mathrm{T}}, \tag{35}$$

and the size of matrices $[\mathbf{H}^\dagger]$ and $[\mathbf{S}^\dagger]$ is $(2N\times 2)$. The total number of unknowns in $\{\mathbf{d}\}$ and $\{\mathbf{d}^\dagger\}$ is $2N+2$, whereas the number of equations in Eq. (34) is $2N$. The remaining two equations are obtained from the boundary conditions. When eigenfunction approximations (32) are substituted into Eq. (29), the boundary conditions are expressed as follows:

clamped boundary condition:

$$\sum_{n=1}^{N+1} a_nT_{2n-1}(1)=0, \quad \sum_{n=1}^{N+1} b_nT_{2n-2}(1)=0, \tag{36}$$

simply supported boundary condition:

$$\sum_{n=1}^{N+1} a_n\{T'_{2n-1}(1)+\nu T_{2n-1}(1)\}=0, \quad \sum_{n=1}^{N+1} b_nT_{2n-2}(1)=0, \tag{37}$$

free boundary condition:

$$\sum_{n=1}^{N+1} a_n\{T'_{2n-1}(1)+\nu T_{2n-1}(1)\}=0,$$
$$\sum_{n=1}^{N+1}\left\{a_nT_{2n-1}(1)+\frac{b_n}{R}T'_{2n-2}(1)\right\}=0. \tag{38}$$

Each of the boundary conditions (36)–(38) can be written in the matrix form (21), and the eigenvalue problem (34) is reformulated into the matrix form (23). Computational results with $N=35$ for clamped, simply supported and free boundary conditions are given in Tables 6–8, where the eigenvalues based on the classical theory [13] are added for comparison. The natural frequencies are calculated for different thickness-to-radius ratios from $h/R=0.005$ to 0.25. Results show good agreement with the thin plate results when h/R is small, and deviate

Table 6
Non-dimensionalized frequency parameter λ_i^2 of the axisymmetric vibration of the Mindlin plate (clamped boundary condition, $\nu = 0.3$, $N = 35$)

Mode	Classical theory	h/R							
		0.005	0.01	0.02	0.05	0.1	0.15	0.2	0.25
1	10.2158	10.215	10.213	10.204	10.145	9.9408	9.6286	9.2400	8.8068
2	39.771	39.762	39.733	39.620	38.855	36.479	33.393	30.211	27.253
3	89.104	89.060	88.926	88.401	84.995	75.664	65.551	56.682	49.420
4	158.183	158.05	157.65	156.08	146.40	123.32	102.09	85.571	73.054
5	247.01	246.69	245.74	242.06	220.73	176.42	140.93	115.56	97.198
6	355.568	354.92	353.00	345.64	305.71	232.97	180.99	145.94	117.90
7	483.872	482.69	479.19	466.04	399.32	291.71	221.62	174.97	122.43
8	631.914	629.91	624.05	602.37	499.82	351.82	262.45	178.76	144.42
9	799.702	796.52	787.27	753.72	605.79	412.77	301.11	205.32	148.75
10	987.216	982.42	968.52	919.15	716.07	474.18	305.15	210.53	170.38
11		1187.5	1167.4	1097.7	829.74	535.81	336.52	237.46	181.05
12		1411.7	1383.6	1288.4	946.07	597.43	345.59	248.18	195.12
13		1654.8	1616.7	1490.5	1064.5	657.61	380.88	268.60	216.40
14		1916.7	1866.3	1702.9	1184.5	662.37	388.16	290.67	220.58
15		2197.4	2131.8	1924.8	1305.7	698.63	425.43	299.71	243.02

Table 7
Non-dimensionalized frequency parameter λ_i^2 of the axisymmetric vibration of the Mindlin plate (simply supported boundary condition, $\nu = 0.3$, $N = 35$)

Mode	Classical theory	h/R							
		0.005	0.01	0.02	0.05	0.1	0.15	0.2	0.25
1	4.977	4.9349	4.9345	4.9335	4.9247	4.8938	4.8440	4.7773	4.6963
2	29.76	29.716	29.704	29.656	29.323	28.240	26.715	24.994	23.254
3	74.20	74.131	74.054	73.752	71.756	65.942	59.062	52.514	46.775
4	138.34	138.23	137.96	136.92	130.35	113.57	96.775	82.766	71.603
5		221.99	221.30	218.65	202.81	167.53	136.98	113.87	96.609
6		325.37	323.89	318.28	286.79	225.34	178.23	145.13	108.27
7		448.29	445.52	435.05	380.13	285.44	219.86	166.29	121.50
8		590.71	585.93	568.13	480.94	346.83	261.51	176.28	131.65
9		752.54	744.82	716.60	587.65	408.91	291.55	191.38	146.17
10		933.71	921.89	879.54	698.97	471.31	303.05	207.23	163.30
11		1134.1	1116.8	1056.0	813.85	533.80	318.34	227.28	170.65
12		1353.6	1329.1	1245.0	931.50	596.23	344.39	237.98	194.94
13		1592.1	1558.5	1445.6	1051.2	649.28	359.27	268.49	198.98
14		1849.6	1804.5	1657.0	1172.6	658.55	385.53	269.03	219.11
15		2125.7	2066.7	1878.2	1295.1	677.58	408.98	298.91	236.92

Table 8
Non-dimensionalized frequency parameter λ_i^2 of the axisymmetric vibration of the Mindlin plates (free boundary condition, $\nu = 0.3$, $N = 35$)

Mode	Classical theory	h/R							
		0.005	0.01	0.02	0.05	0.1	0.15	0.2	0.25
1	9.084	9.0028	9.0017	8.9976	8.9686	8.8679	8.7095	8.5051	8.2674
2	38.55	38.436	38.416	38.335	37.787	36.041	33.674	31.111	28.605
3	87.80	87.715	87.609	87.189	84.443	76.676	67.827	59.645	52.584
4	157.0	156.70	156.37	155.04	146.76	126.27	106.40	90.059	76.936
5	245.9	245.35	244.53	241.31	222.38	181.46	146.83	120.57	99.545
6	354.6	353.61	351.89	345.31	308.98	239.98	187.79	149.63	114.53
7	483.1	481.42	478.24	466.27	404.44	300.38	228.39	171.18	126.34
8	631.0	628.70	623.31	603.32	506.96	361.73	267.32	183.36	138.59
9	798.6	795.37	786.79	755.54	615.01	423.41	297.08	199.04	154.77
10	986.0	981.36	968.36	922.00	727.37	484.92	310.03	217.13	166.06
11		1186.5	1167.7	1101.7	843.04	545.74	330.92	231.82	182.35
12		1410.8	1384.3	1293.8	961.26	604.75	351.70	251.78	197.61
13		1654.1	1617.9	1497.3	1081.4	653.92	372.16	268.69	208.73
14		1916.2	1868.1	1711.3	1202.9	667.41	397.54	285.12	228.95
15		2197.0	2134.3	1934.9	1325.5	695.93	416.63	308.16	238.49

considerably from those of the Kirchhoff plate as h/R grows larger. The numbers given in Tables 6–8 are non-dimensionalized frequency parameters λ_i^2 defined as

$$\lambda_i^2 = \omega_i \frac{R^2}{\sqrt{D/\rho h}}. \tag{39}$$

4. Conclusions

A pseudospectral method using the Chebyshev polynomials as the basis functions is applied to the free vibration analysis of Timoshenko beams and radially symmetric Mindlin plates. Because the formulation is so simple and efficient, allowing the process of calculating weighting coefficients and characteristic polynomials to be avoided, this method has merits over other semi-analytic methods. Rapid convergence, good accuracy as well as the conceptual simplicity characterize the pseudospectral method. The results from this method agree with those of Bernoulli–Euler beams and Kirchhoff plates when the thickness-to-length (radius) ratio is very small, however, deviate considerably as the thickness-to-length (radius) ratio grows larger.

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