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A natural classification of vibration modes of polygonal ducts based on group theoretic analysis

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Abstract

In this work, the natural frequencies and mode shapes of polygonal ducts are obtained by a finite element analysis via a group theoretic approach. We show that the group theoretic techniques provide a natural setting for the solution of the problem via optimum utilization of the symmetry in the problem. The simplifications and insights due to the inherent symmetry is intuitively obvious in simple symmetric structures. However, a systematic exploitation of this symmetry in more complicated structures is less obvious and is possible perhaps only via useful theorems from group representation theory and the associated projection operator theory. We illustrate all the main steps with the help of an example of dihedral groups. This example is directly applicable to the problem addressed in this paper.

The group theoretic approach splits up the original problem into independent subproblems and thereby affects significant computational savings. More importantly, the approach captures the role of symmetry in the problem and provides insights which are otherwise not obvious. The multiplicity of the natural frequencies, arising out of the inherent symmetry is determined a priori and neatly separated into different subproblems. This suggests a neat physical classification of the mode shapes on a symmetry basis which is unique to this approach. The values of the natural frequencies and the mode shapes for triangular, square and pentagonal ducts are shown to be in good agreement with the existing results reported in the literature. The classification of mode shapes on this symmetry basis is used in the analysis of the polygonal ducts with number of sides varying up to 16.

1. Introduction and overview

The free vibration of polygonal ducts has been studied with several different approaches. Polygonal ducts are engineering structures which are used in heating, ventilating or air

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conditioning applications where the vibrations are a major noise source. Therefore, vibration analysis of these structures has received significant attention. Azimi et al. [1] used receptance methods, Yamada and Kobayashi [2] used a transfer matrix method, while Lee [3] used a Raleigh–Ritz method to compute the natural frequencies and mode shapes. All these authors present results for polygonal ducts with 3, 4 and 5 sides and their results are in good agreement. Several important but well-justified assumptions are a necessary part of the analyses reported in literature. Thus, (1) the duct creases are assumed to act like fixed knife edges across which bending moments and slope deformations connect panel to panel and (2) only transverse deflections of the plates that make the duct are considered to be important and that in-plane deflections of the plates are neglected as small. Azimi et al. [1] consider that these assumptions satisfactory for the prediction of the lower natural frequencies of ducts upto 8 sides. The inherent symmetry of the problem gives rise to repeated natural frequencies and mode shapes. It is also noted by Azimi et al. [1] that an obvious neat physical classification for arrangement of these mode shapes does not suggest itself.

In this paper, the natural frequencies and mode shapes are obtained by the finite element method (FEM) augmented with group theoretic routines. Adequate discretization enables the FEM to handle complex geometries to required accuracy. The SIMO element (see Ref. [4–7]) is used to formulate the FE equilibrium equations of the thin panels. The formulation takes into account the contribution of membrane, bending and shear strain energies. This is required as the number of sides of the duct increase beyond eight. However, the analysis entails heavy discretization of the FE mesh and thereby large problem sizes.

We show that group theory provides the natural setting for the solution of the problem. Group theoretic subroutines make optimal utilization of the symmetry in the problem. Finding the natural frequencies of vibration of a structure using finite element discretization reduces to the generalised linear eigenvalue problem (see Ref. [8]).

$$\mathbf{K}\Phi = \omega^2\mathbf{M}\Phi, \tag{1}$$

where $\mathbf{M}$ is the $n \times n$, symmetric, positive semi-definite mass matrix, $\mathbf{K}$ is the $n \times n$, positive definite, symmetric stiffness matrix, ω is one of the natural frequencies and Φ the corresponding mode shape. To exploit the symmetry of the structure, group representation theory can be used to construct an $n \times n$ orthogonal matrix $\mathbf{T}$ such that,

$$\tilde{\mathbf{K}} = \mathbf{T}^{\mathrm{t}}\mathbf{K}\mathbf{T} \quad \text{and} \quad \tilde{\mathbf{M}} = \mathbf{T}^{\mathrm{t}}\mathbf{M}\mathbf{T} \tag{2}$$

each have the same block diagonal form (see Ref. [9]). Thus, the original eigenvalue problem is split into independent subproblems. The heavy discretization necessary for the analysis of polygonal ducts leads to large problem sizes. The block diagonalization using the group theoretic approach yields significant computational advantages. More importantly, the computational effort does not increase with increase in number of sides of the duct because, though the problem size increases, the symmetry of the problem also increases correspondingly. The frequencies of ducts upto 16 sides have been calculated to study the vibration characteristics as the geometry of the duct merges with one of a perfect cylinder. The authors believe that these results are being reported for the first time.

We also show that there are two keys issues in the analysis of free vibration of polygonal ducts.

- The search for lower natural frequencies leads to a finite number of cluster of closely spaced modes which need to be determined.

- There are repeated values of natural frequencies due to the inherent symmetry in the problem. A suitable set of orthogonal eigenvectors needs to be chosen for the eigenspace determined by these repeated natural frequencies.

The group theoretic approach offers unique physical insights on both these issues. It was noted by Azimi et al. [1] that there is no obvious neat physical classification for arrangement of the mode shapes. We show that the symmetry basis due to the group theoretic approach provides the physical classification of the mode shapes of the polygonal ducts.

The layout of the paper is as follows. In Section 2, we discuss the group theoretic approach for the generalised eigenvalue problem of symmetric structures with the help of the example of dihedral groups. The group theoretic results presented here are directly applicable to the free vibration analysis of polygonal ducts. In Section 3, we compare the results of ducts with 3, 4 and 5 sides with the results reported in the literature and show that there is excellent agreement. The analysis is continued for ducts with sides upto 16 to quantitatively understand the vibration characteristics as the geometry merges with that of a perfect cylinder. In Section 4, the physical classification of the mode shapes on a symmetry basis is discussed. We end with some concluding remarks in Section 5.

2. The group theoretic approach for the generalised eigenvalue problem

In this section, we discuss the group theoretic approach for the generalised eigenvalue problem arising in the search for the normal modes of linear free vibration of symmetric elastic structures. A few well-known results from group representation theory and the associated projection operator theory form the basis of this approach. All the main steps are illustrated with the example of dihedral groups (the group of symmetries of a regular polygon) and the reader is directed to relevant literature for complete details.

2.1. The generalised eigenvalue problem

The equation of motion of a finite dimensional approximation to a continuous linear elastic structure leads to a second order linear ordinary differential equation of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{3}$$

We look for special solutions of Eq. (3), referred to as normal co-ordinates where all the points in the structure vibrate harmonically with the same frequency and simultaneously pass through the equilibrium position. Any vibration of the structure is a linear combination of these normal modes. This leads to the generalised eigenvalue problem

$$\mathbf{K}\mathbf{x_0} = \omega^2\mathbf{M}\mathbf{x_0}. \tag{4}$$

The eigenvalues are the natural frequencies and the corresponding eigenvectors are the mode shapes.

There exist n eigenvalues and corresponding eigenvectors that satisfy

$$\mathbf{K}\mathbf{x}_l = \omega_l^2 \mathbf{M}\mathbf{x}_l \quad \text{(no sum)}, \tag{5}$$

where $l = 1, 2, \ldots, n$ denotes the mode number. Further the positive definiteness of $\mathbf{K}$ and $\mathbf{M}$ leads to the result,

$$0 < \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_n \tag{6}$$

and the symmetry of $\mathbf{K}$ and $\mathbf{M}$ implies

$$\begin{aligned} \mathbf{x}_k^{\mathrm{T}} \mathbf{M}\mathbf{x}_l &= \delta_{kl} \quad (\mathbf{M} \text{ orthonormality}), \\ \mathbf{x}_k^{\mathrm{T}} \mathbf{K}\mathbf{x}_l &= \omega_l^2 \delta_{kl} \quad (\mathbf{K} \text{ orthogonality}), \end{aligned} \tag{7}$$

where δ_{kl} is the Kronecker delta. See Ref. [10] for a proof of the orthonormality and orthogonality.

It is important to note that the transformations $\mathbf{X}^{\mathrm{T}}\mathbf{M}\mathbf{X}$ and $\mathbf{X}^{\mathrm{T}}\mathbf{K}\mathbf{X}$ that completely block diagonalise the stiffness and the mass matrix are *not* orthogonal transformations. Thus, the transformation does not have the simple interpretation of a change of basis. In fact, this is the essential difference between the generalised eigenvalue problem and a typical eigenvalue problem where the transformations for the matrix form a change of basis. The transformation in the simple eigenvalue problem is a similarity transformation while the generalised eigenvalue problem requires a congruence transformation. The theorems relating to the conditions for existence of solutions shown in Eq. (7) are discussed in Ref. [11, pp. 466–468].

As explained by Hughes [12, pp. 570–571], an engineer is typically interested in lower modes in considerations of dynamic response. In fact, the higher modes of finite element discretizations are spurious artifacts of the discretization process. Thus, the computational algorithms are aimed at extracting $\{\omega_l^2, \mathbf{x}_l\}$, $1 \leqslant l \leqslant n_{modes}$, where $n_{modes} \ll n$ is the desired number of eigen pairs. In most practical problems, n could be very large ($\sim 20,000$) and, therefore, the solution of the eigenvalue problem, even for a few eigen pairs, may entail extensive and costly calculation. An algorithm which utilises both the Cholesky factorization and the symmetric QR factorization is discussed in Ref. [11, pp. 469–471]. This requires around $14n^3$ *flops* (floating point operations).

The group theoretic approach and the symmetry adapted basis of transformation mentioned in Section 1 (Eqs. (1) and (2)) depend on a key theorem from group representation theory. We discuss this theorem below, in particular its applicability and urge the reader to find the details (including some key definitions and examples of dihedral groups) in the appendix.

2.2. The fundamental theorem

The fundamental theorem addresses the following question,

What do the linear operators $\mathbf{K}$ and $\mathbf{M}$ (Eqs. (1) and (2)) look like in a symmetry adapted basis?

Let L be a linear operator, mapping an n dimensional vector space V into itself. L is said to have the symmetry of the group representation H of the group G if it commutes with every representing matrix, i.e.,

$$LH(g) = H(g)L, \quad \forall g \in G.$$

Theorem 1. *Let G be a finite group with $q(G)$ inequivalent irreducible representations, each of dimension $n_1, n_2, \ldots, n_{q(G)}$. Then there exists a basis for the subspaces, V_μ, of the standard decomposition in which a symmetric linear operator L decomposes into a direct sum of*

n_1 identical square matrices L_1
n_2 identical square matrices L_2
$\vdots$
$n_{q(G)}$ identical square matrices $L_{q(G)}$.

This is the block diagonalisation of the linear operator L. Thus it is clear that the original problem is split up into independent subproblems on a symmetry basis (Figs. 1 and 2).

2.2.1. Application

The basic idea here is that the linear operators **K** and **M** operate on the vector space of displacements which can be decomposed into several subspaces and, correspondingly, the linear operators can be decomposed to give a block diagonal matrix form. Thus, the group theoretic approach decouples the problem into several independent subproblems on a symmetry basis. In particular, for the generalised eigenvalue problem, the subproblems group together vibration modes which share the same symmetry. The main advantages of the approach are:

(1) There are significant savings in computational effort. As discussed in Section 1.2, the aim is to calculate the lower modes of vibration. This is precisely facilitated by calculating the lower modes in each block (i.e., in each subproblem).
(2) There is a priori information on the number of symmetry blocks and the problem sizes of the subproblems.
(3) The finite number of admissible mode shapes (i.e., the symmetry of the subspaces in the standard decomposition) are known a priori. The blocks with the repeated eigenvalues are also known a priori but it is to be noted that repeated eigenvalues may correspond to different mode shapes depending on the symmetry of the associated block.
(4) It also presents an option of parallel processing because the subproblems are independent.

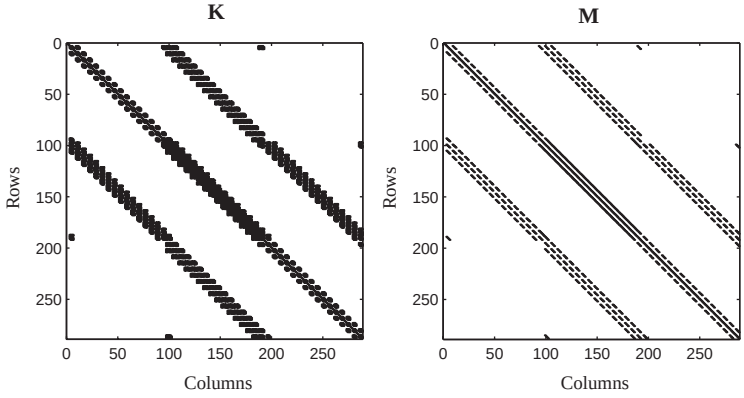

Fig. 1. The banded form of the stiffness and mass matices in the standard basis.

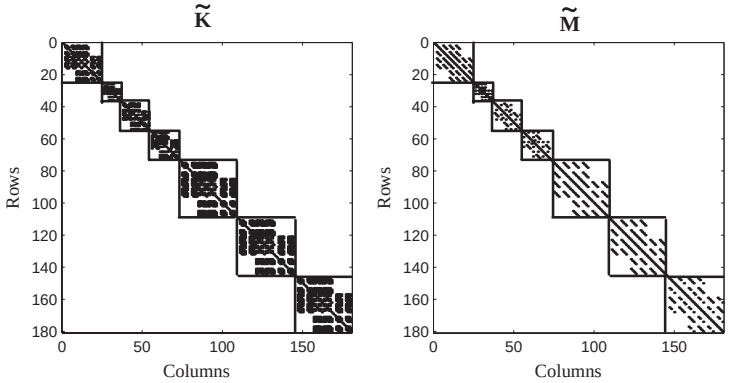

Fig. 2. The identical block diagonalisation of the stiffness and mass matrices in the symmetry adapted basis.

Many important symmetric structures lend themselves to the group theoretic analysis discussed in this article. See Ref. [8] for a discussion on symmetric trusses and Ref. [13] for results on cylindrical shells. In this paper, we discuss free vibration of polygonal ducts where the mode shapes are in no way obvious. We also study the qualitative behaviour as the geometry merges from a structure of low symmetry to one with perfect symmetry (the cylinder). The group theoretic method and the associated decoupling suggest that this is a natural way of dealing with such a problem. In the next section, we present the free vibration analysis of polygonal ducts using this approach.

3. Results and discussion

3.1. Ducts with 3, 4 and 5 sides

In this section, we present the results of the free vibration analysis for polygonal ducts of 3, 4 and 5 sides. The group theoretic results and the results for the loss of symmetry in the finite number of lower vibration modes fall out very neatly for these lower symmetry problems. These mode shapes for the three sided and four sided duct and the associated symmetry are shown in Figs. 3 and 4. Thus, all the lower mode shapes of interest can be obtained in a systematic way and neatly classified on the basis of symmetry. The confusion of missing certain mode shapes for which the connecting moments between plates are zero (this occurs in the receptance method, see Ref. [2]) does not arise. All these aspects, which are due to the group theoretic approach, are discussed in detail for the mode shapes of the pentagonal (see Figs. 5 and 6) duct. The pentagonal duct has rich enough block structure and the discussion is directly applicable to the results of the three sided and square ducts.

The FE analysis, using group theory, of a pentagonal duct is presented here. The a priori information from the group representation theory results discussed in Section 2 and the appendix are as follows,

(1) As per Eq. (A.7) there are exactly six independent subproblems (two associated with one dimensional irreducible representations and four associated with two dimensional irreducible representations).

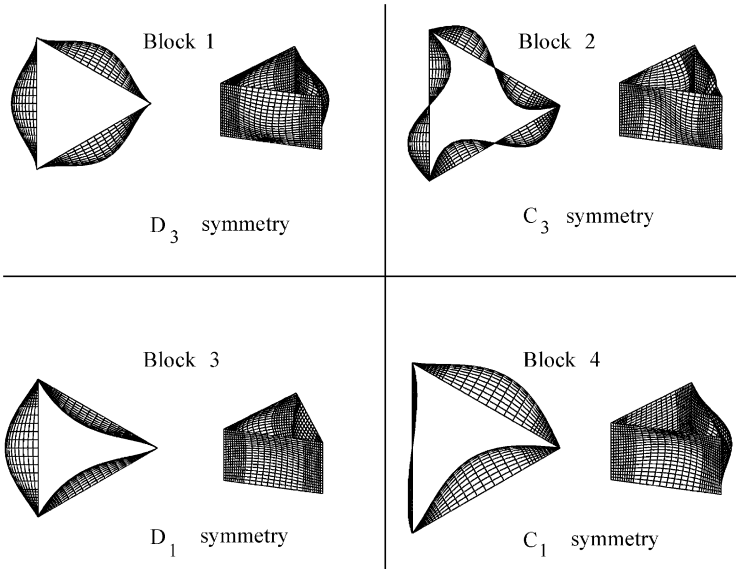

Fig. 3. The mode shapes of the first natural frequencies in the four symmetry blocks for a triangular duct.

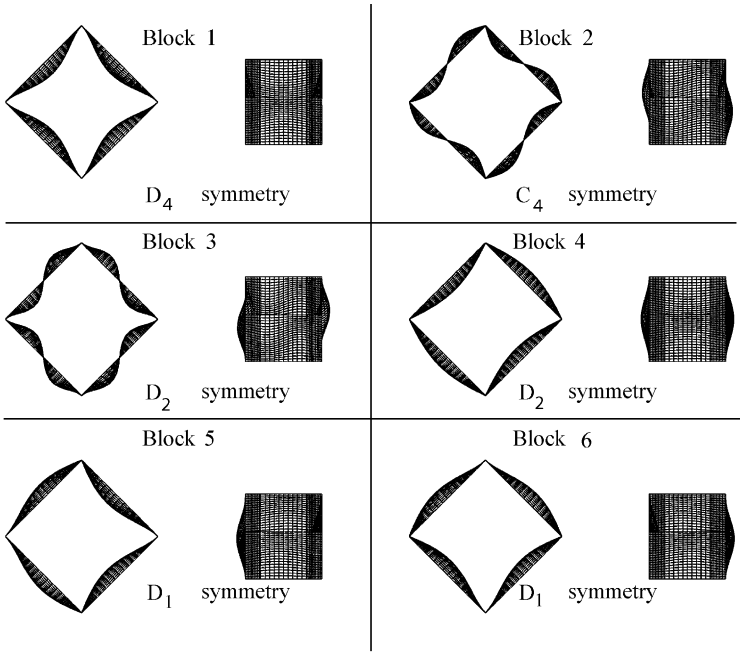

Fig. 4. The mode shapes of the first natural frequencies in the six blocks for a square duct.

(2) The symmetry associated with these finite number of subproblems is given by Eq. (A.13). Thus, in the search for the first few natural frequencies, there is one vibration mode shape with D_5 symmetry, one with C_5 symmetry, two different mode shapes with just reflection

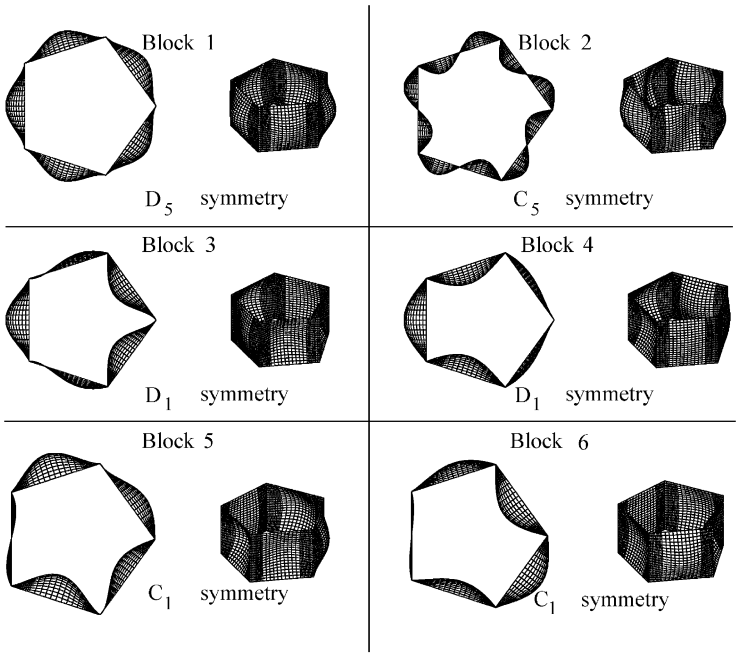

Fig. 5. The mode shapes of the first natural frequencies in the six blocks for the pentagonal duct.

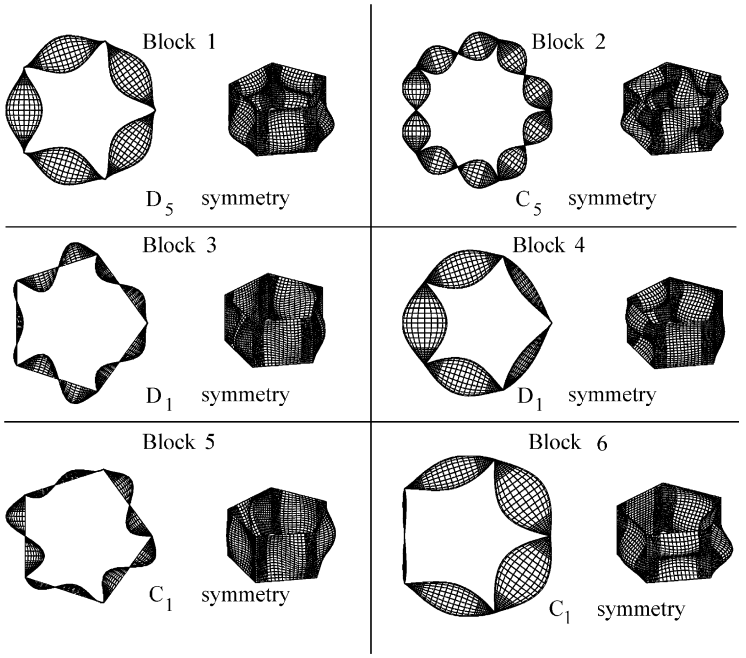

Fig. 6. The mode shapes of the second natural frequencies in the six blocks for the pentagonal duct.

symmetry (D_1 symmetry) and two different completely asymmetric mode shapes (C_1 symmetry). There are two pairs of repeated eigenvalues with different mode shapes (see Figs. 5 and 6).

(3) The problem sizes of these independent subproblems is known a priori. Thus the original discretized problem size is $18\,900 \times 18\,900$ while the largest subproblem size is 3780×3780, respectively (see Ref. [14] for a detailed tabulation of formulae for subproblem sizes).

The subproblem sizes, natural frequencies and symmetries of the mode shapes are tabulated in Table 1. The results are in excellent agreement with those of Azimi et al. [1]. The mode shapes of the first natural frequencies in each of the six blocks are shown in Fig. 5 and those with the second natural frequencies in Fig. 6. As is to be expected the first natural frequency in each block corresponds to a mode shape with a half axial wave and the second with one axial wave. Thus, it is clear that only the first natural frequency in each block needs to be determined in the search for the lower natural frequencies of the structure.

The strain energy distribution in each of the vibration modes is shown in Fig. 7. The contributions due to shear and membrane stretching are negligible and the bending strain energy is dominant. This is in complete agreement with the assumptions made in Ref. [1]. The identical blocks associated with each two dimensional group representation have different mode shapes but identical natural frequencies and strain energies.

Similarly, there are exactly four subproblems associated with the free vibration of a triangular duct and six subproblems associated with a duct of square cross-section. The mode shapes and the symmetry of the configuration are shown in Figs. 3 and 4.

3.2. Ducts with increasing symmetry

The finite element computations via block diagonalisation were carried out for polygonal ducts with 5, 6, 8, 12 and 16 sides and compared with the results of a cylinder of identical dimensions to study the effect of increasing symmetry. The strain energy distribution into bending, membrane and shear along with the mode shape of the fundamental frequency for all the ducts are shown in Figs. 8–10.

It is clear from the results that:

(1) The cluster of mode shapes at lower natural frequencies increases with symmetry (increase in number of blocks in the block diagonalisation).

(2) All mode shapes in all blocks except block 2 have negligible shear strain energy. Block 2 groups together all the torsional modes of vibration, i.e., modes shapes with cyclic symmetry. The fundamental mode shape in this block for polygonal ducts with 5 and 8 sides is shown in Fig. 11.

(3) For polygonal ducts upto 8 sides, the strain energy is dominated by bending and the creases act like knife edges without any deformation. This is in good agreement with the assumptions of Ref. [1].

(4) There is significant contribution from both bending and membrane strain energies in ducts with sides 12 and 16. The deformation of the edges is seen in the mode shapes. The behaviour

Table 1
Free vibration analysis of ducts
Non-dimensional frequency: $\bar{\omega} = \omega \times a^2 \times \sqrt{12\rho(1-\mu^2)/Eh^2}$
Material properties: $E = 20.6 \times 10^4$ N/mm^2, $\rho = 7.85 \times 10^{-9}$ N s^2/mm^4, $\mu = 0.3$
FEM data: No. of elements per side = 30, no. of longitudinal rings = 21.

Block no.	Block size	Isotropy subgroup	First natural frequency 1/2 axial wave	Second natural frequency 1 axial wave	Azimi et al. [1]
Analysis of a pentagonal duct[a]					
1	1932 × 1932	D_5	28.82	54.86	28.95/54.74
2	1848 × 1848	C_5	49.57	79.23	not available
3	3780 × 3780	D_1	24.96	52.71	not available
4	3780 × 3780	D_1	20.39	50.03	20.43/49.76
5	3780 × 3780	C_1	24.96	52.71	not available
6	3780 × 3780	C_1	20.39	50.03	20.43/49.76
Analysis of a square duct[b]					
1	1932 × 1932	D_4	28.90	54.98	28.95/54.74
2	1848 × 1848	C_4	49.56	79.21	49.35/78.96
3	1890 × 1890	D_2	69.54	94.93	69.33/N.A.
4	1890 × 1890	D_2	19.69	49.63	19.74/49.35
5	3780 × 3780	D_1	23.59	51.93	23.65/51.67
6	3780 × 3780	D_1	23.59	51.93	23.65/51.67
Analysis of a triangular duct[c]					
1	1932 × 1932	D_3	28.91	55.02	28.95/54.74
2	1848 × 1848	C_3	49.44	79.03	not available
3	3780 × 3780	D_1	21.52	50.67	21.60/50.45
4	3780 × 3780	C_1	21.52	50.67	21.60/50.45

[a] Geometrical data: side $a = 117.55$ mm, thickness $h = 2$ mm, length $L = 117.55$ mm; $d.o.f. = 18\,900$; boundary condition: *simply supported*.

[b] Geometrical data: side $a = 141.42$ mm, thickness $h = 2$ mm, length $L = 141.42$ mm; $d.o.f. = 15\,120$; boundary condition: *simply supported*.

[c] Geometrical data: side $a = 173.21$ mm, thickness $h = 2$ mm, length $L = 173.21$ mm; $d.o.f. = 11\,340$; boundary condition: *simply supported*.

of energy distribution for larger number of sides (12 and 16) appears complicated. The bending strain energy displays a minima with increasing number of circumferential waves. The vibration modes with lower symmetry have a significant contribution from membrane strain energy because such lower symmetry modes are only possible by significant translation of the duct crease. This contribution of the membrane strain predictably tends to zero with increase in number of circumferential waves (i.e., increase in symmetry) where the mode shapes of free vibration are possible without distortion of the duct crease.

(5) The trend in the distribution of strain energy for the perfect cylinder is clear. The membrane deformation is dominant and bending negligible at lower number of circumferential waves while the trend is exactly opposite at higher number of circumferential waves.

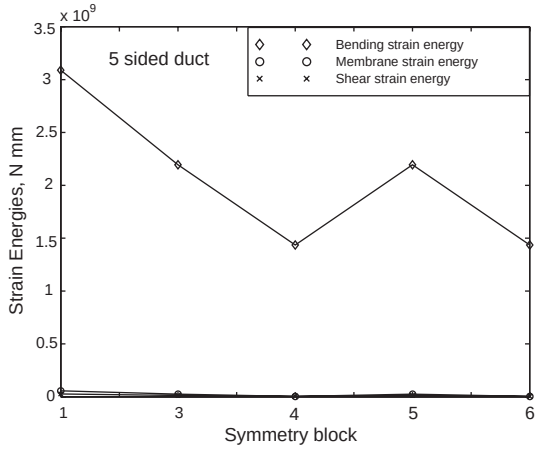

Fig. 7. The distribution of the strain energies in the vibration mode in each symmetry block. The main contribution is by bending strain energy.

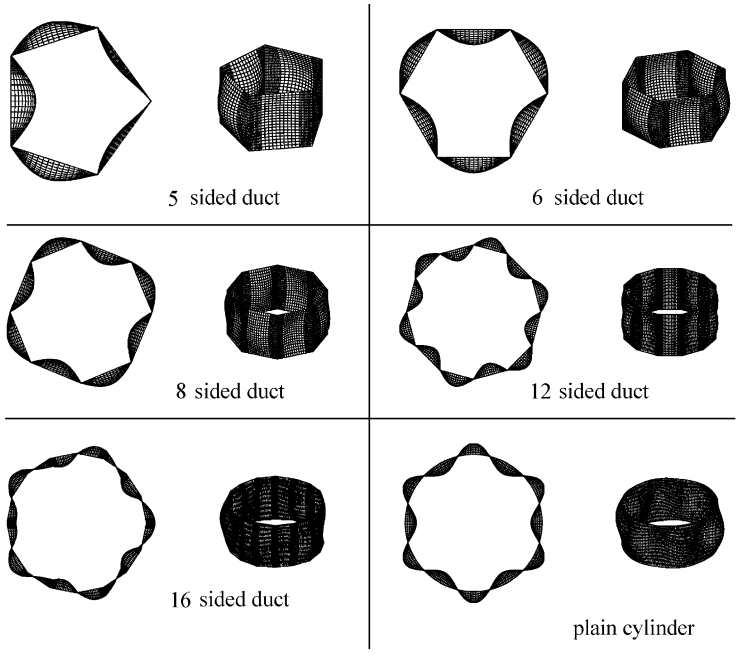

Fig. 8. The fundamental mode for ducts with 5, 6, 8, 12, 16 sides and the plain cylinder.

(6) There is very close agreement in the value of the fundamental natural frequency of the 16 sided duct with that of the plain cylinder. (1.174×10^4 rad/s for the plain cylinder and 1.172×10^4 rad/s for the 16 sided duct). The masses of both the geometries are nearly the same and the FEM discretization for both the structures results in nearly identical stiffness.

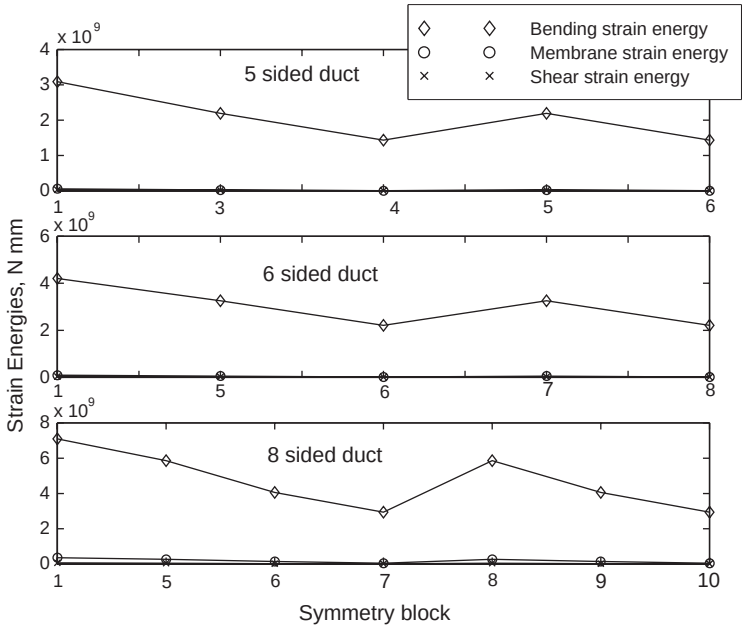

Fig. 9. Distribution of strain energy for ducts with 5, 6, 8 sides.

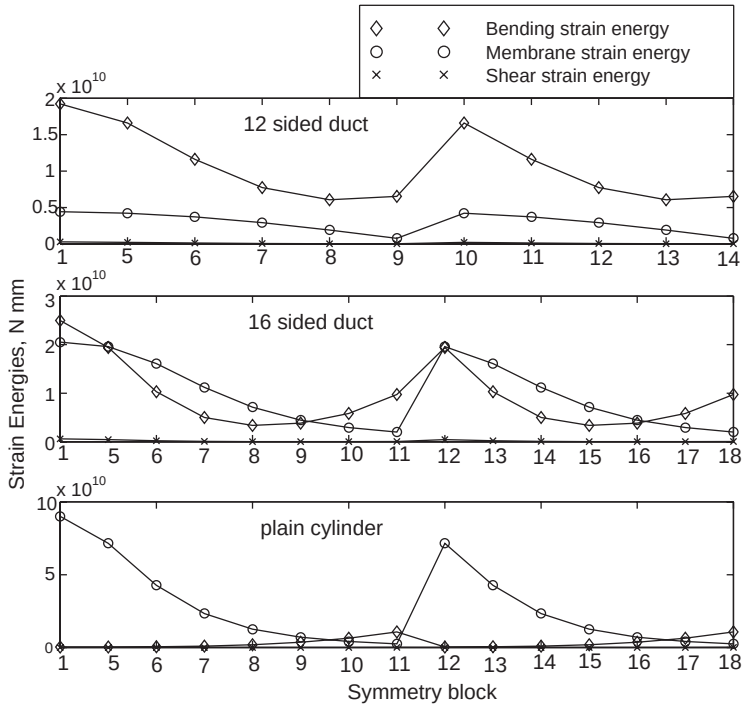

Fig. 10. Distribution of strain energy and fundamental mode for ducts with 12, 16 sides and the plain cylinder.

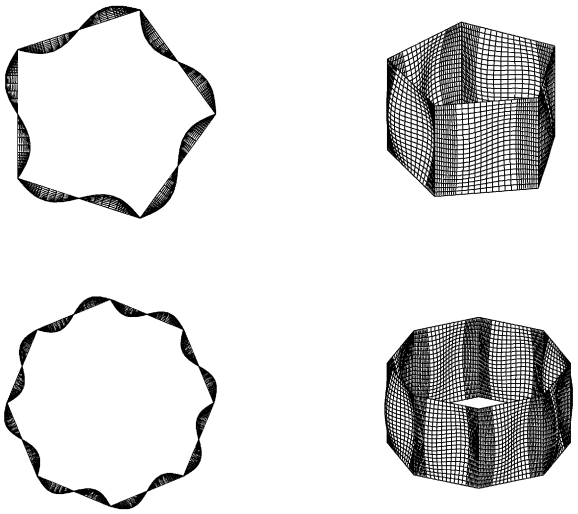

Fig. 11. The mode shapes with cyclic symmetry, i.e., torsional mode.

4. The physical insights due to the symmetry basis

In this section, we highlight the results which are unique to the group theoretic approach of the analysis to the problem of free vibration of polygonal ducts.

4.1. The cluster of natural frequencies

The symmetry basis of the block diagonalisation predicts a priori the number of blocks, i.e., the number of lower natural frequencies in the cluster. These frequencies are decoupled into separate blocks. Thus, the problem is reduced to determine the lowest natural frequency in each block. As explained in Section 3, each block has the identity of the symmetry of a subgroup. Thus, we know the number of important lower natural frequencies and distinguish them with the symmetry of the mode shapes (see Figs. 3 and 5).

4.2. Repeated eigenvalues and associated orthogonal modes

Determining orthogonal modes shapes for the eigenspace of repeated eigenvalues by the receptance method results in a fair bit of extra computation and an intelligent guess of the mode shapes. This guess is not always easy in complicated symmetric systems. The issue is addressed in detail by Johnson and Bishop [15] for a specific symmetric structure of a turbine blade. It is to be noted that this calculation is important in forced response predictions to avoid incomplete solutions. The block diagonalisation ensures that the repeated eigenvalues are decoupled into separate orthogonal blocks. Thus, an orthogonal set of eigenvectors associated with repeated eigenvalues are easily determined in the group theoretic approach. However, we hasten to add here that this is not a serious issue in standard finite element codes which use the *Lanczos Algorithm*. Theoretically, the Lanczos algorithm can never compute the second copy of a multiple

eigenvalue. Fortunately, this argument only holds in exact arithmetic. In finite precision, roundoff errors comes to the rescue though a second copy of the repeated natural frequency will converge some steps after the first has converged. See Ref. [12] for details.

4.3. Solution for ducts with small asymmetry

Finally, we note that though symmetry is required for the analysis carried out in this work, slight discrepancy from the symmetrical shape can also be analyzed by the group theoretic technique. The problem is treated as a symmetric problem with a slight perturbation and one proceeds by standard perturbation techniques. The starting point is the solution of the perfectly symmetric problem and the solution of the perturbed problem is obtained by successive approximations. Each iteration can be reduced to a linear symmetric problem (see Ref. [15, pp. 64–65]). Thus, the solution of the asymmetric problem can be obtained by a series of iterations where each iteration possesses the symmetry of the problem and can be solved with exactly the same procedure as presented in this paper.

5. Conclusions

A complete, accurate and efficient analysis of free vibration analysis of polygonal ducts is facilitated by the finite element method augmented by group theoretic subroutines.

(1) The mode shapes of polygonal ducts upto 8 sides is dominated by transverse bending strain energies. Both bending and inplane stretching are significant in vibration of ducts upto 16 sides. The shear strain energy needs to be considered only in the torsional mode of vibration (i.e., in the mode with cyclic symmetry).
(2) The lower modes of vibration are characterised by a finite cluster of natural frequencies depending on the inherent symmetry of the polygonal duct. This finite number can be calculated a priori and the corresponding mode shapes determined systematically by the group theoretic approach. Thus, the symmetry basis suggests a natural scheme for a physical classification of the mode shapes. The significance of the classification is that it predicts the correct finite number of lower mode shapes that should be considered for a linear vibration analysis of the structure. Thus, it ensures completeness, i.e., no significant mode shape is omitted in the analysis.
(3) Free vibration analysis predicts several pairs of repeated eigenvalues. These are neatly decoupled in the block diagonalisation by the orthogonal similarity transformation and a suitable set of orthogonal eigenvectors can be easily determined.

Acknowledgements

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Appendix A. The group theoretic approach

There are several lucid texts with details of applications of group theory in engineering problems (see Refs. [16,17]). In this section, we outline the main steps in the application of group theory to the free vibration problem with the help of an example of dihedral groups.

A.1. Group representations

The set of symmetries of a regular n sided polygon consisting of rotations and reflections has an algebraic structure of a group, namely the dihedral group (D_n). This is an example of a finite group of order (number of elements of the set) $2n$.

$$D_n = \{e, r_\alpha, \ldots, r_{k\alpha}; s, sr_\alpha, \ldots, sr_{k\alpha}\}, \quad \text{where } r_{n\alpha} = s^2 = (sr_\alpha)^2 = e. \tag{A.1}$$

Here $r_{k\alpha}$ denotes a rotation by an angle of $k\alpha$ (where $\alpha = 2\pi/n$, k assumes integer values from 1 to $n-1$ and s denotes a reflection). Thus, the group is the set of all geometric operations which take the regular polygon into inself.

For applications to solutions of applied problems with symmetry, we consider the vector space in which the solutions of a mathematical problem lie. In the problem of free vibration of structures, we look for periodic solutions. Thus, the solutions lie in the infinite dimensional vector space V of all continuously differentiable functions with period 2π. The first step in applying group theoretic results to exploit the symmetry of the problem is to represent each element of the group (Eq. (A.1)) as a linear transformation acting on a vector space. This group representation is defined below.

Let G be an abstract group, $|G|$ be the number of elements of G, g a typical element of G (i.e., $g \in G$) and $GL(V)$ be the group of all linear transformations of V onto itself. For the problem considered here, G is the dihedral group of Eq. (A.1), i.e., D_n, $|G| = 2n$, a typical element of the group is $g = r_\alpha$ or sr_α and the set of linear transformations ($GL(V)$) representing the group in the vector space are combinations of rotation and reflection matrices as given in Eq. (A.5) below. A representation of G on V is a homomorphism $H : G \to GL(V)$, i.e.,

$$H(gh) = H(g)H(h) \quad \forall g, h \in G. \tag{A.2}$$

Thus, the representation of the group is a mapping from the elements of the group to linear transformations of the vector space such that matrix multiplication of the linear transformations are consistent with the group multiplication operation. Consider the following action of the dihedral group D_n on V, the infinite dimensional vector space of all continuously differentiable functions with period 2π.

$$\begin{aligned} (H(r_{k\alpha})f)(\phi) &= f(\phi + k\alpha), \\ (H(sr_{k\alpha})f)(\phi) &= f(-\phi - k\alpha), \end{aligned} \tag{A.3}$$

where $f(\phi) \in V$. The domain of the periodic functions $f(\phi)$ is a circle $[0, 2\pi]$. Thus, it is clear from Eq. (A.3) that the linear transformations corresponding to rotations ($H(r_{k\alpha})$), rotate the domain of the periodic function by the same angle and the linear transformations corresponding to reflections ($H(sr_{k\alpha})$) reflect the domain of the periodic function along the same axis. It can be easily verified that the set $\{H(r_{k\alpha}), H(sr_{k\alpha}), (k = 1, 2, 3, \ldots, n),\ \alpha = 2\pi/n\}$ is a linear representation

of the group D_n on V (i.e., $H(gh) = H(g)H(h)\ \forall g, h \in D_n$). For example, let us consider elements of the group, g and h, as a reflection s and rotation r_α, respectively. The linear transformation $H(gh)$, i.e., $H(sr_\alpha)$ maps the function $f(\phi)$ to $f(-\phi-\alpha)$ (Eq. (A.3)). The linear transformation given by the product $H(g)H(h)$, i.e., $(H(s)(H(r_\alpha f))(\phi)$ also maps the function $f(\phi)$ to $f(-\phi-\alpha)$.

We now fix basis for V (for our case, the set of all continously differentiable functions with period 2π), a natural choice for the vector space being the Fourier modes $\{1, \cos\phi, \sin\phi, \cos 2\phi, \sin 2\phi, \ldots\}$. In this basis, the group representation set $\{H(r_{k\alpha}), H(sr_{k\alpha}), (k = 1, 2, 3, \ldots, n), \alpha = 2\pi/n\}$ are non-singular orthogonal linear transformations as follows:

$$H(r_{k\alpha}) = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{k\alpha} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{2k\alpha} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{R}_{Nk\alpha} \end{bmatrix}, \tag{A.4}$$

$$H(sr_{k\alpha}) = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}\mathbf{R}_{k\alpha} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}\mathbf{R}_{2k\alpha} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{J}\mathbf{R}_{Nk\alpha} \end{bmatrix}, \tag{A.5}$$

where $\mathbf{R}_{jk\alpha} = \begin{bmatrix} \cos jk\alpha & \sin jk\alpha \\ -\sin jk\alpha & \cos jk\alpha \end{bmatrix}$, $\mathbf{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{J}\mathbf{R}_{jk\alpha} = \begin{bmatrix} \cos jk\alpha & \sin jk\alpha \\ \sin jk\alpha & -\cos jk\alpha \end{bmatrix}$, $j = 1, 2, \ldots N$, where N is the number of sine (and cosine) Fourier terms considered in the calculation.

A.2. Standard decomposition of the vector space

We now compile some crucial results from group representation theory. The reader can refer to Serre [18] for a detailed exposition and complete proofs and to Fujii et al. [19] for a discussion on dihedral groups.

(1) The vector space V can be uniquely decomposed into mutually orthogonal and invariant subspaces (i.e., invariant under the action of all the linear transformations of the group representations). The number of subspaces is equal to the number of mutually inequivalent, irreducible representations of the group ($q(G)$) (see Ref. [18] for definitions and theorems on irreducible representations which are also referred to as the building blocks of any completely reducible representation). Thus,

$$V = \oplus_{\mu=1}^{q(G)} V_\mu^G. \tag{A.6}$$

A finite group has a finite number of irreducible representations and these are well known and tabulated in the group theory literature. The $q(G)$ non-equivalent irreducible representations

of the dihedral group D_n can be indexed as,

$$R(D_n) = \{(1,j)\,|\,j = 1,2,3,4\} \cup \{(2,j)\,|\,j = 1,\ldots,(n-2)/2\} \text{ for } n \text{ even},$$
$$R(D_n) = \{(1,j)\,|\,j = 1,2\} \cup \{(2,j)\,|\,j = 1,\ldots,(n-1)/2\} \text{ for } n \text{ odd},$$

where the first component d of the index (d,j) indicates the dimension of the representation.

The one dimensional irreducible representations $\tau^{(1,j)}$ of D_n are given by

$$\tau^{(1,1)}(r) = 1, \quad \tau^{(1,1)}(s) = 1,$$

$$\tau^{(1,2)}(r) = 1, \quad \tau^{(1,2)}(s) = -1,$$

$$\tau^{(1,3)}(r) = -1, \quad \tau^{(1,3)}(s) = 1,$$

$$\tau^{(1,4)}(r) = -1, \quad \tau^{(1,4)}(s) = -1.$$

The two dimensional irreducible representations $\tau^{(2,j)}$ of D_n are given by

$$\tau^{(2,j)}(r) = \mathbf{R}^j, \quad \tau^{(2,j)}(s) = \mathbf{S},$$

where

$$\mathbf{R} = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We note that r and s are the generators of the group D_n and that the representations are by 1×1 and 2×2 orthogonal matrices.

(2) Thus the standard (unique) decomposition of the vector space V under the action of the dihedral group D_n is,

$$V = \begin{cases} V_{(1,1)} \oplus V_{(1,2)} \oplus V_{(1,3)} \oplus V_{(1,4)} \oplus (\oplus_{l=1}^{n/2-1}\ V_{(2,l)}) & n = \text{even}, \\ V_{(1,1)} \oplus V_{(1,2)} \oplus (\oplus_{l=1}^{(n-1)/2}\ V_{(2,l)}) & n = \text{odd}. \end{cases} \tag{A.7}$$

Further, consider each subspace associated with the two dimensional irreducible representation, $V_{(2,l)}$. This can be split up into two (as it is associated with a two dimensional irreducible representation) invariant, mutually orthogonal subspaces by a suitable choice of basis (referred to as symmetry adapted basis in literature). Thus,

$$V_{(2,l)} = V_{(2,l^+)} \oplus V_{(2,l^-)}. \tag{A.8}$$

To summarise, the group representations splits up the vector space into mutually orthogonal subspaces invariant under the action of each of the linear transformation of the group representation. It turns out that each of these subspaces is associated with certain *symmetries*. The next step is to determine these subspaces and their associated symmetry. The group theory results compiled in Sections A.2.1 and A.2.2 and the formulae of projection operator theory are all that are required.

A.3. Projection operator theory and isotropy subgroups

In this section we address the following questions:

What are the formulae that determine the subspaces in the standard decomposition of the vector space V?

What are the symmetries associated with each of these subspaces?

The configuration vector space V is resolvable into a set of $q(G)$ mutually orthogonal, invariant subspaces $V = \oplus_{\mu=1}^{q(G)} V_\mu^G$. V_μ^G associated with a one dimensional irreducible representation is the range of the projection operator

$$P^\mu = \frac{1}{|G|} \sum_{i=1}^{|G|} tr(\tau^\mu(g_i)) H(g_i), \tag{A.9}$$

where $tr(\tau^\mu(g_i))$ is the trace of the irreducible representation matrix (tabulated in Appendix A), $g \in G$, $H(g_i)$ is the orthogonal group representation as given in Eqs. (A.4) and (A.5). The similar formulae for the span of the two subspaces associated with the two dimensional irreducible representation are a bit more complicated and are not reproduced here. The reader is referred to Ref. [17]. The main point is that the information of Eqs. (A.4) and (A.5) and Appendix A suffice to determine all these subspaces.

We continue with the example of the action of the dihedral group (D_n) on the action of the vector space of all continously differential functions (V). For the case when n is even, the projection operators are

$$\begin{aligned}
P^1 &= \frac{1}{2n} \sum_{k=1}^{n} [H(r_{k\alpha}) + H(sr_{k\alpha})], \\
P^2 &= \frac{1}{2n} \sum_{k=1}^{n} [H(r_{k\alpha}) - H(sr_{k\alpha})], \\
P^3 &= \frac{1}{2n} \sum_{k=1}^{n} (-1)^k [H(r_{k\alpha}) + H(sr_{k\alpha})], \\
P^4 &= \frac{1}{2n} \sum_{k=1}^{n} (-1)^k [H(r_{k\alpha}) - H(sr_{k\alpha})], \\
P^{4+h} &= \frac{2}{2n} \sum_{k=1}^{n} \cos(lk\alpha) [H(r_{k\alpha}) + H(sr_{k\alpha})],
\end{aligned} \tag{A.10}$$

where $l = 1, 2, \ldots, (n-2)/2$, $k = 1, 2, \ldots, n$ and $\alpha = 2\pi/n$. Using these formulae for the projection operators, the following mutually orthogonal, invariant subspaces of V are

$$\begin{aligned}
V_{(1,1)} &= \text{span}\{1, \cos n\alpha, \cos 2n\alpha, \ldots\}, \\
V_{(1,2)} &= \text{span}\{\sin n\alpha, \sin 2n\alpha, \ldots\}, \\
V_{(1,3)} &= \text{span}\left\{\cos\frac{n}{2}\alpha, \cos\frac{3n}{2}\alpha, \ldots\right\},
\end{aligned}$$

$$V_{(1,4)} = \text{span}\left\{\sin\frac{n}{2}\alpha, \sin\frac{3n}{2}\alpha, \ldots\right\},$$
$$V_{(2,l^+)} = \text{span}\{\cos m\alpha\},$$
$$V_{(2,l^-)} = \text{span}\{\sin m\alpha\}, \tag{A.11}$$

where $m = l, n-l, n+l, 2n-l, 2n+l, \ldots, N$.

The notion of symmetry group of these subspaces plays a crucial role in obtaining physical insights which are unique to the group theoretic approach. For the subspace $V_\mu \subset V$, $G[V_\mu]$ is the *symmetry group* of V if

$$G[V_\mu] \stackrel{\text{def}}{=} \{g \in D_n \mid H(g)f(\phi) = f(\phi) \ \forall f(\phi) \in V_\mu\}. \tag{A.12}$$

The following results are easily verified by inspection and with the help of results in Eqs. (A.3) and (A.11)

$$G[V_{(1,1)}] = D_n,$$
$$G[V_{(1,2)}] = C_n,$$
$$G[V_{(1,3)}] = D_{n/2},$$
$$G[V_{(1,2)}] = C_{n/2} \quad (n = \text{even}),$$
$$G[V_{(2,l)}] = C_{gcd[n,l]},$$
$$G[V_{(2,l^+)}] = D_{gcd[n,l]},$$
$$G[V_{(2,l^-])}] = C_{gcd[n,l]}, \tag{A.13}$$

where $gcd[n, l]$ denotes the greatest common divisor of n and l and $l = 1, 2, \ldots, (n-2)/2$.

We should note that so far the discussion has been focused completely on the dihedral group D_n and its action on the vector space V of all continuously differentiable functions. Our main goal is to find the solution of the generalised eigenvalue problem discussed in Section 1 (Eqs. (1) and (2)). The fundamental theorem discussed in Section 2.2 is applicable and leads to the identical block diagonalization of **K** and **M**. See Ref. [8] for a simple proof of the result that the linear operators **K** and **M** do commute with the entire set of linear transformations representing the group.

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