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On the effect of wall undulations on the acoustics of ducts with flow

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Abstract

The acoustic wave equation for quasi-one-dimensional propagation is obtained, along a duct with a small wall sinusoidal perturbation and containing a low Mach number mean flow. The motivation is the study of the effect of wall roughness on the propagation of sound in a duct, and also of the effect of repeated reflections at periodic changes in cross-sectional area. The ray approximation, which holds only for wavelengths which are short compared with the length scales of the variation of the cross-section and mean flow velocity, is used as a factor to reduce the wave equation to a Schrödinger form. The exact solutions are obtained, without restriction, as power series expansions around the middle of the duct; since this solution fails to converge at the two ends of the duct it is matched to the other solutions there. In this way it is possible to calculate everywhere reduced potential, (unreduced) potential, velocity and pressure perturbations. These are plotted as a function of the longitudinal co-ordinates along the duct for several values of the three dimensionless parameters in the problem, viz., (1) the relative height of wall corrugations, (2) the Mach number of the mean flow at the central section and (3) the wavenumber made dimensionless multiplying by the periodicity of corrugations.

1. Introduction

The propagation of waves in ducts with wall undulations has been considered both in the context of acoustic [1,2] and electromagnetic [3,4] waves. For sound of high frequency, the acoustic wave equation is solved together with a boundary condition at the undulated duct wall. If the wavelength is larger than the transverse dimensions of the duct, then only the fundamental longitudinal mode exists. If the changes in cross-section are not too rapid, so that the wavefronts

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remain approximately plane, then a quasi-one-dimensional representation can be used, in which the wave field depends only on time and the longitudinal distance along the duct axis. In the case of the acoustics of ducts without a mean flow this corresponds to the theory of horns [5–8]; an important extension is the acoustics of nozzles, allowing for the presence of a mean flow [9–12]. The present paper addresses the quasi-one-dimensional acoustics of ducts with sinusoidal wall corrugations, both in the absence and presence of a mean flow, of low Mach number in the latter case.

The basic underlying assumption in quasi-one-dimensional acoustic propagation theory, implies that the height of corrugations relative to the transverse dimensions of the duct be small, in the sense $\varepsilon^2 \ll 1$ or $\varepsilon < 0.3$, and includes two cases of practical engineering interest, viz., the case (1) of very small corrugations say $\varepsilon < 0.1$, and the case (2) of moderate periodic changes of cross-section $0.1 < \varepsilon < 0.3$. The first case (1), of small corrugations $\varepsilon < 0.1$, demonstrates the effect of wall roughness on the propagation of sound in a duct; even if this effect is small over a wavelength, the question arises of what is the propagation distance, expressed in number of wavelengths, for which it becomes non-negligible. The second case (2) of moderate change in cross-section $0.1 < \varepsilon < 0.3$ is known to lead to wave reflection and transmission at the first contraction; the question arises as to the possible additional effect of reflection and transmission at similar periodic undulations of the wall shape. An engineering example, is a flexible inlet duct of an engine; this example also points to the restrictions of the present theory, implied in the assumptions of quasi-one-dimensional acoustic propagation: the effects of flow and acoustic boundary layers near the duct walls are neglected, as are those of vortex shedding by the corrugations; coupling to flexible wall modes is also not considered.

After presenting briefly in the introduction (Section 1) the background to the present problem, the solution of the convected nozzle wave equation (Section 2) is considered for a duct with sinusoidal undulations (Section 3). The acoustic wave field is represented by the potential, velocity and pressure perturbations (Section 4), which depend on three parameters, namely, the longitudinal wavenumber, the Mach number and the height of corrugations (Section 5). It is possible to introduce a reduced potential, which is independent of the Mach number, and thus is the same for a horn (in the sense of a duct not carrying a mean flow) or a nozzle containing a low Mach number mean flow. The four acoustic variables, viz., the reduced potential, the (unreduced) potential, the velocity and pressure perturbations, are plotted for one period of the duct length, for several combinations of the parameters (Figs. 1–7). The acoustic wave field has two components, which can be expanded as power series around the middle position; since these series do not converge near the edges of one duct period, two other series representations are obtained, valid near the edges (Appendix A) and then matched to the 'central' solution (Appendix B).

2. Quasi-one-dimensional propagation in a low Mach number nozzle

Starting from the classical wave equation for the acoustic potential ϕ in an inhomogeneous medium [13]

$$c^{-2}\partial^2\rho/\partial t^2 - \rho^{-1}\nabla \cdot (\rho\nabla\phi) = 0, \tag{1}$$

where c is the sound speed and ρ the mass density, for quasi-one-dimensional propagation of the fundamental longitudinal mode, the convected nozzle wave equation can be obtained [14] by (1) replacing the mass density ρ per unit volume by the mass density per unit length of duct ρS, where $S(x)$ is the cross-sectional area, leading to the horn [15] wave operator:

$$\rho^{-1}(\nabla \cdot \rho \nabla \phi) \to S^{-1}(\partial/\partial x)(S\partial\phi/\partial x) = \phi'' + (S'/S)\phi', \tag{2}$$

where the mass density is taken a constant ($\rho = const$) and prime denotes derivative with respect to the longitudinal co-ordinate x along the duct axis, viz., $\phi' \equiv \partial\phi/\partial x$; (2) the effect of convection of sound by the mean flow with velocity $U(x)$ is accounted for replacing [16] the local derivative by the material derivative

$$\partial^2\phi/\partial t^2 \to \mathrm{d}^2\phi/\mathrm{d}t^2 \equiv (\partial/\partial t + U\partial/\partial x)^2\phi = \ddot{\phi} + 2U\dot{\phi}' + U(U\phi')', \tag{3}$$

where dot denotes derivative with respect to time ($\dot{\phi} \equiv \partial\phi/\partial t$). The substitution (3) is valid in the low Mach number approximation, in which case the last term on the r.h.s of (3) can be omitted when replacing Eq. (3) together with Eq. (2) in Eq. (1)

$$U^2 \ll c^2: \qquad \ddot{\phi} + 2U\dot{\phi}' - c^2[\phi'' + (S'/S)\phi'] = 0 \tag{4}$$

to lead to the convected nozzle wave equation, specifying the acoustic potential for quasi-one-dimensional sound propagation in a duct of varying cross-section containing a low Mach number mean flow. By quasi-one-dimensional sound propagation it is meant [7] that only the fundamental longitudinal acoustic mode is considered, whose wavelength must be larger than the transverse dimensions of the duct $\lambda > d \sim \sqrt{S}$ (otherwise transverse modes would exist), and the changes in cross-section should be moderate, so that the wavefronts remain approximately flat, and thus the wave variables can be considered as constant over the cross-section (no flow or acoustic boundary layers near the wall), and depend only on the longitudinal co-ordinate x along the axis of the duct (and on time t).

A one-dimensional flow $U(x)$ is always potential, and at low Mach number $U^2 \ll c^2$ the sound speed c and mass density ρ are constant, and the conservation of the volume flux $U(x)S(x) \sim const$, which implies that the lengthscale of variation of cross-sectional area $S(x)$ is the same, apart from sign, as the lengthscale of variation of the mean flow velocity $U(x)$:

$$L \equiv S/S' = -U/U'. \tag{5}$$

Since the mean state properties do not depend on time, it is convenient to use a Fourier decomposition

$$\phi(x,t) = \int_{-\infty}^{\infty} \Phi(x;\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{6}$$

where Φ is the spectrum of the acoustic potential for a wave of frequency ω at position x. Substitution of Eq. (6) into Eq. (4) leads to a second order linear ordinary differential equation

$$\Phi'' + (1/L + 2\mathrm{i}kM)\Phi' + k^2\Phi = 0 \tag{7}$$

whose coefficients involve

$$k \equiv \omega/c, \quad M(x) \equiv U(x)/c \tag{8a, b}$$

the wavenumber (8a) which is constant, and the Mach number (8b) and length-scale (5) which are generally functions of position: (1) the Mach number (8a) would be constant only for a uniform flow in a duct of constant cross-section and (2) the lengthscale (5) would be constant only for an exponential duct.

It is well known [17] that the coefficient of the first derivative Φ' in Eq. (7) can be eliminated by the change of independent variable

$$\Phi(x;\omega) = \exp\left\{-\int^{x} (1/2L + \mathrm{i}kM)\,\mathrm{d}\xi\right\}\Psi(x;\omega), \tag{9}$$

which leads to a differential equation in 'Schrodinger' form

$$\Psi'' + [k^2 - (1/2L)' - 1/4L^2]\Psi = 0, \tag{10}$$

in which all the terms in the square brackets involving the Mach number:

$$M^2 \ll 1: \qquad k^2M^2 + \mathrm{i}kM' + \mathrm{i}kM/L = k^2M^2 \ll k^2, \tag{11}$$

cancel on account of $L = -M/M'$ in Eq. (5), or are negligible at low Mach number. In the ray approximation of wavelength λ much smaller than the lengthscale L of changes in cross-section ($\lambda^2 \ll L^2$) (but still larger than the dimension of the cross-section $\lambda > d$, to preserve quasi-one-dimensional propagation), Eq. (10) simplifies to $\Psi'' + k^2\Psi = 0$, and its solution

$$1 \ll k^2L^2 = (2\pi L/\lambda)^2: \qquad \Psi_\pm(x;\omega) \sim \mathrm{e}^{\pm \mathrm{i}kx}, \tag{12}$$

is a superposition of plane waves, propagating in the positive Ψ_+ and negative Ψ_- x-direction. The transformation (9) shows

$$\Phi(x;\omega) = \Phi(0;\omega)\sqrt{S(0)/S(x)}\exp\left\{-\mathrm{i}k\int_0^x M(\xi)\,\mathrm{d}\xi\right\}\Psi(x;\omega) \tag{13}$$

that (1) the wave amplitude scales like the inverse square root of the cross-sectional area $\Phi \sim S^{-1/2}$, so that the acoustic energy flux $\rho Sc\,|\,\Phi\,|^2 \sim const$ is conserved, (2) the mean flow causes a phase shift specified by the integrated Doppler shift [12,18], viz. the last term on the right hand side of Eq. (13). If the restriction $k^2L^2 \gg 1$ of the ray approximation is not made, then Eq. (13) may still be used to specify a reduced acoustic potential Ψ, which in general is still a plane wave (12), but is now specified by the exact solution of Eq. (10). Since the latter does not involve the Mach number, the reduced acoustic potential is the same for a duct with or without a low Mach number mean flow.

3. Duct with sinusoidal wall undulations

The linear, second order, ordinary differential equation in 'Schrodinger' form (10), which specifies the reduced acoustic potential, will be solved exactly for a duct with sinusoidal wall undulations, so that the cross-sectional area

$$0 < \varepsilon < 1: \qquad S(x) = S_0[1 + \varepsilon\cos(x/l)] \tag{14a}$$

varies around the mean value S_0, with a longitudinal period $2\pi l$, viz., $S(x + 2\pi l) = S(x)$, and ε is the relative height of the corrugations. Note that the lengthscale (5) of variations in cross-section:

$$-L(x) = l\{\cot(x/l) + (1/\varepsilon)\, csc(x/l)\}, \tag{14b}$$

takes the minimum value (in modulus) $|L_{min}| = l/\varepsilon$ for $x/l = n\pi \pm \pi/2$ corresponding to the average cross-section $S(n\pi \pm \pi/2) = S_0$, and becomes infinite $L(n\pi) = \infty$ for $x/l = n\pi$ corresponding to the minimum or maximum cross-sections $S(n\pi) = S_0[1 + \varepsilon(-1)^n]$. Substitution of Eq. (14a,b) into Eq. (5) leads Eq. (10) to the differential equation

$$\begin{aligned} &l^2[1 + \varepsilon\cos(x/l)]^2\Psi'' + \{k^2l^2[1 + \varepsilon\cos(x/l)]^2 \\ &\quad (\varepsilon/2)\cos(x/l)\,[1 + \varepsilon\cos(x/l)] + (\varepsilon/2)^2\sin^2(x/l)\}\Psi = 0, \end{aligned} \tag{15}$$

for the reduced acoustic potential. The change of independent variable,

$$y \equiv \cos(x/l\,), \quad F(y) \equiv \Psi(x;\omega), \tag{16a, b}$$

transforms the coefficients from sinusoidal functions in Eq. (15) to polynomials in

$$\begin{aligned} &4(1 - y^2)(1 + \varepsilon y)^2 F'' - 4y(1 + \varepsilon y)^2 F' \\ &\quad + [4k^2l^2(1 + \varepsilon y)^2 + \varepsilon^2 + 2\varepsilon y + \varepsilon^2y^2]F = 0, \end{aligned} \tag{17}$$

where prime now denotes derivative with regard to y, viz., $F' \equiv \mathrm{d}F/\mathrm{d}y$.

Since the shape (14) of the duct is periodic, it is sufficient to consider a longitudinal section $0 \leqslant x/l \leqslant 2\pi$, corresponding Eq. (16a) to $-1 \leqslant y \leqslant +1$. Note that the differential equation (17) has [17] three regular singularities:

$$y = -1/\varepsilon, \pm 1, \tag{18}$$

viz.: (1) one regular singularity $y = -1/\varepsilon < -1$ for $\varepsilon < 1$ lies outside the interval $-1 \leqslant y \leqslant 1$ of interest, (2) the other two regular singularities $y = \pm 1$ lie at the ends of the interval of interest. The simplest solution [17] is to expand in a power series around the central position $y = 0$, which is a regular point

$$F_\sigma(y) = \sum_{n=0}^{\infty} a_n(\sigma) y^{n+\sigma}; \tag{19}$$

substitution of Eq. (19) into Eq. (17) leads to the recurrence formula for the coefficients:

$$\begin{aligned} 0 = {} & 4(n + \sigma)(n + \sigma - 1)a_n(\sigma) + 8\varepsilon(n + \sigma - 1)(n + \sigma - 2)a_{n-1}(\sigma) \\ & + \{4(n + \sigma - 2)[(\varepsilon^2 - 1)(n + \sigma - 3) - 1] + 4K^2 + \varepsilon^2\}a_{n-2}(\sigma) \\ & + 4\varepsilon[1 + 4K^2 - 4(n + \sigma - 3)^2]a_{n-3}(\sigma) \\ & + \varepsilon^2[1 + 4K^2 - 4(n + \sigma - 4)^2]a_{n-4}(\sigma) \end{aligned} \tag{20}$$

which involves two dimensionless parameters: (1) the relative height ε of undulations in (13) and (2) the dimensionless wavenumber

$$K \equiv kl = \omega l/c, \tag{21}$$

obtained multiplying the wavenumber (8a) by l which specifies the spatial periodicity $2\pi l$ of the duct.

Setting $n = 0$ in Eq. (20) yields the indicial equation:

$$n = 0: \qquad 4\sigma(\sigma - 1)a_0 = 0. \tag{22}$$

The case $a_0(\sigma) = 0$, would lead by Eq. (20) to $a_n(\sigma) = 0$ for all $n = 1, 2, \ldots, \infty$, and hence to a trivial solution $F_\sigma(y) = 0$. Thus a non-trivial solution is possible only for $\sigma = 0, 1$. Setting $n = 1$ in Eq. (20) yields the indicial equation:

$$4\sigma(1 + \sigma)a_1(\sigma) + 8\varepsilon\sigma(\sigma - 1)a_0(\sigma) = 0, \tag{23}$$

which should be considered for the two cases in $\sigma = 0, 1$: (1) if $\sigma = 1$, it follows from Eq. (23) that

$$\sigma = 1: \qquad a_1(1) = 0 \neq a_0(1), \tag{24a}$$

which specifies by Eq. (19) a solution $F_1(y)$ of Eq. (17); (2) if $\sigma = 0$, then Eq. (23) can be satisfied by

$$\sigma = 0: \qquad a_1(0) \neq 0 = a_0(0), \tag{24b}$$

which leads by Eq. (19) to a solution $F_0(y)$, which is linearly independent of the preceding. The general solution is a linear combination of the preceding

$$F(y) = B_0 F_0(y) + B_1 F_1(y), \tag{25}$$

where the arbitrary constants of integration B_0, B_1 are determined from boundary conditions. The two boundary conditions could be the acoustic velocity and pressure perturbations at the duct entrance, for example.

4. Acoustic potential, velocity and pressure

Substituting Eq. (16a,b) in Eq. (19) the two components of the reduced acoustic potential are:

$$\Psi_\sigma(x; \omega) = \sum_{n=0}^{\infty} a_n(\sigma) \cos^{n+\sigma}(x/l), \tag{26}$$

which apply to a duct without flow (horn); the mean flow Mach number

$$M(x)/M_0 = U(x)/U_0 = 1/[1 + \varepsilon \cos(x/l)], \tag{27}$$

appears (9) in the (unreduced) acoustic potential

$$\begin{aligned} \Phi_\sigma(x; \omega) &= [1 + \varepsilon \cos(x/l)]^{-1/2}\, \Psi_\sigma(0; \omega) \\ &\exp\left\{-\left(2\mathrm{i}KM_0/\sqrt{1 - \varepsilon^2}\right) \arctan\left[\sqrt{(1 - \varepsilon)/(1 + \varepsilon)}\, \tan(x/2l)\right]\right\}, \end{aligned} \tag{28}$$

which applies to a duct with flow (nozzle). Bearing in mind the periodicity of wall undulation, the acoustic fields need only be considered in a longitudinal section of the duct

$$0 \leqslant x/l \leqslant \pi, \tag{29}$$

between the maximum $S_{max} = S(0) = S_0(1 + \varepsilon)$ and minimum $S_{min} = S(\pi l) = S_0(1 - \varepsilon)$ cross-sections.

The acoustic velocity v and pressure p perturbations, or their respective spectra V, P

$$v, p(x, t) = \int_{-\infty}^{+\infty} V, P(x; \omega) e^{-i\omega t} \, d\omega, \tag{30a, b}$$

are related to the spectrum of the acoustic potential Φ: (1) for the acoustic velocity, by the gradient

$$\mathbf{v} = \nabla\phi, \; V_\sigma(x; \omega) = d\Phi_\sigma(x; \omega)/dx; \tag{31a, b}$$

(2) for the acoustic pressure by the material derivative:

$$p = -\rho d\phi/dt = -\rho(\dot{\phi} + U\phi'), \tag{32}$$

or equivalently in terms of spectra

$$P_\sigma(x; \omega) = i\rho\omega\Phi_\sigma(x; \omega) - \rho U(x) V_\sigma(x; \omega), \tag{33}$$

where Eq. (31b) was used.

The acoustic field depends on three dimensionless parameters, namely (1) the relative height of corrugations ε in Eq. (14a), (2) the mean flow Mach number (8b) at the average cross-section $M_0 = M(x/l = \pi/2)$ in Eq. (27) and (3) the dimensionless (21) wave number (8a). One way to assess the influence of these three parameters is to select a set:

$$\varepsilon = 0.2, \qquad M_0 = 0.1, \qquad K = 1, \tag{34a–c}$$

of baseline values (34a–c), and then to vary each in turn as follows. The relative height of undulations ε satisfies $0 \leqslant \varepsilon < 1$, and is restricted to $\varepsilon^2 \ll 1$ for quasi-one-dimensional propagation; it is given the baseline (34a) value $\varepsilon = 0.2$, plus

$$\varepsilon = 0.0, 0.1, 0.2, 0.3, \tag{35a–d}$$

smaller $\varepsilon = 0.1$ and larger $\varepsilon = 0.3$ values, with the uniform duct $\varepsilon = 0.0$ also included for reference. The Mach number $M_0 = M(\pi l/2)$ at the average cross-section $x = \pi l/2$, $S(\pi l/2) = S_0$ is given the baseline (34b) value $M_0 = 0.1$, consistent with low Mach number mean flow $(M_0)^2 \ll 1$, as well as

$$M_0 = 0.01, \; 0.1, \; 0.2, \tag{36a–c}$$

a lower value, and a very small value corresponding to almost negligible flow. The dimensionless wavenumber (21) is given a baseline value $K = 1$ corresponding to a longitudinal periodicity $2\pi l = 2\pi/k = \lambda$ equal to the wavelength; besides,

$$K = 0.1, 1, 10, \tag{37a–c}$$

is given a much larger value corresponding to the ray approximation $K^2 \gg 1$, and a smaller value corresponding to the opposite limit of compact scattering $K \ll 1$. The highest value of the Mach number at the average section $M_0 = 0.2$ still satisfies the low Mach number approximation in the narrowest section $M^2 \leqslant M_{max}^2 = [M_0/(1-\varepsilon)]^2 \leqslant [0.2/(1-0.3)]^2 = 0.082 \ll 1$; the largest height of undulations $\varepsilon = 0.3$ is limited by the approximation $\varepsilon^2 \ll 1$ of plane wavefronts $\varepsilon^2 \leqslant 0.3^2 = 0.09 \ll 1$; the largest dimensionless wavenumber $10 = K = \omega l/c = 2\pi l/\lambda$, for a wavelength $\lambda = \pi l/5$ is compatible with quasi-one-dimensional propagation if the transverse lengthscale of the cross-section $d \sim \sqrt{S}$ is smaller $d < \lambda$, i.e., for a tube of small transverse dimension relative to the periodicity of undulations $d < \pi l/5$. Further restrictions in the case of a duct with mean flow relate to the neglect of acoustic and flow boundary layers, and omission of vortex shedding from the wall corrugations.

5. Effect of wavenumber, Mach number and height of corrugations

The two components ($\sigma = 0, 1$) of the four acoustic fields (reduced potential Ψ_σ, acoustic potential Φ_σ, velocity V_σ and pressure P_σ), are plotted next over one-half the longitudinal period (29), varying in turn each of the dimensionless parameters (height of undulations ε, Mach number M_0 and wavenumber K). The plots of the two components (even and odd) of the acoustic fields are based on the series expansion (19) or (26) around the middle position $x = \pi l/2$ or $y = 0$, which fail to converge at the widest $x = 0$, $y = 1$ and narrowest $x = \pi l$, $y = -1$ sections, because $y = \pm 1$ are singularities (18) of the differential equation (17). The point $y = 0$ is a regular point, and the solution in power series about this point converges for $|y| < 1$ or $-1 < y < +1$. It is possible to obtain (Appendix A) around the regular singularities at $y = \pm 1$, Frobenius–Fuchs expansions in powers of $(y \mp 1)$, which can be matched (Appendix B) to solution (19), to specify the acoustic field in the entire duct $|y| \leqslant 1$ or $0 \leqslant x \leqslant \pi l$, as required for the plots in Figs. 1–7.

The simplest acoustic field is the reduced potential $\Psi_\sigma(x;\omega)$, because: (1) it is real, i.e., it is a standing wave, which has as amplitude and no phase; (2) it is independent of the Mach number, and thus affected only by the wavenumber K and height of undulations ε. The reduced acoustic potential for a duct of constant cross-section $\varepsilon = 0$ would be (12) a plane wave, given by a sinusoidal function of kx. The reduced acoustic potential $\Psi_0(x;\omega)$ is plotted in Fig. 1 versus the longitudinal co-ordinate $y = \cos(x/l)$, with $\sigma = 0$ on the right and $\sigma = 1$ on the left, and varying ε at the top and varying K at the bottom. It is seen (Fig. 1 top left) that for $1 = K = kl$ and a duct of constant cross-section $\varepsilon = 0$, $\Psi_1 = y = \cos(x/l) = \cos(kx) = -\sin(x/l - \pi/2)$ is a sinusoidal wave odd relative to the mid-position; towards the smallest section $x = \pi l$, $y = -1$, $S = S_0(1 - \varepsilon)$ the value of Ψ_1 is smaller in absolute value for larger height of undulations; towards the wider section of the duct $x = 0$, $y = 1$, $S = S_0(1 + \varepsilon)$ the amplitude is reduced relative to a plane wave, although it resumes the normalized value $\Psi_1 = 1$ at $x = 0$. In the middle section $x = \pi l/2$, $y = 0$, $S = S_0$ the amplitude is independent of the height of undulations. It is seen in Fig. 1 bottom left that the oscillatory character of the wave field Ψ_1 becomes evident for larger wavenumber $K = 10$, and is not noticeable for small wavenumber $K = 0.1$. The other component of the reduced potential (Fig. 1 top right) for $K = 1$ and a duct of constant cross-section is $\Psi_0(x;\omega) = \sqrt{1 - y^2} = \sin(x/l) = \sin(kx) = \cos(x/l - \pi/2)$ also a sinusoidal wave, which is even relative to the mid-position; as the height of undulations ε becomes larger, the amplitude decreases towards the narrow end $x = \pi l$ and increases towards the wider end $x = 0$. As for Ψ_0, also for Ψ_1, the oscillatory character of the wave field (Fig. 1 bottom right) becomes evident for larger wavenumber.

Relative to the reduced potential Ψ, the acoustic potential Φ introduces (13) an amplitude change due to variations in cross-section and a phase shift due to the Doppler effect of the mean flow. It follows that the acoustic potential is a complex quantity, and it depends on the Mach number, in addition to the height of undulations and the wavenumber. The first component of the acoustic potential Φ_0 is plotted in Fig. 2, separating the amplitude $|\Phi_0|$ on the left from the phase $\arg(\Phi_0)$ on the right, and varying in turn the height of undulations (top), the Mach number

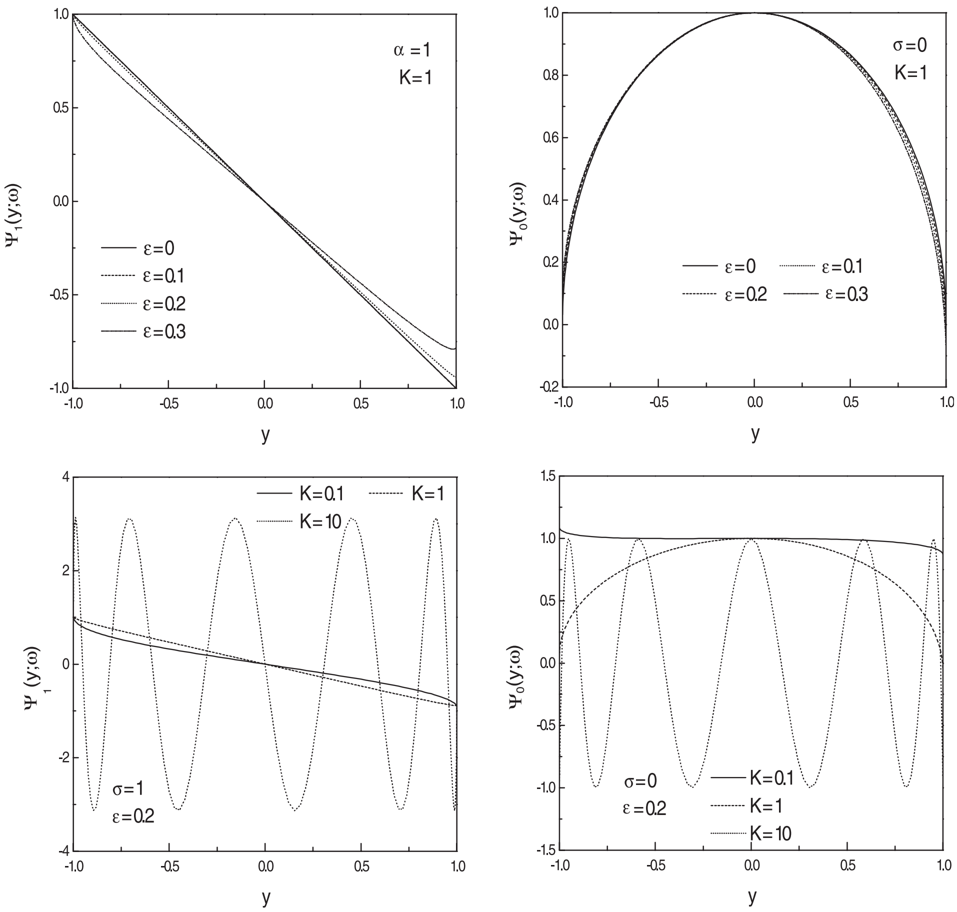

Fig. 1. Reduced acoustic potential $\Psi_\sigma(x;\omega)$ versus longitudinal co-ordinate $y \equiv \cos(x/l)$ over one-half period of the duct $0 \leqslant x \leqslant \pi l$ or $1 \geqslant y \geqslant -1$, for (right/left) component $\sigma = 0/\sigma = 1$ of the wave field; (top/bottom) varying height of undulations / varying wavenumber.

(middle) and the wavenumber (bottom). Fig. 2 top left shows that the acoustic potential would be symmetric relative to the mid-position for a duct of constant cross-section $\varepsilon = 0$, but as the height of undulations increases the amplitude is higher in the narrower section $x = \pi l$, peaks away from the middle and the node moves away from the widest section $x = 0$; this is confirmed (Fig. 2 top right) by the phase jump of 180° at the node moving away from the widest section, with phase variations being otherwise small, due to weak Doppler effects for $K = 1$ and $M_0 = 0.1$. A phase jump of π corresponds ($e^{i\pi} = -1$) to a change of sign of the acoustic field, as it goes through a zero. The amplitude of the first component of the acoustic potential (Fig. 2 middle left) is little affected by the Mach number, whereas the phase (Fig. 2 middle right) shows an increased but still weak Doppler effect for $K = 1$. For larger wavenumber $K = 10$ the oscillatory character is more evident in the amplitude (Fig. 2 bottom left), which has more nodes, and in the phase (Fig. 2 bottom right), which has more phase jumps. Whereas the first component of the acoustic potential Φ_0 is basically an 'even' function (Fig. 2), which peaks near the middle section, the second component Φ_1 is basically an 'odd' function which (Fig. 3) vanishes at the middle section, and

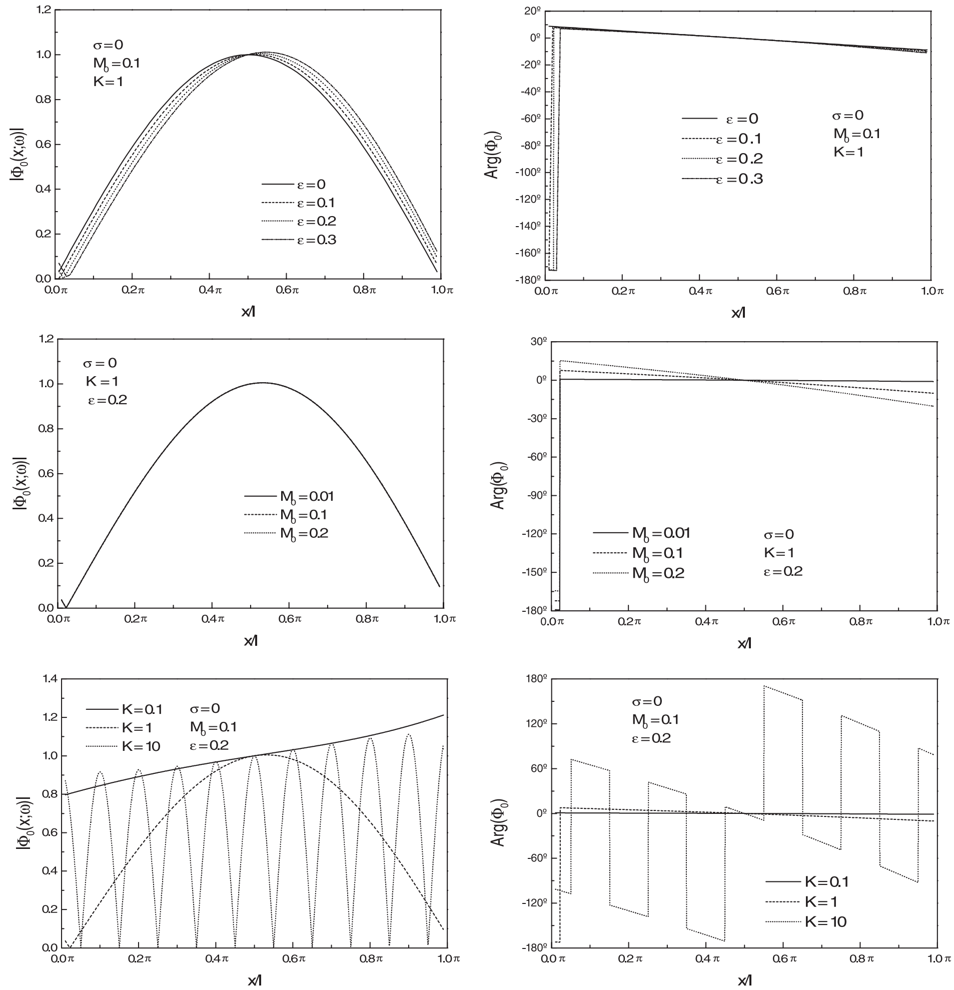

Fig. 2. Acoustic potential corresponding to first wave component $\Phi_0(x;\omega)$ versus dimensionless longitudinal distance x/l over one-half period of the duct $0 \leqslant x/l \leqslant \pi$, for (left/right) amplitude/phase; (top/middle/bottom) varying height of undulations/Mach number/wavenumber.

becomes more 'distorted' with larger amplitudes (Fig. 3 left) and phase (Fig. 3 right) variations toward the narrow sections as the undulations increase in relative height (Fig. 3 top), the Mach number of the mean flow increases (Fig. 3 middle) and the dimensionless wavenumber increases (Fig. 3 bottom). Although the acoustic field can be represented by the unreduced potential alone, its implications concerning the acoustic velocity and pressure perturbations deserve discussion.

6. Discussion

The acoustic velocity of the first component of the wave field (Fig. 4) is basically an even function, whose amplitude (Fig. 4 top left) vanishes in the middle section, and increases towards

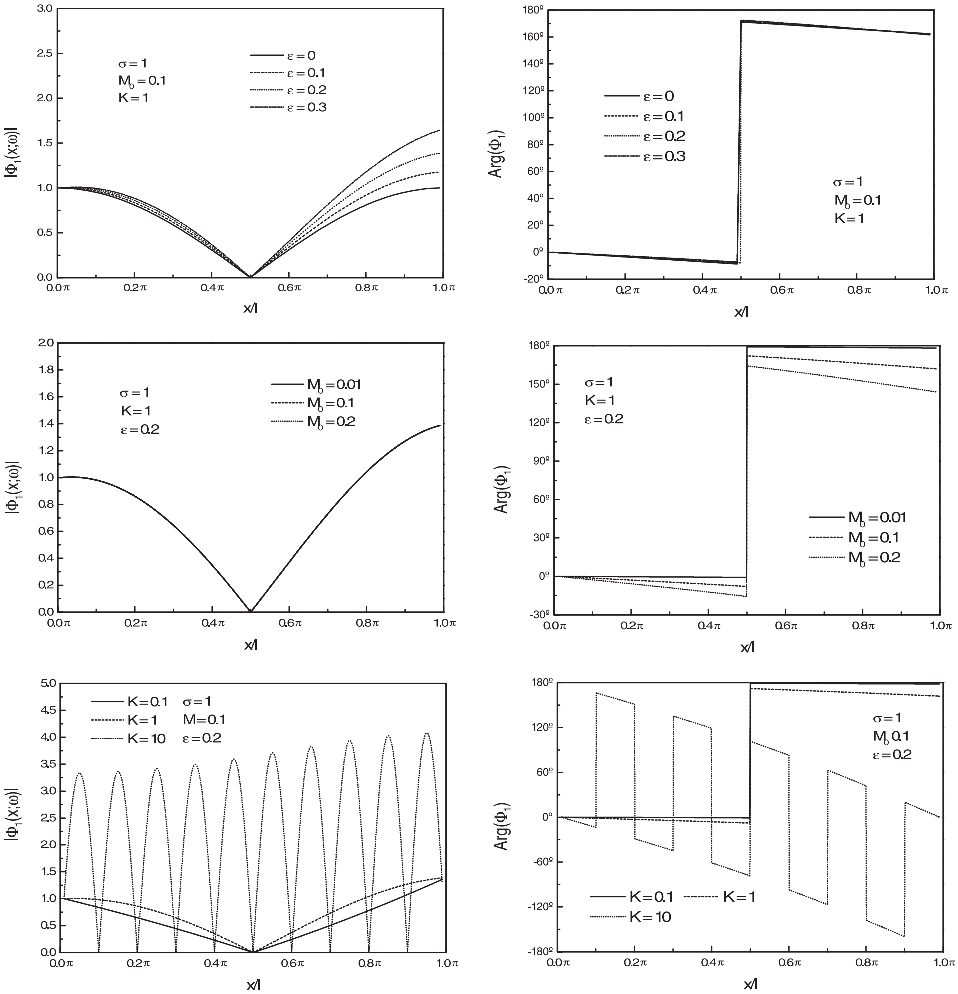

Fig. 3. As Fig. 2 for second wave component of acoustic potential $\Phi_1(x;\omega)$.

the widest and narrowest sections, more at the latter; the phase (Fig. 4 top right) deviates more from linear near the center section and for larger height of undulations. The Mach number has little effect on amplitude (Fig. 4 middle left) and a very pronounced effect on phase (Fig. 4 middle right), which is far from linear; in fact, the phase has a minimum in the mid-section, and then rises again before decaying towards the narrowest section. As the wavenumber increases the oscillatory character becomes more evident in the nodes of the amplitude (Fig. 4 bottom left) and jumps of the phase (Fig. 4 bottom right). The second component of the acoustic velocity (Fig. 5) is basically an odd function, as it concerns amplitude (Fig. 5 left), with asymmetries which become more marked for larger height of undulations (Fig. 5 top left) and larger wavenumber (Fig. 5 bottom left), but are not affected by Mach number (Fig. 5 middle left). The phase (Fig. 5 right) consists mostly of phase jumps (Fig. 5 top right) with a continuous Doppler shift noticeable for larger but still low Mach number (Fig. 5 middle right), and rounded-off jump edges for larger wavenumbers (Fig. 5 bottom right). The first component P_0 of the acoustic pressure (Fig. 6), is basically an even

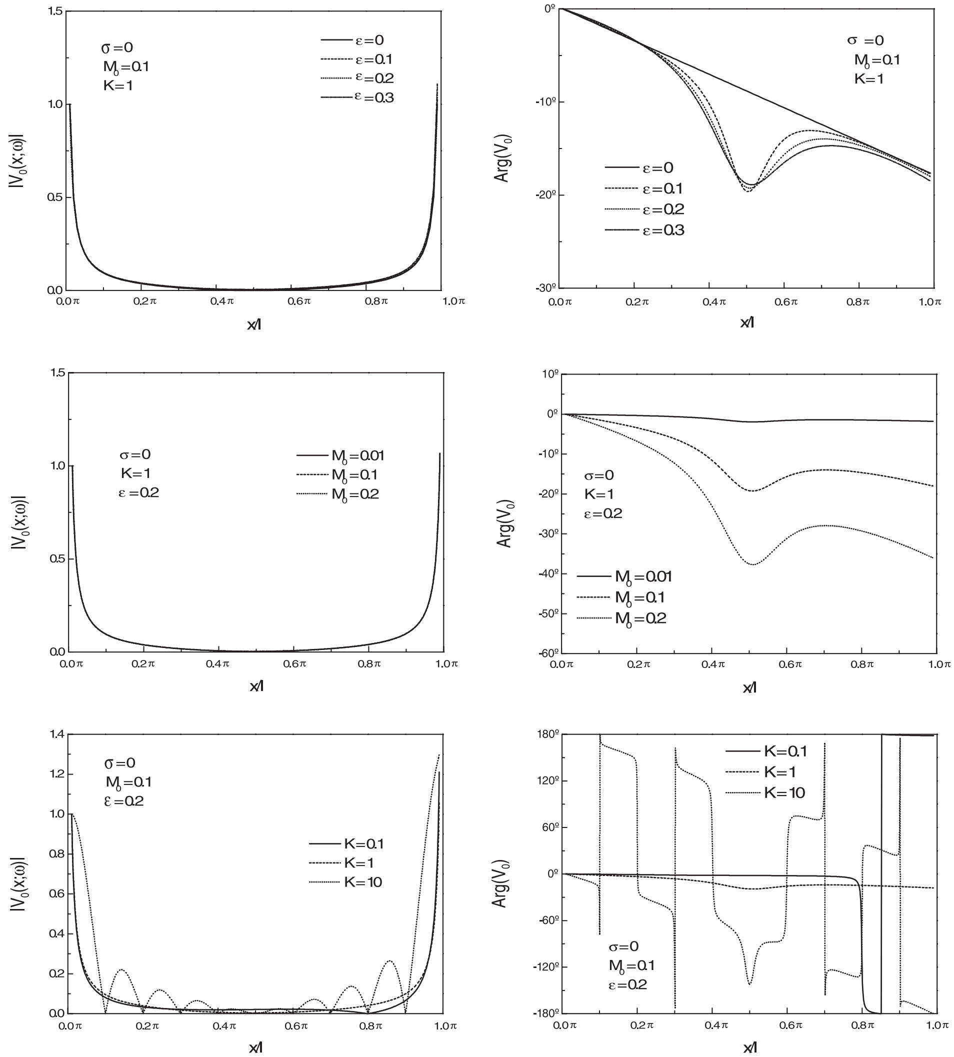

Fig. 4. As Fig. 2 for first wave component of acoustic velocity $V_0(x;\omega)$.

function for amplitude (Fig. 6 left) with maxima at the widest and narrowest sections and a secondary, lower maximum near the middle; these asymmetries become more noticeable for larger height of undulations (Fig. 6 top left), larger Mach number (Fig. 6 middle left) and larger wavenumber (Fig. 6 bottom left). The phase (Fig. 6 right) shows similar deviations from symmetry (Fig. 6 top and middle right), except for larger wavenumbers (Fig. 6 bottom right) when it becomes an odd function. The second component P_1 of the acoustic pressure (Fig. 7) is an odd function, but otherwise shows similar trends for amplitude (Fig. 7 left) and phase (Fig. 7 right) as regards variations of the height of undulations (Fig. 7 top), Mach number of mean flow (Fig. 7 middle) and wavenumber (Fig. 7 bottom).

The propagation of sound in a duct with undulated walls can be considered as a particular case of 'horn' or duct of varying cross-section, in the absence of mean-flow and as a particular case of

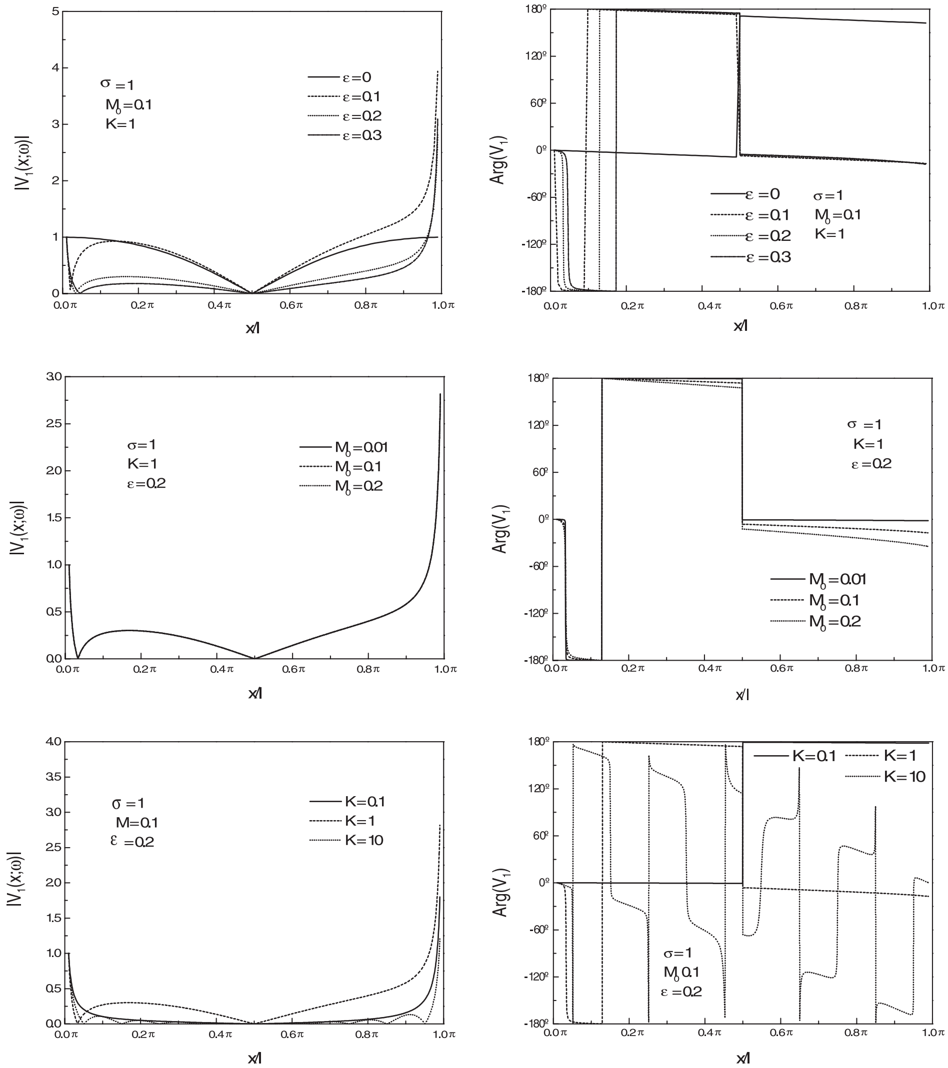

Fig. 5. As Fig. 2 for second wave component of acoustic velocity $V_1(x;\omega)$.

'nozzle' in the presence of mean flow, since the methods of analysis are similar. The acoustics of horns of varying cross-section is simplest for five shapes which allow solutions in terms of elementary functions, viz., the exponential [19], catenoidal [20], sinusoidal [21] and inverses [22]. The acoustics of low Mach number nozzles of corresponding shapes involve special functions, e.g., confluent hypergeometric functions for the exponential shape [23] and modified Mathieu functions for the catenoidal, sinusoidal and inverse shapes [24,25]. For other duct shapes, special functions are needed even for horns, e.g., Bessel function for power law ducts [26], and Hermite functions for Gaussian horns [27]; the extension from horns to nozzles can be made most simply via a transformation [28] which has been applied to all the preceding shapes. A different transformation applies to the extension from the acoustics of horns to the longitudinal vibrations

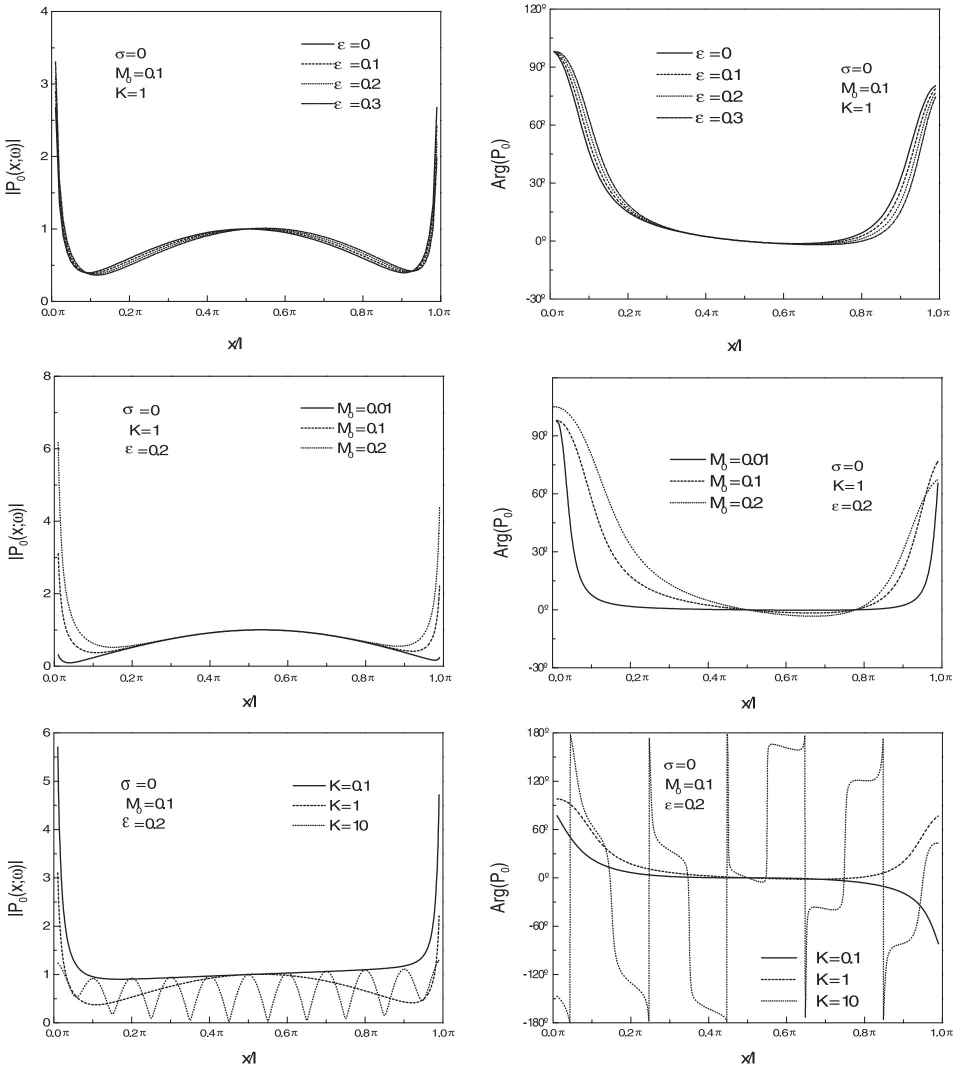

Fig. 6. As Fig. 2 for first wave component of acoustic pressure $P_0(x;\omega)$.

of visco-elastic rods [15,29]. Other horn shapes like multi-parameter ducts [30] have been studied. The present paper concerns a duct with periodic wall undulations, whose relative magnitude ε should remain consistent $\varepsilon^2 \ll 1$ with quasi-one-dimensional propagation.

The case of very small corrugations $\varepsilon < 0.1$ represents the effect of surface roughness on sound propagation in a duct. It can be seen from Figs. 4–7 that the effect on the amplitude of the acoustic velocity and pressure is small $< 1\%$ over an undulation scale. Bearing in mind that the radius of the duct is much larger, by a factor of ten or more, and the wavelength is larger than the dimensions of the cross-section for quasi-one-dimensional propagation, the effect over one wavelength may be non-negligible, and become significant over many wavelengths. The phase effects of the undulations are more pronounced than amplitude effects, e.g., the phase minimum

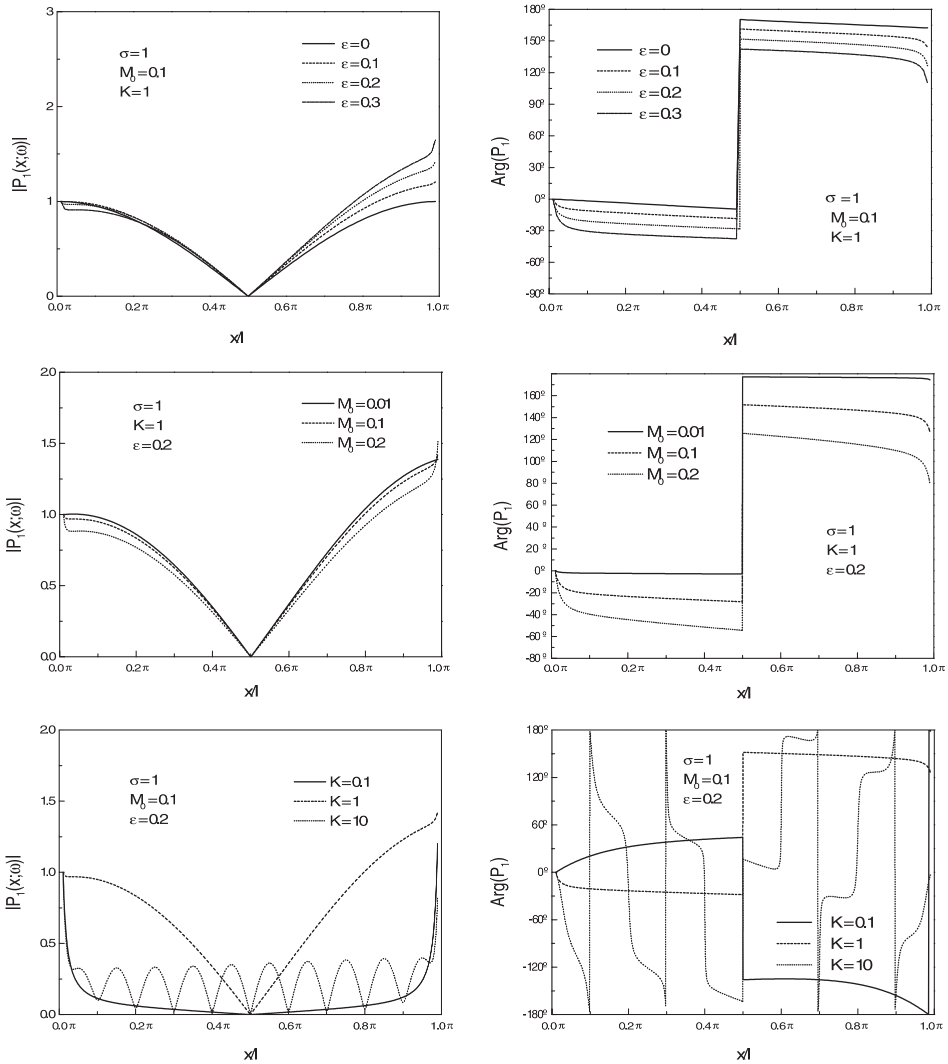

Fig. 7. As Fig. 2 for second wave component of acoustic pressure $P_1(x;\omega)$.

in Fig. 4 top right suggests that waves are reflected from the undulations towards the wider section of the duct. These reflections will be more important for moderate undulations $0.1<\varepsilon<0.3$, when there are noticeable differences between the narrowest and widest sections of the duct e.g., in Fig. 7 top left. Since each of the periodic undulations produces a reflection, their cumulative effect could be significant. In the presence of low Mach number mean flow $M^2 \ll 1$, the Mach number at the average section was restricted to $M_0<0.2$ to keep below $M_{max}<0.3$ at the narrowest section. The effect of mean flow is most pronounced on the phase of sound, and could enhance interference in the case of a sound wave with broad spectrum. The quasi-one-dimensional approximation assumes uniform flow over the cross-section, and thus neglects the effects of acoustic and flow boundary layers, in causing a shear flow profile and also the effects of vortex shedding from the corrugations.

Appendix A. Solutions around the maximum and minimum cross-section

The cross-sectional area of the duct (14) has a maximum $S_{max} = S_0(1+\varepsilon)$ [minimum $S_{min} = S_0(1-\varepsilon)$] at $x = 0$ $(x = \pi l)$, corresponding (16a) to $y = 1$ $(y = -1)$ which is a regular singularity of the differential equation (17). The change of independent variable

$$z = 1 \pm y, \quad G^{\mp}(z) \equiv F(y), \tag{A.1, A.2}$$

places the regular singularity at $z = 0$, and transforms the differential equation (27) to

$$\begin{aligned} &4[1-(z-1)^2](1 \pm \varepsilon z \mp \varepsilon)^2\, G''_{\mp} - 4(z-1)(1 \pm \varepsilon z \mp \varepsilon)^2 G'_{\mp} \\ &\quad + \{4K^2(1 \pm \varepsilon z \mp \varepsilon)^2 + \varepsilon^2 \pm 2\varepsilon z \mp 2\varepsilon^2 (z-1)^2\}\, G_{\mp} = 0. \end{aligned} \tag{A.3}$$

Since $z = 0$ is a regular singularity, solutions exist as Frobenius–Fuchs series

$$G_v^{\pm} = \sum_{n=0}^{\infty} b_n^{\pm}(v) z^{n+v}, \tag{A.4}$$

with recurrence formula for the coefficients:

$$\begin{aligned} &4(\varepsilon^2 \mp 2\varepsilon + 1)(n+v)(1-2n-2v) b_n^{\mp}(v) \\ &= 2\{2(n+v-1)[(n+v-2)(-5\varepsilon^2 \pm 6\varepsilon - 1) - 3\varepsilon^2 \pm 4\varepsilon - 1] \\ &+ 4K^2 \mp 2(4K^2+1)\varepsilon + (4K^2+2)\varepsilon^2\} b_{n-1}^{\mp}(v) \\ &+ \{4(n+v-2)[2\varepsilon(n+v-3)(2\varepsilon \mp 1) + \varepsilon(3\varepsilon \mp 2)] \\ &- 2\varepsilon(4K^2+1)(\varepsilon \mp 1)\} b_{n-2}^{\mp}(v) \\ &+ \varepsilon^2\{1 + 4K^2 - 4(n+v-3)^2\} b_{n-3}^{\mp}(v), \end{aligned} \tag{A.5}$$

is obtained by substituting, Eq. (A.4) in Eq. (A.3).

Setting $n = 0$ in (A.5) leads to the indicial equation

$$4v(\varepsilon^2 \mp 2\varepsilon + 1)(2\sigma - 1) b_0^{\pm}(v) = 0, \tag{A.6}$$

whose roots are $v = 0, 1/2$; these roots lead to independent solutions

$$v = 0: G_0^{\pm}(z) = \sum_{n=0}^{\infty} b_n^{\pm}(0) z^n, \tag{A.7}$$

$$v = 1/2: G_{1/2}^{\pm}(z) = \sum_{n=0}^{\infty} b_n^{\pm}(1/2) z^{n+1/2} \tag{A.8}$$

which are finite at $z = 0$, or $x = 0, \pi l$. The solution $\Psi_v^{\pm}(x;\omega)$ has to be matched to $\Psi_v(x;\omega)$ to specify the acoustic field in the whole duct.

Appendix B. Matching the three pairs of solutions

The acoustic field is specified by the following pairs of solutions (Table 1) or alternatively using the variable $y = \cos(x/l)$, so that matching can be done at the points

$$0 < x_1^-,\ x_2^- < \pi l/2 < x_1^+,\ x_2^+ < \pi l, \tag{B.1}$$

Table 1
Acoustic fields

Reduced acoustic potential	Index values	Region of convergence
$\Psi_\sigma(x;\omega)$	$\sigma = 0, 1$	$0<x<\pi l$
$\Psi_\sigma^+(x;\omega)$	$\sigma = 0, 1/2$	$-\pi l/2<x<\pi l/2$
$\Psi_\sigma^-(x;\omega)$	$\sigma = 0, 1/2$	$\pi l/2<x<3\pi l/2$

Table 2
Pairs of solutions

Independant variables	Series expansion around	Particular integral
$y = \cos(x/l)$	$x = \pi l/2$	$F_\sigma(y)$
$y = 1 + \cos(x/l)$	$x = \pi l$	$G_\sigma{}^+(z)$
$y = 1 - \cos(x/l)$	$x = 0$	$G_\sigma{}^-(z)$

or equivalently

$$1 > y_1^-,\ y_2^- > 0 > y_1^+,\ y_2^+ > -1, \tag{B.2}$$

using the functions presented in Table 2.

Since $F_\sigma(y)$ is a solution valid at the points y_1^+, y_2^+ in the region of overlap with $G_\nu^+(1-y)$, and at the points y_1^-, y_2^- in the region of overlap with $G_\nu^-(1+y)$,

$$F_0(y_1^\pm) = A_\pm G_0^\pm(1\mp y_1^\pm) + B_\pm G_{1/2}^\pm(1\mp y_1^\pm), \tag{B.3}$$

$$F_1(y_2^\pm) = A_\pm G_0^\pm(1\mp y_2^\pm) + B_\pm G_{1/2}^\pm(1\mp y_2^\pm). \tag{B.4}$$

Thus the four matching constants $A_\pm$, $B_\pm$ are determined by

$$\begin{aligned}\{A_\pm, B_\pm\}&[G_0^\pm(1\mp y_1^\pm)G_{1/2}^\pm(1\mp y_2^\pm) - G_0^\pm(1\mp y_2^\pm)G_{1/2}^\pm(1\mp y_1^\pm)]\\ &= \{G_{1/2}^\pm(1\mp y_2^\pm)F_0(y_1^\pm) - G_{1/2}^\pm(1\mp y_1^\pm)F_1(y_2^\pm),\\ &\quad - G_0^\pm(1\mp y_2^\pm)F_0(y_1^\pm) + G_0^\pm(1\mp y_1^\pm)F_1(y_2^\pm)\}.\end{aligned} \tag{B.5, B.6}$$

In this way the acoustic fields may be calculated for $0<x<\pi l$ using Eq. (19), for $0\leqslant x<\pi l/2$ using Eq. (B.3) and (B.4) with lower signs and for $\pi l/2<x\leqslant\pi l$ using upper lower signs. This procedure was used to compute the plots in Figs. 1–7.

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