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Letter to the Editor

Quadratic non-linear oscillators

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Quadratic non-linear oscillators (QNO) provide useful models for both the testing of perturbation procedures [1,2] and the analysis of various phenomena in the physical and engineering sciences [1,2]. In general, these differential equations are special cases of the second order equation

$$\ddot{x} + x + \alpha x^2 + \beta x\dot{x} + \gamma(\dot{x})^2 = 0, \tag{1}$$

where (α, β, γ) are parameters. In particular, this note is concerned with the three equations:

$$\ddot{x} + x + \varepsilon x^2 = 0, \tag{2}$$

$$\ddot{x} + x + \varepsilon x\dot{x} = 0, \tag{3}$$

$$\ddot{x} + x + \varepsilon \dot{x}^2 = 0. \tag{4}$$

In these expressions, the relevant parameter has been set equal to zero and, without loss of generality, it is assumed that ε is positive. The main goal of the work to be presented below is to compare the solution behaviors of these three QNO differential equations. While Eqs. (2) and (3) have been previously studied, respectively, in Refs. [3,4], the behavior of the solutions to Eq. (4) have not been studied in detail.

To begin, consider Eq. (2) which corresponds to a non-linear conservative system. Its system equations are

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x(1 + \varepsilon x). \tag{5}$$

Examination of Eq. (5) shows that it has two fixed points or equilibrium solutions [2,5] located at

$$(\bar{x}^{(1)}, \bar{y}^{(1)}) = (0, 0), \quad (\bar{x}^{(2)}, \bar{y}^{(2)}) = \left(-\frac{1}{\varepsilon}, 0\right). \tag{6}$$

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The trajectories, $y(x)$, in the two-dimensional (x, y) phase space are determined by the following first order differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x(1+\varepsilon x)}{y}. \tag{7}$$

Since Eq. (7) is separable, it can be integrated to give the energy function or first integral,

$$\frac{y^2}{2} + \frac{x^2}{2} + \frac{\varepsilon x^3}{3} = E. \tag{8}$$

Using standard procedures from the qualitative theory of differential equations [2;5;6, Appendix I], the following conclusions can be reached:

(1) For initial values (x_0, y_0) satisfying the conditions

$$-\left(\frac{1}{\varepsilon}\right) < x_0 < x^*, \tag{9}$$

$$0 \leqslant \frac{y_0^2}{2} + \frac{x_0^2}{2} + \frac{\varepsilon x_0^3}{3} < \frac{1}{6\varepsilon^2}, \tag{10}$$

where x^* is the real and positive root of

$$z^3 + \left(\frac{3}{2\varepsilon}\right) z^2 - \left(\frac{1}{2\varepsilon^3}\right) = 0, \tag{11}$$

the corresponding solutions to Eq. (2) are bounded and periodic.

(2) All other values for the initial conditions give rise to unbounded solutions. In particular, if $y_0 > 0$ and $x_0 < 0$, then the trajectory in phase space comes in toward the origin and eventually turns around and goes out to regions of the phase space where both x and y can monotonically take on arbitrarily large negative values. In summary, Eq. (2) has oscillatory solutions only for initial conditions that have sufficiently small values in a neighborhood of the fixed point $(\bar{x}, \bar{y}) = (0, 0)$.

The second case, Eq. (3), can be rewritten as

$$\ddot{x} + (1 + \varepsilon \dot{x}) x = 0, \tag{12}$$

and corresponds to what has been called a "generalized harmonic oscillator" [4]. From the system equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -(1 + \varepsilon y)x, \tag{13}$$

it follows that there is only one fixed point or equilibrium state located at $(\bar{x}, \bar{y}) = (0, 0)$. The trajectories in phase space are solutions of the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(1+\varepsilon y)x}{y}. \tag{14}$$

Note that since Eq. (14) is separable and, consequently, easily integrated, the following first-integral exists for Eq. (12):

$$\frac{y}{\varepsilon} - \left(\frac{1}{\varepsilon^2}\right) \ln(1 + \varepsilon y) + \frac{x^2}{2} = E. \tag{15}$$

Note that for x and y "small," this latter equation reduces to

$$\frac{y^2}{2} + \frac{x^2}{2} = E + O(y^3), \tag{16}$$

which, to terms of $O(y^3)$, is the energy function for the linear harmonic oscillator [2]. It has been shown, see Mickens [4], using phase space techniques, that the following results hold for the solutions of Eq. (12):

(1) All trajectories that begin in the half-plane region of phase space such that $y > -(1/\varepsilon)$ are periodic.

(2) All trajectories in the half-plane region $y < -(1/\varepsilon)$ are non-periodic and correspond to unbounded motions.

(3) The boundary between the behaviors given in (1) and (2) is an exact solution to the differential equation and is given by the expression

$$y = -\left(\frac{1}{\varepsilon}\right) \quad \text{or} \quad x(t) = -\left(\frac{t}{\varepsilon}\right) + x_0. \tag{17}$$

This corresponds to uniform motion to the left in phase space with constant velocity.

The third case, Eq. (4), is the quadratic velocity non-linear oscillator equation. It is known that for sufficiently small initial values (x_0, y_0), this equation has periodic solutions. In fact, for $|\varepsilon| \ll 1$, application of the perturbation method gives [2]

$$\begin{aligned} x(\theta, \varepsilon) = {} & A\cos\theta + \varepsilon\left(\frac{A^2}{6}\right)(-3 + 4\cos\theta - \cos 2\theta) \\ & + \varepsilon^2\left(\frac{A^3}{3}\right)\left[-2 + \left(\frac{61}{24}\right)\cos\theta - \left(\frac{2}{3}\right)\cos 2\theta + \left(\frac{1}{8}\right)\cos 3\theta\right] \\ & + O(\varepsilon^3), \end{aligned}$$

where

$$\theta = \omega t, \quad \omega(\varepsilon) = 1 - \varepsilon^2\left(\frac{A^2}{6}\right) + O(\varepsilon^3). \tag{18}$$

The system equations for Eq. (4) are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - \varepsilon y^2, \tag{19}$$

and, consequently,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{x + \varepsilon y^2}{y}\right). \tag{20}$$

Only one fixed point or equilibrium solution exists and it is located at $(\bar{x}, \bar{y}) = (0, 0)$. Also, observe that Eq. (20) is invariant under the transformation

$$x \to x, \quad y \to -y, \tag{21}$$

which corresponds to reflection through the x-axis. In other words for any given trajectory, its mirror image, on reflection through the x-axis, is also a trajectory of Eq. (20). However, a more significant feature of this equation is that it can be exactly solved. This means that Eq. (4) has a known first integral. It can be calculated by using the change of variables

$$u = y^2, \tag{22}$$

which converts Eq. (20) to the form

$$\frac{du}{dx} + 2\varepsilon u = -2x. \tag{23}$$

The general solution to Eq. (23) is

$$u(x) = Ae^{-2\varepsilon x} + \left(\frac{1}{2\varepsilon^2}\right) - \left(\frac{1}{\varepsilon}\right)x, \tag{24}$$

where A is an arbitrary integration constant. In terms of y, this equation is

$$2\varepsilon^2 y^2 = 2\varepsilon^2 A e^{-2\varepsilon x} + 1 - 2\varepsilon x. \tag{25}$$

A further simplification occurs if new variables (x', y') are used. Making these substitutions into Eq. (25), defining

$$C = 2\varepsilon^2 A, \tag{26}$$

and then *dropping* the primes, gives

$$y^2 = Ce^{-x} + 1 - x. \tag{27}$$

It will now be demonstrated that closed curves, for Eq. (27), only exist for values of C such that

$$-1 < C < 0. \tag{28}$$

First, consider the case where $C = 0$. Thus, Eq. (27) becomes

$$y^2 = 1 - x. \tag{29}$$

Examination of this equation shows that it is a parabola, symmetric with respect to the x-axis, having it maximum x value equal to $x_m = 1$. Therefore, its vertex is located in the (x, y) phase-plane at $(1, 0)$. It is clear that Eq. (29) corresponds to an unbounded trajectory that comes in from the second quadrant in the (x, y) plane, passes through $(1, 0)$ and then eventually becomes unbounded in the fourth quadrant.

Second, it is easy to see that trajectories corresponding to $C > 0$ are also unbounded. This follows from the fact that from Eq. (27),

$$[y(x, C = 0)]^2 > [y(x, C > 0)]^2, \tag{30}$$

where $y(x, C)$ denotes the value of y in Eq. (27) for a given (non-negative) value of C. Thus, the trajectory $y(x, C > 0)$ lies entirely outside of and to the right of the trajectory for $y(x, C = 0)$. Since $y(x, C = 0)$ is bounded, then $y(x, C = 0)$ is also unbounded.

Now, let $C < 0$ which can be written as $C = -|C|$. For this situation, Eq. (27) becomes

$$y^2 + |C|e^{-x} = 1 - x. \tag{31}$$

It is sufficient to only consider the initial conditions

$$x_0 < 1, \quad y_0 = 0. \tag{32}$$

The existence of periodic solutions follows from an analysis of the equation

$$|C|\mathrm{e}^{-x_0} = 1 - x_0. \tag{33}$$

A periodic solution occurs for values of C such that Eq. (33) has two real solutions, say $x_0^{(1)}$ and $x_0^{(2)}$, for which one is positive and the other is negative. Since $(\bar{x}, \bar{y}) = (0, 0)$ is the only fixed point, any periodic trajectory must have this property, i.e., the trajectory crosses the x-axis at the two points $(x_0^{(1)}, 0)$ and $(x_0^{(2)}, 0)$, where

$$x_0^{(1)} < 0 < x_0^{(2)}. \tag{34}$$

A way of resolving this issue is to ask do the two curves

$$y_1(x_0) = |C|\mathrm{e}^{-x_0}, \quad y_2(x_0) = 1 - x_0 \tag{35}$$

cross and how many times do they do this? A simple graphical analysis, i.e., sketching $y_1(x_0)$ and $y_2(x_0)$ versus x_0, with $|C|$ varying, gives the following results:

(1) If $|C| > 1$, then $y_1(x_0)$ and $y_2(x_0)$ do not intersect; therefore, no periodic solutions are possible.

(2) For $|C| = 1$, $y_1(x_0)$ and $y_2(x_0)$ are tangent at $x_0 = 0$. Since $y_0 = 0$, it follows that this case corresponds to the fixed point $(x_0, y_0) = (\bar{x}, \bar{y}) = (0, 0)$.

(3) Lastly, for $0 < |C| < 1$, the two curves intersect in two points; one is at a negative value of x_0, while the other is for a positive x_0. These two values can be identified with $x_0^{(1)}$ and $x_0^{(2)}$ in Eq. (24). The conclusion is that Eq. (27) has solutions which are closed curves in the (x, y) space only if C satisfies the condition

$$-1 < C < 0. \tag{36}$$

It should be noted that all of the above results have been obtained under the assumption that the parameter ε is possible. However, it is not difficult to demonstrate that similar conclusions are reached for $\varepsilon < 0$.

The major conclusion is that all of the three quadratic oscillator equations (2)–(4) have certain regions in the (x, y) phase space for which periodic solutions exist. There also exist other regions, that enclose the regions where periodic solutions occur, for which only unbounded motions take place. The next step is to investigate the conditions under which the full equation (1) has periodic solutions and the regions in phase space where they exist.

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