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Letter to the Editor

Vibrations of a free two-mass system with quadratic non-linearity and a constant excitation force

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1. Introduction

In many papers, the problem of the non-linear vibration of a two-mass system is discussed. Usually, the free vibrations are analyzed. In Refs. [1,2], the systems which contain two masses connected by a spring with strong non-linear properties are considered. The vibrations of the system are described with a system of two coupled second order non-linear differential equations and the exact solutions are obtained in the form of the Jacobi elliptic function. In Refs. [3,4], the model considered is more complex: the non-linear elastic properties of the springs which connect the masses to fixed supports are also taken into consideration. The concept of normal modes is assumed for solving the differential equations of motion of the system with cubic non-linearity. In Ref. [5], the non-similar normal modes of free oscillations of a coupled non-linear oscillator are examined. Beside cubic non-linearities, quadratic non-linearities are also investigated. For all of these considerations, it is common that only the free vibrations of the system are treated. In the paper by Vakakis and Caughey [6], forced vibrations are also discussed.

The aim of the present paper is to determine the forced vibration properties of a free system which contains two masses connected by a spring with quadratic non-linear properties and on which a constant force acts. The vibration of a one-degree-of-freedom mechanical system with a quadratic non-linearity [7] and constant excitation force [8] was investigated previously. The second order non-linear differential equation with a strong square non-linearity is exactly solved applying Jacobi elliptic function. The influence of the excitation on the amplitude, frequency and period of forced vibrations are determined. In this paper, the methodology for solving this problem is extended to a system of two non-homogenous coupled differential equations with a strong square non-linearity.

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2. Model of the system

The physical model of the system is shown in Fig. 1. It contains two masses, m_1 and m_2, which are connected with a spring with a non-linear elastic property. The non-linearity is of the quadratic type. A constant excitation force F acts on the mass m_1. The kinetic and the potential energies of the system are

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2,$$
$$V = \frac{1}{2} c(x-y)^2 + \frac{1}{3} b^2 (\pm)|x-y|(x-y)^3, \tag{1}$$

where x and y are the deflections of the masses, c and b^2 are constant coefficients. In the relation for the potential energy, the *absolute* function is used. The force in the spring is a quadratic function of deformation and the condition of antisymmetry is evident. If the spring is extended, a force which has the tendency to relax and to put the spring in the previous state appears. In that case, the deformation is positive. When the spring is pressed, the deformation is negative. The signs in the bracket $(\pm)$ indicate the type of the spring, i.e., the plus sign in the bracket is for hard spring and the minus sign in the bracket is for soft spring.

Using relations (1) and the fact that the force F acts, the mathematical model of the system is

$$m_1 \ddot{x} + c(x-y) + b^2(\pm)|x-y|(x-y)^2 = F,$$
$$m_2 \ddot{y} - c(x-y) - b^2(\pm)|x-y|(x-y)^2 = 0, \tag{2}$$

with respect to the initial conditions

$$x(0) = y(0) = 0, \quad \dot{x}(0) = \dot{y}(0) = 0. \tag{3}$$

3. Solving procedure

Introducing the new variable

$$X = x - y \tag{4}$$

in Eqs. (2), the transformed equations of motion are

$$m_1 \ddot{x} + cX + (\pm) b^2 |X| X^2 = F,$$

$$m_2 \ddot{x} - m_2 \ddot{X} - cX - (\pm) b^2 |X| X^2 = 0. \tag{5}$$

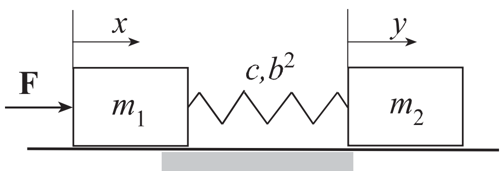


Fig. 1. Model of the two-mass system connected with a spring.

Eliminating $\ddot{x}$ in Eq. (5), the following differential equation is obtained:

$$\ddot{X} + c\frac{M}{m_1 m_2}X + (\pm)b^2\frac{M}{m_1 m_2}|X|X^2 = \frac{F}{m_1}, \tag{6}$$

where the total mass of the system is $M = m_1 + m_2$. Differential equation (6) is a single non-homogenous second order differential equation with a quadratic non-linearity. For the special case when the elastic property of the spring is linear, Eq. (6) transforms to

$$\ddot{X} + c\frac{M}{m_1 m_2}X = \frac{F}{m_1}. \tag{7}$$

Summarizing Eqs. (5) it is

$$M\ddot{x} = m_2\ddot{X} + F, \tag{8}$$

and after integration and using the initial conditions (3) it is

$$x = \frac{m_2}{M}X + \frac{Ft^2}{2M}. \tag{9}$$

From Eqs. (4) and (9) it is

$$y = -\frac{m_1}{M}X + \frac{Ft^2}{2M}. \tag{10}$$

Eqs. (9) and (10) are the solutions of the system of differential equations (2) with the initial conditions (3), where X is the solution of the differential equation (6) for the non-linear case and of the differential equation (7) for the linear case.

Now introduce the following parameters:

$$c^* = \frac{\omega^2}{\mu_1\mu_2}, \quad b^{2*} = \frac{(\omega^*)^2}{\mu_1\mu_2}, \quad a = \frac{F}{\mu_1 M}, \tag{11}$$

where

$$\mu_1 = \frac{m_1}{M}, \quad \mu_2 = \frac{m_2}{M}, \quad \omega^2 = \frac{c}{M}, \quad (\omega^*)^2 = \frac{b^2}{M}.$$

Substituting Eq. (11) into Eq. (6), the following differential equation is obtained:

$$\ddot{X} + c^*X + (\pm)b^{2*}|X|X^2 = a, \tag{12}$$

with the initial conditions which are according to Eq. (3)

$$X = 0, \quad \dot{X} = 0. \tag{13}$$

In Ref. [8], the solution of differential equation (12) for the initial conditions (13) is considered. It is concluded that the solution is of the oscillatory type and the function X is non-negative for all values of time t.

Analyzing relations (9) and (10) which describe the motion of the masses and using the previously mentioned comment on the solution X given in Ref. [8], it is obvious that the motion of both masses is a simple superposition of two separate motions: one oscillatory motion, as the solution of Eq. (12) is of oscillatory type, and a non-oscillatory motion which is a quadratic time function. The solutions have the identical form for the linear and non-linear cases. As the function X is non-negative, the motion $x(t)$ of the mass m_1 (9) on which the force F acts is also positive as the oscillatory motion is added to the non-oscillatory motion. The motion $y(t)$ of the mass m_2 (10) is obtained by subtraction of the oscillatory motion of the non-oscillatory motion. It is evident that the total motion depends on the oscillations and it is of special interest to analyze them. The systems with linear spring, hard spring and with soft spring are considered separately.

3.1. System with hard spring

For the system with hard spring properties the upper sign in Eq. (12) is evident. According to the results shown in the Ref. [8], the particular solution of Eq. (12) for Eq. (13) is

$$X = \frac{A_h}{1 + \frac{b^{2*} A_h^2}{3a}} \mathrm{sd}^2 \left(\Omega_h t \sqrt{1 + \frac{b^{2*} A_h^2}{3a}}, \frac{\frac{b^{2*} A_h^2}{3a}}{\left(1 + \frac{b^{2*} A_h^2}{3a}\right)} \right), \tag{14}$$

where

$$\Omega_h = \sqrt{\frac{a}{2A_h}}, \tag{15}$$

and

$$A_h = \frac{3}{4} \frac{c^*}{b^{2*}} \left[\sqrt{1 + \frac{16}{3} \frac{b^{2*} a}{(c^*)^2}} - 1 \right]. \tag{16}$$

It is an oscillatory solution which is a function of the Jacobi elliptic function sd [9]. Applying the symbols (11) and the physical sense of the terms in relations (14)–(16), the amplitude of vibration is

$$A_h^* = \frac{A_h}{1 + ((\omega^*)^2 M A_h^2 / 3\mu_2 F)}, \tag{17}$$

and the frequency of vibration is

$$\Omega_h^* = \Omega_h \sqrt{1 + \frac{b^{2*} A_h^2}{3a}} = \sqrt{\frac{1}{2A_h \mu_1} \frac{F}{M} + \frac{A_h}{6} \frac{(\omega^*)^2}{\mu_1 \mu_2}}, \tag{18}$$

and the modulus of the Jacobi elliptic function is

$$k_h^2 = \frac{\frac{(\omega^*)^2 A_h^2}{3\mu_2} \frac{M}{F}}{\left[1 + \frac{(\omega^*)^2 A_h^2}{3\mu_2} \frac{M}{F}\right]}, \tag{19}$$

where

$$A_h = \frac{3}{4} \left(\frac{\omega}{\omega^*}\right)^2 \left[\sqrt{1 + \frac{16}{3} \left(\frac{\omega^*}{\omega}\right)^2 \left(\frac{F}{M}\right) \frac{\mu_2}{\omega^2}} - 1 \right]. \tag{20}$$

Substituting Eq. (14) into Eqs. (9) and (10), the motion of the two-mass system is

$$x_h = \mu_2 A_h^* \,\mathrm{sd}^2(\Omega_h^* t, k_h^2) + \frac{Ft^2}{2M},$$
$$y_h = -\mu_1 A_h^* \,\mathrm{sd}^2(\Omega_h^* t, k_h^2) + \frac{Ft^2}{2M}. \tag{21}$$

In Fig. 2a, the frequency Ω_h^*–μ_1 (mass ratio) diagrams for various values of non-linearities ω^* and constant excitation ratio ($F/M = 10$) and in Fig. 2b for various excitation ratios F/M and constant non-linearity ($\omega^* = 1$) are plotted. It is assumed that $\omega = 1$. The frequency Ω_h^* has a tendency to decrease from infinity for $\mu_1 = 0$ to a minimal value at a certain value of mass ratio which is higher than $\mu_1 = 0.5$, and after that the frequency increases from this minimal value to infinity for $\mu_1 = 1$ by increasing of the mass ratio μ_1. The frequency of vibration depends on the excitation parameter F/M and on the coefficient of non-linearity: the higher the value of the excitation the higher the value of the frequency. The higher the value of the coefficient of non-linearity the higher the value of the frequency. The mass ratio μ_1 for the minimal value of the frequency also depends on the value of the excitation ratio F/M and coefficient of non-linearity ω^*: the smaller the value of F/M the corresponding mass ratio is closer to $\mu_1 = 0.5$. The same is true for the smaller values of the coefficient of non-linearity.

From Eq. (19), it is evident that the modulus of the Jacobi elliptic function depends on the mass ratio μ_1, on the excitation ratio F/M and the non-linearity: the higher the value of the non-linearity the modulus of the Jacobi elliptic function is higher and the smaller the excitation the smaller the value of the modulus.

In Fig. 3a, the amplitudes of vibrations of both the masses: $A_{hx} = \mu_2 A_h^*$ and $A_{hy} = -\mu_1 A_h^*$ as functions of the mass ratio μ_1 for various values of non-linearities ω^* and constant excitation ratio ($F/M = 10$) and in Fig. 3b for various excitation ratios F/M and constant non-linearity ($\omega^* = 1$) are plotted. The amplitude A_{hx} has a tendency to monotonic decrease from a maximal

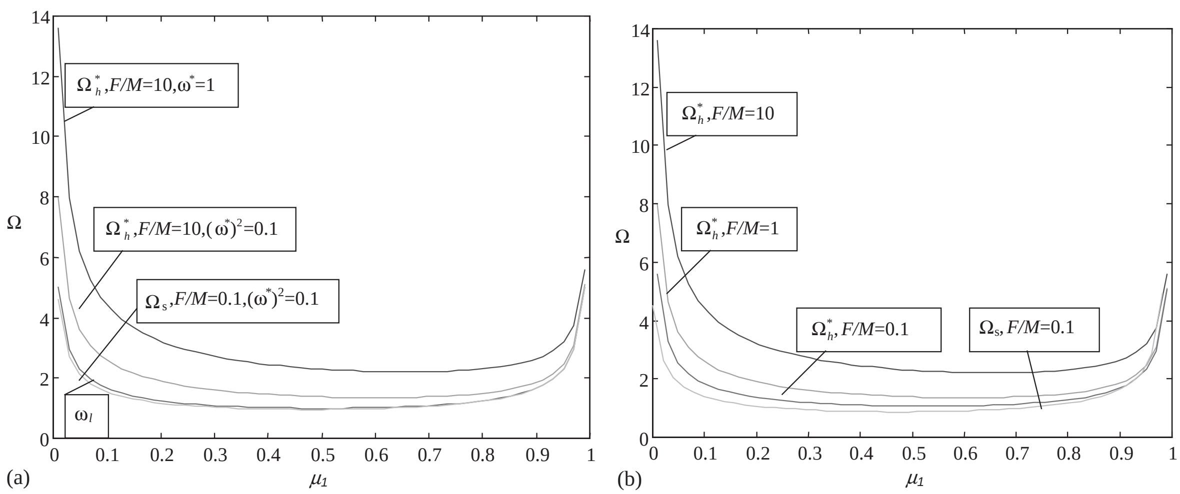
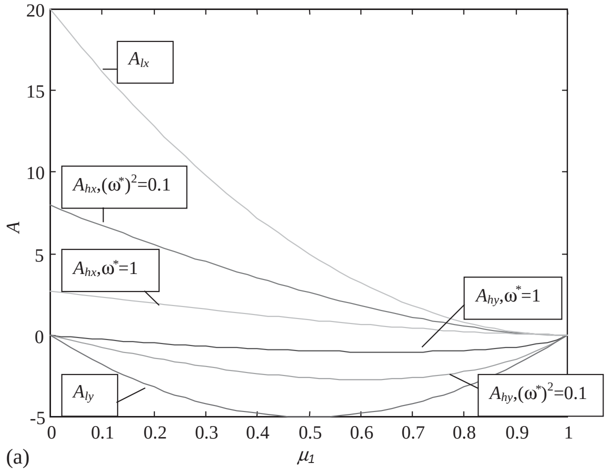


Fig. 2. Ω–μ_1 diagrams for (a) linear, hard and soft springs; (b) $\omega^* = 1$ and various values of F/M.



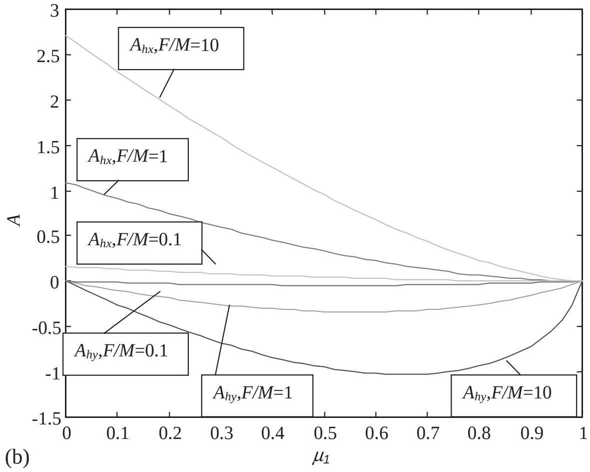


Fig. 3. A–μ_1 diagrams for the (a) linear and hard springs when $F/M = 10$; (b) hard spring when $\omega^* = 1$ and F/M has various values.

value $(A_{hx})_{max}$ for $\mu_1 = 0$ to zero for $\mu_1 = 1$. The amplitude A_{hy} is zero for $\mu_1 = 0$ and $\mu_1 = 1$ and has a minimal value for μ_1 in the interval $0.5 < \mu_1 < 1$. The amplitudes of both masses depend on the excitation ratio and coefficient of non-linearity, too: the amplitudes are higher for higher values of the excitation ratio and also for the higher values of the coefficient of non-linearity.

3.2. System with soft spring

For the mechanical system with soft spring, the lower sign in Eq. (12) is regular and its complete solution for the initial conditions (13) is according to [8]

$$X = A_s \,\mathrm{sn}^2\left(\sqrt{\frac{a}{2A_s}}\,t, \frac{b^{2*}A_s^2}{3a}\right), \tag{22}$$

where

$$A_s = \frac{3}{4}\frac{c^*}{b^{2*}}\left[1 - \sqrt{1 - \frac{16}{3}\frac{ab^{2*}}{(c^*)^2}}\right]. \tag{23}$$

The solution has the form of the Jacobi elliptic function sn [9]. The solution is of the oscillatory type. Analyzing relation (23) it is evident that there is a limitation for the oscillatory motion of the system with soft spring. Namely, the motion is real if the following relation is satisfied:

$$\frac{16}{3}\frac{ab^{2*}}{(c^*)^2} \leqslant 1. \tag{24}$$

Introducing relations (11) into Eq. (22), the frequency of vibration is

$$\Omega_s = \sqrt{\frac{F}{2M\mu_1 A_s}}, \tag{25}$$

the modulus of the Jacobi elliptic function

$$k_s^2 = \frac{(\omega^*)^2 M A_s^2}{3\mu_2 F}, \tag{26}$$

and the amplitude of vibration

$$A_s = \frac{3}{4}\left(\frac{\omega}{\omega^*}\right)^2 \left[1 - \sqrt{1 - \frac{16}{3}\left(\frac{\omega^*}{\omega}\right)^2 \frac{F}{M}\frac{\mu_2}{\omega^2}}\right]. \tag{27}$$

Substituting Eq. (22) into Eqs. (9) and (10) the motion of the two-mass system is

$$x_s = \mu_2 A_s \,\mathrm{sn}^2(\Omega_s t, k_s^2) + \frac{Ft^2}{2M},$$
$$y_s = -\mu_1 A_s \,\mathrm{sn}^2(\Omega_s t, k_s^2) + \frac{Ft^2}{2M}. \tag{28}$$

The frequency Ω_s–μ_1 (mass ratio) diagrams are plotted in Fig. 2a and b for the excitation ratio $F/M = 0.1$ and the non-linearity $(\omega^*)^2 = 0.1$ and $\omega^* = 1$, respectively. In Fig. 4a, the amplitudes of vibrations of the both bodies: $A_{sx} = \mu_2 A_s$ and $A_{sy} = -\mu_1 A_s$ as functions of the mass ratio μ_1 for various values of non-linearities ω^* and constant excitation ratio ($F/M = 0.1$) and in Fig. 4b for various excitation ratios F/M and constant non-linearity ($\omega^* = 1$) are plotted. It is assumed that $\omega = 1$. The discussion given for the system with hard spring are the same as for the soft spring, as the forms of the curves are the same for the both types of non-linearity. The only difference is that the minimal value of A_{sy} and Ω_s is for μ_1 which is in the interval $0<\mu_1<0.5$. For smaller values of the excitation force and the coefficient of non-linearity the minimal value is for μ_1 closer to $\mu_1 = 0.5$.

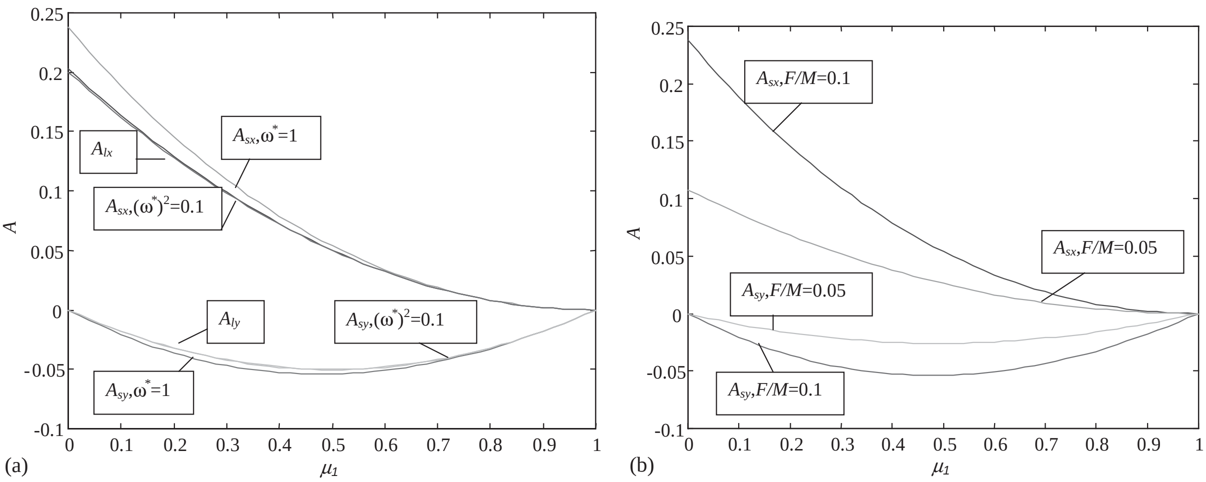


Fig. 4. A–μ_1 diagrams for the (a) linear and soft springs when $F/M = 0.1$; (b) soft spring when $\omega^* = 1$ and F/M has various values.

For the linear case when $b^2 = 0$ the modulus of Jacobi elliptic function (26) is zero and according to Ref. [10], solution (22) transforms to

$$X = \frac{2a}{c^*} \sin^2\left(\frac{\sqrt{c^*}}{2} t\right). \tag{29}$$

Substituting Eq. (29) into Eqs. (9) and (10), applying relations (11) it is

$$x = A_{lx} \sin^2(\omega_l t) + \frac{Ft^2}{2M}, \quad y = A_{ly} \sin^2(\omega_l t) + \frac{Ft^2}{2M}, \tag{30}$$

where the amplitudes of the linear system are

$$A_{lx} = 2\frac{F}{M}\frac{(1-\mu_1)^2}{\omega^2}, \quad A_{ly} = -2\frac{F}{M}\frac{\mu_1(1-\mu_1)}{\omega^2}, \tag{31}$$

and $\omega_l = 0.5\omega/\sqrt{\mu_1(1-\mu_1)}$ is the frequency of vibration. In Figs. 2a and 3a, the ω_l–μ_1, A_{lx}–μ_1 and A_{ly}–μ_1 curves are plotted. The frequency of vibration depends on the mass ratio and does not depend on the excitation force. Analyzing the curves it is evident that the amplitudes of vibrations depend on the mass ratio μ_1 and on the value of excitation ratio F/M.

4. Comparison of the results and the conclusions

Now we compare the solutions obtained for the system with a linear spring, a non-linear hard spring and a non-linear soft spring. It can be concluded:

1. For the two-mass system connected with a linear or non-linear spring with quadratic non-linear properties on which a constant excitation force acts the motion of the masses is a simple superposition of a non-vibrational and vibrational motion. The oscillatory motions of both the masses in the system have the same periods of vibration and both the masses have a simultaneous oscillatory motion.
2. The frequency of vibration and the period of vibration of the masses m_1 and m_2 depend on the rigidity properties of the spring, on the total mass M of the system and its distribution and also on the excitation force. For the systems with the same mass ratio but different types of springs the intensity of frequency of vibration differs: for the system with linear spring the frequency is smaller than for the system with hard spring and higher than for the system with soft spring. For all of types of springs it is common that for $\mu_1 = 0$ and $\mu_1 = 1$ the frequency of vibration is indefinitely high and the oscillatory motion disappears: the system moves translatory with constant acceleration F/M.
3. The amplitudes of vibrations of the both masses, depend on the mass distribution in the system, on the non-linear properties of the spring and also on the excitation force. For all types of springs it is evident that for $m_1 < m_2$, the absolute value of the amplitude of vibration of the mass m_1 is larger than for the mass m_2. For $m_1 = m_2$, the absolute values of the amplitudes of vibrations of the both masses are equal but in opposite directions. For $m_1 > m_2$, the absolute value of the amplitude of vibration of the mass m_1 is smaller than the amplitude of vibration of the mass m_2. For the systems with the same mass ratio but different springs the amplitude of vibrations differ. For the system with linear spring the amplitude of vibration of the masses m_1

and m_2 is higher than for the system with hard spring and smaller than for the system with soft spring.

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