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Letter to the Editor

On the definition of parametric excitation for vibration problems

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Parametric excitation is a common phenomenon in science and engineering [1–4], and its study is mathematically related to second order periodic ordinary differential equations of the following form [5]:

$$a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = 0, \tag{1}$$

where the coefficients $a(t)$, $b(t)$ and $c(t)$ vary periodically with time. For stochastic systems the coefficients are random processes with assumed correlation function and probability distribution [6–8]; in this article the examples used are largely confined to the deterministic case. For a single-degree-of-freedom (s.d.o.f.) oscillator these coefficients are the mass, stiffness and damping parameters.

This note points out that the standard definition of parametric excitation, given in most vibration texts, may have inadvertently created a little confusion since the periodic variation of all parameters of a mechanical system does not necessarily lead to parametric excitation. In particular, it is emphasized that physical concepts have to be applied to justify mathematical calculations for periodic variation of damping, since it does not lead to parametric excitation.

The application of parametric excitation to physical problem is a classical one and goes back to Faraday and Rayleigh. It is known that Faraday (ca.,1831) was the first to demonstrate the effect of parametric excitation so that it is by no means a recent idea. Rayleigh [9] was the first to study a systematic mathematical analysis and provide a theoretical basis for these observations. Recent studies in engineering mechanics have used parametric excitation to model variety of physical processes such as vortex-induced vibration [10], motion of electrically conducting structural member in magnetic field [11], air-inflated cylindrical membrane [12] inducement of chaotic motion under parametric excitation [13], etc.

For mechanical vibration problems, Eq. (1) can be simply extended to a s.d.o.f. oscillator, where the mass (m), damping (c) and stiffness (k) are assumed to be time-dependent periodic

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functions as follows:

$$m(t)\ddot{x}(t) + c(t)\dot{x}(t) + k(t)x(t) = F(t). \tag{2}$$

In general, for a mechanical system, parametric excitation refers to amplification of oscillation due to the time-dependent variation of parameters such as inertia and/or stiffness [2–4,14]. When the variation is periodic in stiffness—which is most often the case—it is widely represented by the Mathieu equation [3], for which there are zones of linear instability defined by system parameters and frequencies.

In existing vibration and structural dynamics literature the *definition* of parametric excitation appears in different forms relating it to the s.d.o.f. oscillator (Eq. (2)). Some authors prefer to define it as an excitation due to "the periodic variation of mass, stiffness and damping" [1,7]. Some others prefer to define it as one due to "periodic variation of stiffness" [14–18] and others as "periodic variation of stiffness and mass" [19]. Still, others define it, more generally, as a vibration with time periodic changes in one or more parameters of the system [2,20,21].

The extension to include periodic variation of damping—beyond stiffness and inertia—in the definition is sometimes done [1,7]. This note addresses the appropriateness of that extension from a physical point of view. It is argued that parametric excitation is only possible when an energy-storing parameter like inertia and/or stiffness (and not damping) changes periodically.

As mentioned before parametric excitation is well known in a particular form, when the stiffness varies periodically, at twice the natural frequency (over its own natural frequency). This is known as the *Mathieu equation* and is given by

$$\ddot{x} + \omega_0^2(1 + h\cos 2\omega t)x = 0. \tag{3}$$

For $h = 0$, the solution $x = x_0\cos(\omega t + \theta)$ is periodic. When $h \neq 0$, the solution of the Mathieu equation is of the following form (in mathematics this is termed the *Floquet solution*):

$$x = \mathrm{e}^{\mu t}P(t), \tag{4}$$

where $P(t)$ is a periodic function, and μ is a constant parameter. The stability of solution $x(t)$ is dictated by the sign of μ: If $\mu = 0$, then the solution is periodic, $\mu < 0$ the solution is damped and finally for $\mu > 0$ the solution is exponentially rising confined to some instability zones.

The excitation case when there is periodic variation in damping is given by

$$\ddot{x} + (\gamma + h\cos 2\omega t)\dot{x} + \omega_0^2 x = 0. \tag{5}$$

The steady solution can be obtained by the *method of harmonic balance* by assuming the solution to be $x = \mathrm{e}^{\mu t}\cos(\omega t + \theta)$, where ω is around ω_0, i.e. $\omega_0 \approx \omega$. On substituting the assumed solution in Eq. (5) and on neglecting the terms $\cos(3\omega t\theta)$ and $\sin(3\omega t\theta)$ (which can be shown to have negligible effects) the following is obtained:

$$\begin{aligned}(\omega_0^2 - \omega^2 + \mu^2 + \gamma\mu)\mathrm{e}^{\mu t}\cos(\omega t + \theta) - (2\omega\mu + \gamma\omega)\mathrm{e}^{\mu t}\sin(\omega t + \theta)\\ + \frac{\omega h}{2}\,\mathrm{e}^{\mu t}\sin(\omega t - \theta) + \frac{\mu h}{2}\,\mathrm{e}^{\mu t}\cos(\omega t - \theta) = 0\end{aligned} \tag{6}$$

or

$$[(A + D)\cos\theta - (B + C)\sin\theta]\mathrm{e}^{\mu t}\sin\omega t + [(B - C)\cos\theta - (A - D)\sin\theta]\mathrm{e}^{\mu t}\cos\omega t = 0, \tag{7}$$

where $A = \omega_0^2 - \omega^2 + \mu^2 + \gamma\mu, B = 2\omega\mu + \gamma\omega, C = \omega h/2, D = \mu h/2$.

Equating coefficients of $e^{\mu t}\sin\omega t$ and $e^{\mu t}\cos\omega t$ in Eqs. (6) and (7), the following equations are obtained:

$$[(A+D)\cos\theta-(B+C)\sin\theta]=0,\quad (B-C)\cos\theta-(A-D)\sin\theta=0. \tag{8a, b}$$

Eqs. (8a) and (8b) have solution if the determinant of their coefficients is zero, i.e.

$$(\omega_0^2-\omega^2+\mu^2+\gamma\mu)^2+(2\omega\mu+\gamma\omega)^2-(\omega h/2)^2-(\mu h/2)^2=0. \tag{9}$$

From Eq. (9), the condition for marginal stability ($\mu=0$) is obtained as

$$(\omega_0^2-\omega^2)^2+(\gamma\omega)^2-\left(\frac{\omega h}{2}\right)^2=0. \tag{10}$$

Now, the condition for exact resonance ($\omega=\omega_0$) will be considered which will suffice for the present discussion. For $\omega=\omega_0$, the above relation reduces to $h=2\gamma$ and, so for parametric instability (exponentially rising solution) the condition is $h>2\gamma$. Mathematically, a condition is at hand and a physical interpretation is needed.

The condition $h>2\gamma$ indicates that the net damping variation over a portion of the cycle must be negative (shaded area of Fig. 1). This means that the periodic variation is not necessary for linear instability but the creation of a *negative damping condition* [22,23] is a must. However, such a negative damping condition can be created by constant energy input proportional to velocity as follows (b is positive):

$$\ddot{x}(t)+\gamma\dot{x}(t)+\omega_0^2x(t)=-b\dot{x}(t). \tag{11}$$

When $b>\gamma$, an exponentially instability will occur and the right-hand side represents a non-periodic source of energy.

A physical problem that exemplifies Eq. (11) is the single-degree torsional flutter, where energy input occurs from non-periodic flow through the *aerodynamic function b* [24,25] resulting in

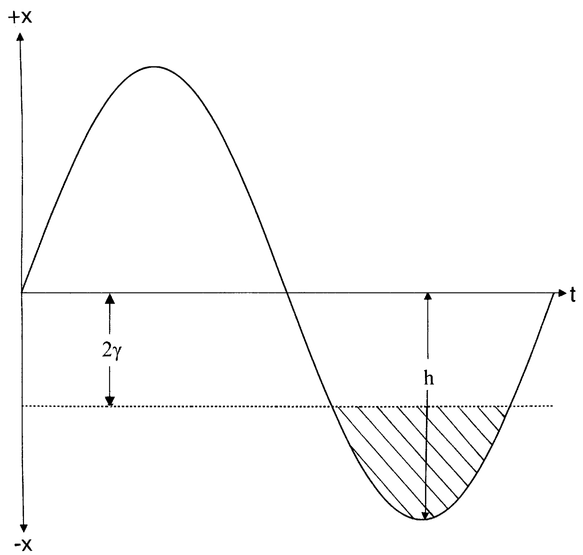

Fig. 1. Condition for parametric instability for a periodic damping variation (the shaded area is negative damping).

self-excited oscillatory instability. On the other hand, by definition the energy input in a parametric excitation system has to be periodic (or stochastic but not uniform). This distinction has to be emphasized since the physical mechanisms for the two excitations are totally different [26,27].

Now the instability condition for Eq. (5) has to be compared to parametric excitation due to periodic variation of stiffness. For this the Mathieu equation with constant damping γ is considered:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + (H\cos 2\omega t)x = 0. \tag{12}$$

At exact resonance $\omega = \omega_0$, the condition for instability can be obtained as before and is given by $H > 2\gamma$. The condition $H > 2\gamma$ does not imply that the net positive stiffness needs to be negative to produce instability. Physically, what happens in this case is that over the cycle, energy is fed by stiffness variation that negates the positive damping effect.

The distinction between the mechanism of periodic variation of stiffness and damping can also be observed on basis of energy input into the oscillator. For this consider a general s.d.o.f. oscillator whose damping and stiffness have a periodic variation:

$$\ddot{x}(t) + [\gamma + p(t)]\dot{x}(t) + [\omega_0^2 + q(t)]x(t) = 0, \tag{13}$$

where $p(t)$ and $q(t)$ are periodic functions.

On multiplying Eq. (13) by $\dot{x}(t)$,

$$\ddot{x}(t)\dot{x}(t) + [\gamma + p(t)]\dot{x}(t)\dot{x}(t) + [\omega_0^2 + q(t)]x(t)\dot{x}(t) = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\dot{x}^2\} + \omega_0^2\frac{\mathrm{d}}{\mathrm{d}t}\{x^2\} + [\gamma + p(t)]\dot{x}(t)\dot{x}(t) + q(t)x(t)\dot{x}(t) = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\{E\} = -[\gamma + p(t)]\dot{x}^2(t) - q(t)x(t)\dot{x}(t),$$

where $E = x^2 + \omega_0^2 x^2$ is the energy of the oscillator. On integrating the above equation over one cycle,

$$\bar{E} = \int_0^T \mathrm{d}E = -\int_0^T \{[\gamma + p(t)]\dot{x}^2(t) + q(t)x(t)\dot{x}(t)\}\,\mathrm{d}t,$$

where $\bar{E}$ is the average energy input into the oscillator over one cycle and when $\bar{E}$ is greater than zero, parametric instability is initiated. Thus the instability conditions in the two cases are as follows:

1. For the stiffness variation only ($p(t) = 0$),

$$\bar{E} = -\int_0^T \{\gamma\dot{x}^2 + q(t)x(t)\dot{x}(t)\}\,\mathrm{d}t > 0,$$

and there is no direct relation between γ and $q(t)$ that can make $\bar{E}$ positive.

2. For damping variation only ($q(t) = 0$),

$$\bar{E} = -\int_0^T \{[\gamma + p(t)]\dot{x}^2(t)\}\,\mathrm{d}t > 0.$$

For $\bar{E}$ to be positive over a cycle the amplitude of $p(t)$ must be absolutely greater than the mechanical damping γ ($\dot{x}^2$ is always positive), i.e. a negative damping condition has to be created.

In conclusion, the periodic variation of damping can change the manner of dissipation but the system will still be dissipative. The important point is that a periodic damping variation cannot cause "negative damping" i.e. supply energy as in the periodic variation of stiffness or inertia. Finally, it is the writer's belief that from a pedagogical point of view it is important not only to exclude periodic damping variation in the definition of parametric excitation but also to explicitly mention that this condition does not lead to parametric excitation.

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