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Letter to the Editor

Transverse vibration and stability of an Euler–Bernoulli beam with step change in cross-section and in axial force

S. Naguleswaran*

Department of Mechanical Engineering, University of Canterbury, Private Bag 4800, Christchurch, New Zealand

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1. Introduction

Several publications are available on the vibration of beams with one-step change in cross-section (not carrying an axial force)—the most important being Jang and Bert [1] who expressed frequency equations for classical end supports as fourth order determinant equated to zero. Naguleswaran [2] expressed the frequency equations as second order determinants equated to zero. Several publications are briefly reviewed in Ref. [2]. Transverse vibration of uniform beams carrying a constant axial force is covered in text books e.g., Ref. [3]. Bokian [4] presented the frequencies (in graphical form) of a uniform beam under axial compressive force and discussed buckling conditions for classical boundary conditions. Bokian [5] extended the work in Ref. [4] to tensile axial loads.

The transverse vibration of one-step Euler–Bernoulli beam under axial force which changes stepwise at the step, is considered in this paper. The system parameters are the ratio of the mass per unit length and the ratio of flexural rigidity of the two portions, the step position and the axial force in the two portions. The frequency equations of 16 combinations of boundary conditions are derived and presented as fourth order determinants equated to zero. For the selected beam parameters, the first three frequency parameters are tabulated for several sets of the axial force in the two portions. From the pattern of the change in frequency parameter with change in the axial forces and from physical considerations, it was concluded that for certain combinations of the two axial forces, one of the modes was past stability.

A zero natural frequency (which initiates onset of instability or Euler buckling), is possible for certain critical combinations of the axial force—at least one of which must be compressive. Timoshenko [6] derived the transcendental equation from which the critical end force of a one-step cantilever may be obtained. The reference also considered the buckling of a simply supported one-step beam under an end force and another force at the step. Girijavallabhan [7] and Schreyer [8] presented methods to obtain lower bounds of the critical end load of one-step cantilever.

*Tel.: +64-3-364-2987x7215; fax: +64-3-364-2078.
E-mail address: s.naguleswaran@mech.canterbury.ac.nz (S. Naguleswaran).

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O'Rouke and Zebrowki [9] used a finite difference based scheme to obtain the lower bound of the critical end force of one-step cantilevers and simply supported beams. The difference between the 'exact' values from Ref. [6] and the corresponding values in Refs. [7–9] were substantial.

The vibration of clamped–free, clamped–clamped, clamped–pinned uniform beams stiffened by one or more rings and under constant conservative or follower axial force was addressed by Dube et al. [10]. Au et al. [11] used modified beam vibration functions to study the vibration and stability of beams with abrupt changes in cross-section and for example calculations/comparison chose the same beams as in Ref. [10]. In Refs. [10,11] step changes in axial force was not allowed for. Fan et al. [12] presented a kind of Gibbs–Phenomenon–Free Fourier series and demonstrated its applications to study the vibration and stability of uniform beams stiffened with rings and beams with open cracks. A beam stiffened by one ring has two-step changes in cross-section. Refs. [10–12] do not have any results on beams with one-step change in cross-section.

In the present paper, the critical axial force combinations for the 16 sets of boundary conditions are tabulated for the selected system parameters.

The theory developed is applicable to any type of step change in cross-section but in the present paper particular attention was paid to three of the types which occur commonly in engineering applications. Type 1 beam is of constant depth and with step changes in breadth, Type 2 is of constant breadth and with step changes in depth and Type 3 is with step changes in depth and breadth i.e., with similar cross-sections—for example, a beam of circular cross-section with step changes in diameter. The 'active' dimension of the three types of beam are, respectively, the breadth, depth and diameter of the beam portion.

The results may be used as bench marks to judge the accuracy of results obtained by any numerical methods.

2. Theory

Fig. 1a shows the Euler–Bernoulli beam $O_1O_0O_2$ with step change in cross-section and in axial force at O_0. The end O_1 is axially restrained and O_2 is axially free. The ends O_1 and O_2 are on classical clamped (*cl*), pinned (*pn*), sliding (*sl*) or free (*fr*) supports. The flexural rigidity, mass per unit length, the length of the portion O_1O_0 are EI_1, m_1 and L_1 and the axial force in the portion is T_1. The co-ordinate systems with origin at O_1, O_2 are in contra directions. The dynamics of each beam portion are treated separately.

2.1. The mode shape of O_1O_0

Using the sign convention in Ref. [3], for free vibration at frequency ω, if the ordinate $y_1(x_1)$ is the amplitude of vibration at abscissa x_1 $(0 \leqslant x_1 \leqslant L_1)$, then the amplitude of bending moment $M_1(x_1)$ and shearing force $Q_1(x_1)$ are

$$M_1(x_1) = EI_1 \frac{d^2 y_1(x_1)}{dx_1^2}, \quad Q_1(x_1) = -EI_1 \frac{d^3 y_1(x_1)}{dx_1^3} + T_1 \frac{dy_1(x_1)}{dx_1},$$

$$EI_1 \frac{d^4 y_1(x_1)}{dx_1^4} - T_1 \frac{d^2 y_1(x_1)}{dx_1^2} - m_1 \omega^2 y_1(x_1) = 0. \tag{1}$$

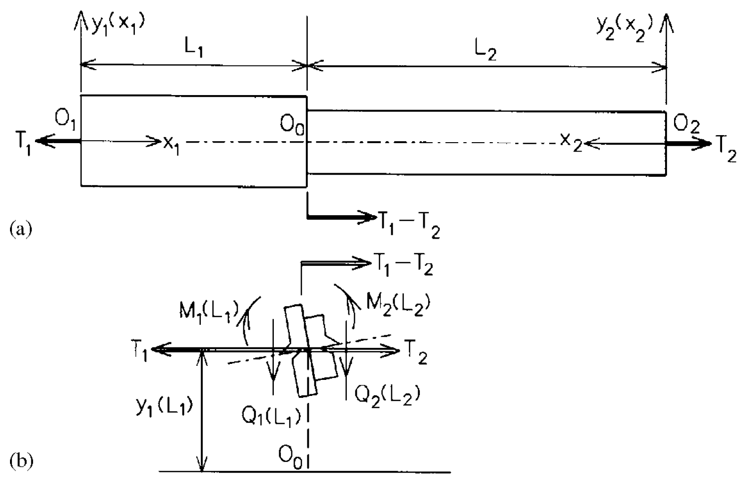

Fig. 1. The one-step beam $O_1O_0O_2$, the axial forces at O_1, O_0 and O_2, the co-ordinate systems and the forces and moments on the element at O_0. The end O_1 is axially restrained and O_2 is axially free.

To express the set of equations (1) in dimensionless form, a beam of flexural rigidity EI_R, mass per unit length m_R and length L is used as 'reference' and one defines the dimensionless abscissa X_1, amplitude $Y_1(X_1)$, step position parameter R_1, the operators D_1, D_1^n, the dimensionless bending moment $M_1(X_1)$, shearing force $Q_1(X_1)$, axial force τ_1, flexural rigidity ratio ϕ_1, mass per unit length ratio μ_1, dimensionless frequency parameters α_R & α_1 as follows:

$$X_1 = \frac{x_1}{L}, \quad Y_1(X_1) = \frac{y_1(x_1)}{L}, \quad R_1 = \frac{L_1}{L}, \quad D_1 = \frac{\mathrm{d}}{\mathrm{d}X_1}, \quad D_1^n = \frac{\mathrm{d}^n}{\mathrm{d}X_1^n},$$

$$M_1(X_1) = \frac{M_1(x_1)L}{EI_R}, \quad Q_1(X_1) = \frac{Q_1(x_1)L^2}{EI_R}, \quad \tau_1 = \frac{T_1L^2}{EI_R},$$

$$\phi_1 = \frac{EI_1}{EI_R}, \quad \mu_1 = \frac{m_1}{m_R}, \quad \alpha_R^2 = \frac{m_R\omega^4L^4}{EI_R}, \quad \alpha_1^4 = \frac{m_1\omega^2L^4}{EI_1} = \left(\frac{\mu_1}{\phi_1}\right)\alpha_R^4, \tag{2}$$

In Eqs. (2), α_R is the natural frequency parameter. The nth natural frequency parameter is denoted by $\alpha_{R,n}$. The 'active' dimension d_1 of Type 1, 2 and 3 beams are the breadth, depth and diameter, respectively, and one has

$$\text{for Type 1 beam,} \quad \mu_1 = d_1/d_R \quad \text{and} \quad \phi_1 = d_1/d_R,$$
$$\text{for Type 2 beam,} \quad \mu_1 = d_1/d_R \quad \text{and} \quad \phi_1 = (d_1/d_R)^3,$$
$$\text{for Type 3 beam,} \quad \mu_1 = (d_1/d_R)^2 \quad \text{and} \quad \phi_1 = (d_1/d_R)^4, \tag{3}$$

where d_R is the 'active' dimension of the 'reference' beam.

Eqs. (1) in dimensionless form are

$$M_1(X_1) = \phi_1 D_1^2[Y_1(X_1)], \quad Q_1(X_1) = -\phi_1 D_1^3[Y_1(X_1)] + \tau_1 D_1[Y_1(X_1)],$$
$$\phi_1 D_1^4[Y_1(X_1)] - \tau_1 D_1^2[Y_1(X_1)] - \mu_1\alpha_R^4 Y_1(X_1) = 0. \tag{4}$$

The solution of the dimensionless mode shape differential equation (4) is

$$Y_1(X_1) = C_{1,1}\sin a_1X_1 + C_{2,1}\cos a_1X_1 + C_{3,1}\sinh b_1X_1 + C_{4,1}\cosh b_1X_1, \quad (5)$$

where $C_{1,1}$ through to $C_{4,1}$ are the four constants of integration and

$$a_1^2 = \frac{\sqrt{\tau_1^2 + 4\mu_1\phi_1\alpha_R^4} - \tau_1}{2\phi_1}, \quad b_1^2 = \frac{\sqrt{\tau_1^2 + 4\mu_1\phi_1\alpha_R^4} + \tau_1}{2\phi_1}. \quad (6)$$

The need for Eq. (5) to satisfy the boundary conditions at O_1 may be used to eliminate two of the constants. The mode shape of the portion O_1O_0 may be expressed as

$$Y_1(X_1) = A_1U_1(X_1) + B_1V_1(X_1), \quad (7)$$

where A_1 and B_1 are constants and the functions $U_1(X_1)$ and $V_1(X_1)$ for *cl*, *pn*, *sl* or *fr* boundary conditions at O_1 are

$$\begin{aligned}
&cl: \quad U_1(X_1) = \sin a_1X_1 - \frac{a_1}{b_1}\sinh b_1X_1, && V_1(X_1) = \cos a_1X_1 - \cosh b_1X_1,\\
&pn: \quad U_1(X_1) = \sin a_1X_1, && V_1(X_1) = \sinh b_1X_1,\\
&sl: \quad U_1(X_1) = \cos a_1X_1, && V_1(X_1) = \cosh b_1X_1,\\
&fr: \quad U_1(X_1) = \sin a_1X_1 + \frac{\phi_1a_1^3 + \tau_1a_1}{\phi_1b_1^3 - \tau_1b_1}\sinh b_1X_1, && V_1(X_1) = \cos a_1X_1 + \frac{a_1^2}{b_1^2}\cosh b_1X_1
\end{aligned} \quad (8)$$

The derivatives of $U_1(X_1)$ and $V_1(X_1)$ are obtained easily by straightforward differentiation.

2.2. The mode shape of portion O_2O_0

The flexural rigidity, mass per unit length, the length of the portion O_2O_0 are EI_2, m_2 and L_2 and axial force in the portion of the beam is T_2. Following the same procedure outlined in previous section, the mode shape of the portion O_2O_0 may be expressed in the form

$$Y_2(X_2) = A_2U_2(X_2) + B_2V_2(X_2), \quad (9)$$

in which A_2 and B_2 are constants and the functions $U_2(X_2)$ and $V_2(X_2)$ for *cl*, *pn*, *sl* or *fr* supports at O_2 are obtained by replacing the subscript 1 with 2 in the set of equations (8) in which the various coefficients are obtained with the same subscript substitution in Eqs. (2)–(6).

3. The frequency equation

The forces and moments acting on the element at O_0 is shown in Fig. 1b. The need to satisfy continuity of deflection and of slope and compatibility of bending moment and of shearing force at O_0 (bearing in mind the contra direction of the co-ordinate axes at O_1 and at O_2 will result in the following equations in dimensionless form:

$$\begin{aligned}
&Y_1(R_1) = Y_2(R_2), \quad D_1[Y_1(R_1)] = -D_2[Y_2(R_2)], \quad \phi_1D_1^2[Y_1(R_1)] = \phi_2D_2^2[Y_2(R_2)],\\
&\phi_1D_1^3[Y_1(R_1)] - \tau_1D_1[Y_1(R_1)] = -\phi_2D_2^3[Y_2(R_2)] + \tau_2D_2[Y_2(R_2)].
\end{aligned} \quad (10)$$

When Eqs. (7) and (9) are substituted into Eq. (10), for non-trivial solution, the coefficient matrix must be singular and one gets the frequency equation;

$$\begin{vmatrix} U_1(R_1) & V_1(R_1) & -U_2(R_2) & -V_2(R_2) \\ D_1[U_1(R_1)] & D_1[V_1(R_1)] & D_2[U_2(R_2)] & D_2[V_2(R_2)] \\ \phi_1 D_1^2[U_1(R_1)] & \phi_1 D_1^2[V_1(R_1)] & -\phi_2 D_2^2[U_2(R_{22})] & -\phi_2 D_2^2[V_2(R_2)] \\ \phi_1 D_1^3[U_1(R_1)] & \phi_1 D_1^3[V_1(R_1)] & \phi_2 D_2^3[U_2(R_2)] & \phi_2 D_2^3[V_2(R_2)] \\ -\tau_1 D_1[U_1(R_1)] & -\tau_1 D_1[V_1(R_1)] & -\tau_2 D_2[U_2(R_2)] & -\tau_2 D_2[V_2(R_2)] \end{vmatrix} = 0. \quad (11)$$

3.1. Natural frequency calculations

In this paper the 'reference' beam in the set of equations (2) was chosen with $EI_R = EI_1$ i.e., $\phi_1 = 1$ and $m_R = m_1$ i.e., $\mu_1 = 1$ and natural frequency parameters were expressed (without loss of generality) via the frequency parameter $\alpha_R = \alpha_1$. Without loss of generality, one may choose

$$R_1 + R_2 = 1. \quad (12)$$

The system parameters are μ_2, ϕ_2, R_1, τ_1 and τ_2. The roots of the frequency equation (11) were determined by a 'search' to bracket an approximate range within which a root is present followed by an iterative procedure based on linear interpolation. The procedure is as follows: $U_1(X_1)$ and $V_1(X_1)$ was chosen from Eq. (8) taking account of the boundary conditions at O_1. A trial frequency parameter ($\alpha_R = 0.1$ say) was assumed and $U_1(R_1)$, $V_1(R_1)$, $D_1[U_1(R_1)]$, $D_1[V_1(R_1)]$, etc. were calculated. For the selected set of system parameters one proceeded to calculate the elements of the first and second columns of the determinant of Eq. (11). Similarly taking account of the type of support at O_2, the elements of the third and fourth columns of the determinant were calculated for the same α_R. The value of the determinant of the frequency equation (11) was calculated by inductive development [13]. The value of α_R was increased in steps of 0.1 and the calculations described were repeated till a sign change in the determinant occurred. The sign change indicated the presence of a root within this range. A 'search' was made within this range but with change of 0.01 in α_R to narrow the range within which the root lies. At this stage an iterative procedure based on linear interpolation was invoked to calculate the root to the pre-set accuracy. The 'search' procedure was continued (from the value of the first root) to locate the second root and so on.

In the following example calculations the parameters of the one-step beams are: 'active' dimension $d_R = 1.0$, $d_1 = d_R$, $d_2 = 0.80\, d_R$, beam portion length parameters: $R_1 = 0.375$ ($R_2 = 0.625$). Naguleswaran [2] tabulated the first three frequency parameters of the example Type 1, 2 and 3 beams *without* axial force for 16 combinations of classical boundary conditions. Note that in absence of axial force, rigid-body rotation is possible for *pn**fr*, *fr**pn* and *fr**fr* beams. In Tables 1–4 in the present paper, the frequency parameters of the example beams are tabulated for various combinations of τ_1 and τ_2.

The beam considered in Table 1 is a one-step tie-bar under constant axial tension $\tau = 10.0$ i.e., $\tau_1 = 10.0$, $\tau_2 = 10.0$. The axial tension 'stiffens' the system and the frequency parameters are greater than the corresponding frequency parameter in Ref. [2] bearing in mind that under axial tension, rigid-body rotation is not possible for *pn**fr*, *fr**pn* and *fr**fr* beams.

Table 1
The first three non-zero frequency parameters of the three types of one-step beams

BC $O_1 \backslash O_2$	Type 1			Type 2			Type 3		
	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl\cl	5.0696	8.1124	11.2094	4.8082	7.6667	10.5270	4.9010	7.7046	10.5970
cl\pn	4.4443	7.3996	10.4627	4.2791	7.0102	9.8636	4.4149	7.0649	9.9370
cl\sl	2.9255	5.9095	8.9225	2.8799	5.6161	8.4450	3.0196	5.7043	8.5013
cl\fr	2.8234	5.4484	8.2878	2.8163	5.2631	7.8933	2.9681	5.4079	7.9802
pn\cl	4.3648	7.3913	10.4579	4.1477	7.0183	9.7915	4.1886	7.0737	9.8599
pn\pn	3.8193	6.6927	9.7236	3.6867	6.3862	9.1464	3.7746	6.4485	9.2319
pn\sl	2.4547	5.2222	8.2020	2.4193	4.9988	7.7806	2.5289	5.0582	7.8623
pn\fr	2.4001	4.8392	7.5966	2.3890	4.7102	7.2746	2.5064	4.8207	7.3863
sl\cl	2.7820	5.8881	8.9116	2.6910	5.5739	8.3598	2.6815	5.6600	8.3897
sl\pn	2.3875	5.2409	8.1941	2.3716	5.0240	7.6990	2.4008	5.1333	7.7527
sl\sl	3.8026	6.7054	9.7107	3.7179	6.3331	9.1355	3.8010	6.4238	9.1714
sl\fr	3.5909	6.1890	9.0578	3.5766	5.9314	8.5540	3.6894	6.0718	8.6244
fr\cl	2.6599	5.3460	8.2510	2.5857	5.0550	7.7890	2.5781	5.1070	7.8158
fr\pn	2.3165	4.7818	7.5557	2.3099	4.5814	7.1412	2.3378	4.6646	7.1801
fr\sl	3.5323	6.1273	9.0410	3.4645	5.7964	8.5567	3.5373	5.8536	8.6057
fr\fr	3.3709	5.6829	8.4157	3.3630	5.4527	8.0058	3.4604	5.5608	8.0823

Beam parameters: $d_1 = d_R = 1.0$, $d_2 = 0.8\, d_R$, $R_1 = 0.375$, $R_2 = 1 - R_1$. Axial forces: $\tau_1 = 10.0$, $\tau_2 = 10.0$.

Table 2
Same as Table 1 but axial forces: $\tau_1 = 10.0$, $\tau_2 = 0.0$

BC $O_1 \backslash O_2$	Type 1			Type 2			Type 3		
	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl\cl	4.9058	7.9094	11.0683	4.5922	7.3977	10.3334	4.6506	7.3834	10.3603
cl\pn	4.1416	7.1341	10.2784	3.8982	6.6612	9.6142	3.9790	6.6496	9.6317
cl\sl	2.6118	5.6257	8.6941	2.5035	5.2485	8.1400	2.5865	5.2784	8.1313
cl\fr	2.1347	4.8642	7.9079	2.0683	4.5498	7.3952	2.1425	4.6085	7.3788
pn\cl	4.1823	7.1806	10.3183	3.9124	6.7395	9.6007	3.9185	6.7416	9.6287
pn\pn	3.4982	6.3940	9.5405	3.2883	5.9978	8.8998	3.3271	5.9846	8.9326
pn\sl	2.2082	4.8829	7.9677	2.1313	4.5660	7.4692	2.2081	4.5540	7.4851
pn\fr	1.8330	4.1653	7.1751	1.8034	3.8972	6.7319	1.8824	3.9099	6.7303
sl\cl	2.4123	5.7112	8.7410	2.2514	5.3358	8.1290	2.1976	5.3798	8.1110
sl\pn	1.6552	4.9271	7.9800	1.5611	4.6220	7.4101	1.5354	4.6626	7.4058
sl\sl	3.3215	6.4683	9.5095	3.1412	6.0177	8.8631	3.1547	6.0492	8.8392
sl\fr	2.5872	5.6978	8.7401	2.4755	5.3171	8.1280	2.5035	5.3559	8.1093
fr\cl	2.2785	5.1894	8.0655	2.1378	4.8416	7.5402	2.0908	4.8566	7.5144
fr\pn	1.5527	4.4936	7.3189	1.4827	4.2118	6.8238	1.4653	4.2362	6.7955
fr\sl	3.0633	5.8860	8.8252	2.9122	5.4741	8.2668	2.9292	5.4701	8.2507
fr\fr	2.3735	5.1896	8.0621	2.3014	4.8352	7.5363	2.3394	4.8465	7.5088

Table 3
Same as Table 1 but axial forces: $\tau_1 = 10.0$, $\tau_2 = -5.0$

BC $O_1\backslash O_2$	Type 1			Type 2			Type 3		
	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl\cl	4.8136	7.8002	10.9943	4.4638	7.2490	10.2297	4.4959	7.2019	10.2315
cl\pn	3.9441	6.9890	10.1805	3.6216	6.4634	9.4785	3.6373	6.4080	9.4624
cl\sl	2.3400	5.4649	8.5710	2.1064	5.0284	7.9711	2.0570	5.0143	7.9217
cl\fr	*4.4395*	*7.6935*	*10.9418*	*3.9573*	*7.1000*	*10.1606*	*3.8879*	*7.0096*	*10.1441*
pn\cl	4.0786	7.0647	10.2452	3.7705	6.5805	9.4984	3.7490	6.5470	9.5026
pn\pn	3.2792	6.2245	9.4428	2.9797	5.7664	8.7643	2.9485	5.6976	8.7645
pn\sl	1.9601	4.6825	7.8387	1.7613	4.2922	7.2913	1.7082	4.2201	7.2634
pn\fr	*3.6239*	*6.9252*	*10.1858*	*3.1445*	*6.3879*	*9.4201*	*3.0211*	*6.2916*	*9.4056*
sl\cl	2.1247	5.6101	8.6517	1.8493	5.1921	8.0051	1.6980	5.2042	7.9589
sl\pn	*4.7291*	*7.8663*	*10.9785*	*4.3447*	*7.2517*	*10.2260*	*4.3161*	*7.2116*	*10.1844*
sl\sl	2.9369	6.3350	9.4039	2.5944	5.8309	8.7171	2.4557	5.8190	8.6582
sl\fr	*5.3690*	*8.5683*	*11.7398*	*4.8560*	*7.8889*	*10.9307*	*4.7687*	*7.8132*	*10.8931*
fr\cl	1.9604	5.0995	7.9671	1.6896	4.7114	7.4048	1.5304	4.6977	7.3470
fr\pn	*4.3059*	*7.1923*	*10.2801*	*3.9454*	*6.6481*	*9.5894*	*3.9038*	*6.5773*	*9.5725*
fr\sl	2.6534	5.7513	8.7101	2.3235	5.2843	8.1082	2.1712	5.2357	8.0522
fr\fr	*4.8529*	*7.8666*	*11.0294*	*4.3599*	*7.2647*	*10.2634*	*4.2444*	*7.1653*	*10.2491*

cl\fr, pn\fr, sl\pn, sl\fr, fr\pn, fr\fr—first mode of Type 1, 2 and 3 beams unstable.

Table 4
Same as Table 1 but axial forces: $\tau_1 = 0.0$, $\tau_2 = -5.0$

BC $O_1\backslash O_2$	Type 1			Type 2			Type 3		
	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$
cl\cl	4.6637	7.7441	10.9243	4.2897	7.1934	10.1629	4.2971	7.1448	10.1632
cl\pn	3.7643	6.9216	10.1155	3.4152	6.3876	9.4201	3.4039	6.3237	9.4047
cl\sl	2.1380	5.3464	8.5160	1.8981	4.8844	7.9212	1.8298	4.8450	7.8728
cl\fr	*4.2670*	*7.6347*	*10.8713*	*3.7343*	*7.0385*	*10.0937*	*3.6212*	*6.9432*	*10.0758*
pn\cl	3.7823	6.9572	10.1372	3.4153	6.4800	9.3818	3.3323	6.4504	9.3779
pn\pn	2.8422	6.1074	9.3304	2.3995	5.6498	8.6481	2.2069	5.5799	8.6436
pn\sl	0.5227	4.4795	*7.7297*	*4.0511*	*7.1879*	*10.0783*	*3.9389*	*7.1624*	*10.0581*
pn\fr	*3.2276*	*6.8130*	*10.0758*	*2.5255*	*6.2788*	*9.3011*	*2.2188*	*6.1827*	*9.2782*
sl\cl	1.9914	5.3996	8.5409	1.7137	4.9626	7.8822	1.5523	4.9561	7.8252
sl\pn	*4.5114*	*7.7279*	*10.9040*	*4.1146*	*7.0930*	*10.1547*	*4.0724*	*7.0338*	*10.1151*
sl\sl	2.7196	6.1424	9.3115	2.3738	5.6118	8.6184	2.2169	5.5753	8.5542
sl\fr	*5.1475*	*8.4518*	*11.6658*	*4.6041*	*7.7559*	*10.8611*	*4.4892*	*7.6645*	*10.8259*
fr\cl	0.5762	4.5447	7.7456	*4.0639*	*7.1776*	*10.1750*	*3.9658*	*7.1174*	*10.1805*
fr\pn	*3.6383*	*6.9119*	*10.1226*	*3.1275*	*6.3457*	*9.4302*	*2.9491*	*6.2542*	*9.4186*
fr\sl	0.7465	5.3014	8.5203	*4.7606*	*7.9171*	*10.8720*	*4.6413*	*7.8644*	*10.8765*
fr\fr	*4.2436*	*7.6310*	*10.8780*	*3.5611*	*7.0134*	*10.1060*	*3.2564*	*6.9004*	*10.0934*

cl\fr, pn\fr, sl\pn, sl\fr, fr\pn, fr\fr—first mode of Type 1, 2 and 3 beams unstable. pn\sl, fr\cl, fr\sl—first mode of Type 2 and 3 beams unstable.

The beam considered in Table 2 had the first portion under axial tension $\tau_1 = 10.0$ while the second portion was not under an axial force i.e., $\tau_2 = 0.0$. The frequency parameters are less than the corresponding frequency parameters in Table 1 but greater that those in Ref. [2].

The beam in Table 3 had the first portion under tension $\tau_1 = 10.0$ while the second portion was under compression $\tau_2 = -5.0$. The frequency parameters here are less than the corresponding frequency parameters in Table 2. From the pattern of frequency parameter change in Tables 1 and 2 it is concluded that *cl**fr*, *pn**fr*, *sl**pn*, *sl**fr*, *fr**pn* and *fr**fr* beams have buckled in the first mode. Recalculation with axial forces $\tau_1 = 10.0$, $\tau_2 = -2.0$ showed that the beams were stable for all the 16 combinations of boundary conditions.

In Table 4, the first portion was not under an axial force i.e., $\tau_1 = 0.0$, while the second portion of the beam was under compression $\tau_2 = -5.0$. The frequency parameters here are less than the corresponding frequency parameters in Table 3. Type 1, 2 and 3 *cl**fr*, *pn**fr*, *sl**pn*, *sl**fr*, *fr**pn* and *fr**fr* beams have buckled in the first mode under this axial force. In addition Type 2 and 3 *pn**sl*, *fr**cl* and *fr**sl* beams have buckled in the first mode.

4. Euler buckling

Evidence from Tables 1–4 and physical considerations suggest that decrease in τ_1 and/or τ_2 will result in a decrease in the frequency parameters. For some combinations of τ_1 and τ_2 a frequency

Table 5
The first two critical axial force $\tau_{c,1}$ and $\tau_{c,2}$ of one-step beam under constant axial compressive force

BC $O_1\backslash O_2$	Type 1		Type 2		Type 3	
	$\tau_{c,1}$	$\tau_{c,2}$	$\tau_{c,1}$	$\tau_{c,2}$	$\tau_{c,1}$	$\tau_{c,2}$
$R_1 = 0.375$						
*cl**cl*	−33.6543	−70.327	−24.3771	−50.801	−20.8214	−42.353
*cl**pn*	−17.1799	−51.201	−12.5579	−36.383	−10.7975	−30.410
*cl**sl*	−8.8109	−33.664	−6.9989	−24.499	−6.1887	−21.053
*cl**fr*	−2.2763	−19.191	−1.8546	−14.576	−1.6367	−12.787
*pn**cl*	−17.5818	−52.424	−12.6994	−40.109	−10.5859	−34.505
*pn**pn*	−8.3176	−34.816	−5.7238	−26.085	−4.6941	−22.082
*pn**sl*	−2.0060	−19.327	−1.3135	−13.960	−1.0593	−11.637
*pn**fr*	−8.3176	−34.816	−5.7238	−26.085	−4.6941	−22.082
$R_1 = 0.5$						
*cl**cl*	−34.9876	−72.442	−26.2391	−58.282	−22.3984	−51.395
*cl**pn*	−17.4319	−53.857	−12.8019	−42.021	−10.9603	−36.109
*cl**sl*	−8.8549	−34.988	−7.2430	−26.247	−6.5820	−22.432
*cl**fr*	−2.3584	−19.350	−2.0839	−14.674	−1.9198	−12.863
*pn**cl*	−18.5199	−52.644	−14.8888	−41.401	−12.9934	−37.089
*pn**pn*	−8.7461	−35.416	−6.5197	−28.760	−5.5066	−25.754
*pn**sl*	−2.0472	−20.255	−1.3811	−16.221	−1.1257	−14.154
*pn**fr*	−8.7461	−35.416	−6.5197	−28.760	−5.5066	−25.754

Beam parameters: $d_1 = d_R = 1.0$, $d_2 = 0.8\, d_R$, R_1 as stated.

Table 6
The first two critical axial force $\tau_{1c,1}$ and $\tau_{1c,2}$ in first portion of one-step beam

BC $O_1\backslash O_2$	Type 1		Type 2		Type 3	
	$\tau_{1c,1}$	$\tau_{1c,2}$	$\tau_{1c,1}$	$\tau_{1c,2}$	$\tau_{1c,1}$	$\tau_{1c,2}$
$\tau_2 = 0.0$						
cl\cl	−68.5273	−178.964	−54.2496	−170.872	−48.1686	−168.098
cl\pn	−47.6525	−174.422	−38.6971	−168.348	−35.0444	−166.206
cl\sl	−23.7528	−164.648	−21.6548	−162.247	−20.8732	−161.387
cl\fr	−17.5460	−157.914	−17.5460	−157.914	−17.5460	−157.914
pn\cl	−29.2003	−103.260	−22.8799	−90.950	−19.7757	−86.483
pn\pn	−16.6273	−91.817	−12.4257	−84.043	−10.5347	−81.220
pn\sl	−2.9310	−76.813	−1.9781	−74.476	−1.6134	−73.631
pn\fr	*−70.1839*	*−280.735*	*−70.1839*	*−280.735*	*−70.1839*	*−280.735*
$\tau_2 = 2.0$						
cl\cl	−70.2119	−179.750	−56.1811	−171.685	−50.2004	−168.914
cl\pn	−50.0799	−175.519	−41.4060	−169.445	−37.8561	−167.295
cl\sl	−25.4269	−166.663	−23.3936	−164.227	−22.6194	−163.334
cl\fr	−22.2976	−162.968	−21.8191	−162.431	−21.5527	−162.134
pn\cl	−30.3975	−104.287	−24.3273	−92.000	−21.3448	−87.518
pn\pn	−18.3312	−93.228	−14.4118	−85.462	−12.6410	−82.619
pn\sl	−3.6647	−78.750	−2.7705	−76.406	−2.4209	−75.538
pn\fr	−2.2741	−75.182	−2.0541	−74.656	−1.9308	−74.365

$\tau_2 = 0$ *pn\fr*—rigid-body rotation possible. τ_2 as shown in table. Beam parameters: as in Table 1.

parameter may be zero. This is Euler buckling or onset of instability. A necessary but not sufficient condition for buckling is one of τ_1 and τ_2 or both must be compressive. Further decrease in τ_1 and/or τ_2 will render the mode unstable. In what follows, some critical combinations of the axial forces are considered.

Consider the buckling of the example one-step beam under critical axial force $\tau_c = \tau_1 = \tau_2$. For the selected set of beam parameters, to calculate τ_c one writes $\alpha_R = 0$ in the frequency equation (11). The 'search and linear interpolation' routine used for frequency parameter calculation was used to calculate the critical axial force τ_c for $R_1 = 0.375$ and for $R_1 = 0.50$. The first two critical forces $\tau_{c,1}$ and $\tau_{c,2}$ are tabulated in Table 5 for *cl\cl* through to *pn\fr* boundary conditions only. It was found that τ_c of *sl\cl*, *cl\sl* and *sl\sl* beams were the same, τ_c of *sl\pn*, *cl\n* and *sl\fr* beams were the same, τ_c of *fr\cl*, *pn\sl* and *fr\sl* beams were identical and τ_c of *fr\pn*, *pn\fr* and *fr\fr* beams were identical. Timoshenko [6, p. 113] derived the formula to calculate the critical compressive force of a one-step cantilever. The values obtained from the formula and the values listed for τ_c of *cl\fr* beam were identical. Timoshenko [6, p. 98] also provided the expression for buckling of a simply supported one-step beam under constant compressive axial force. The τ_c of *pn\pn* beam in Table 5 and the values from the equation in Ref. [6] and were found to be identical.

In the next axial force combinations considered, the first portion was under critical axial compressive force τ_{1c} while the axial force in second portion was constant. The first two critical axial force $\tau_{1c,1}$ and $\tau_{1c,2}$ tabulated in Table 6 are for $\tau_2 = 0.0$ and for $\tau_2 = 2.0$. Note that for

Table 7
The first two critical axial force $\tau_{2c,1}$ and $\tau_{2c,2}$ in second portion of one-step beam

BC $O_1\backslash O_2$	Type 1		Type 2		Type 3	
	$\tau_{2c,1}$	$\tau_{2c,2}$	$\tau_{2c,1}$	$\tau_{2c,2}$	$\tau_{2c,1}$	$\tau_{2c,2}$
$\tau_1 = 0.0$						
cl\cl	−51.2009	−92.022	−34.7967	−62.831	−28.8331	−51.682
cl\pn	−22.1247	−73.048	−15.4717	−49.277	−13.0019	−40.549
cl\sl	−10.8867	−53.272	−8.1190	−36.348	−6.9637	−30.143
cl\fr	−2.4292	−27.422	−1.9313	−19.370	−1.6875	−16.300
pn\cl	−38.0548	−69.727	−25.0537	−49.048	−20.2154	−40.609
pn\pn	−14.4104	−52.920	−9.4089	−36.373	−7.5781	−29.786
pn\sl	−5.0532	−45.479	−3.2341	−29.107	−2.5873	−23.285
pn\fr	*−20.2130*	*−80.852*	*−12.9363*	*−51.745*	*−10.3490*	*−41.396*
$\tau_1 = 2.0$						
cl\cl	−51.8625	−92.795	−35.3483	−63.437	−29.3243	−52.242
cl\pn	−22.5650	−73.822	−15.8266	−49.912	−13.3122	−41.133
cl\sl	−11.2308	−53.893	−8.3510	−36.859	−7.1454	−30.586
cl\fr	−2.5465	−27.941	−2.0028	−19.762	−1.7410	−16.624
pn\cl	−39.8965	−70.515	−26.6213	−49.655	−21.6967	−41.138
pn\pn	−15.5866	−54.319	−10.4628	−37.467	−8.5882	−30.779
pn\sl	−7.0297	−47.643	−5.1005	−31.245	−4.3829	−25.405
pn\fr	−0.9277	−22.339	−0.8503	−15.013	−0.8029	−12.390

$\tau_1 = 0$ *pn\fr*—rigid-body rotation possible. τ_1 as shown in table. Beam parameters: as in Table 1.

$\tau_2 = 0.0$, the critical axial force τ_{1c} of *cl\fr* beams are same for Type 1, 2 and 3 beams. This is to be expected because for *fr* condition of the second portion, buckling will be independent of the dimensions of the second portion and from Ref. [6], $\tau_{1c} = [(2n-1)\pi/2R_1)]^2$ where $n = 1, 2\ldots$. For $\tau_2 = 0.0$ for the same reason the critical axial force of *pn\fr* beams are the same and from Ref. [4] $\tau_{1c} = [n\pi/R_1]^2$, where $n = 0, 1, 2, \ldots$. Note that the first buckling mode of *pn-fr* beam is rigid-body rotation and technically the critical τ_{1c} of *pn\fr* beam (for $\tau_2 = 0.0$) in Table 6 need to be shifted one cell to the right. Recall that τ_{1c} of *cl\fr* and *fr\fr* beams are the same and τ_{1c} of *sl\fr* and *pn\fr* beams are the same. For $\tau_2 = 2.0$ rigid-body rotation is not possible.

In the next case considered, the axial force in first portion was constant while the axial force in second portion was critical τ_{2c}. The first two critical forces $\tau_{2c,1}$ and $\tau_{2c,2}$ are tabulated in Table 7 for $\tau_1 = 0.0$ and for $\tau_1 = 2.0$. For $\tau_1 = 0$, if the first portion is *fr* then, τ_{2c} of *fr\cl*, *fr\pn*, *fr\sl* and *fr\fr* beams should be independent of the dimensions of the first portion. Note that (for $\tau_1 = 0$), τ_{2c} of *fr\cl* (which is the same as τ_{2c} of *pn\sl*) is the same for Type 1, 2 and 3 beams provided the τ_{2c} is normalized relative to the second portion of the beam.

5. Concluding remarks

The frequency equations of Euler–Bernoulli one-step beam under different axial force in the two beam portions and 16 combinations of classical boundary conditions are expressed as fourth order determinants equated to zero. The system parameters are the ratio of the mass per unit length and the ratio of flexural rigidity of the two portions, the step position and the axial force in the two portions. For an example set of beam parameters, the first three frequency parameters are tabulated for various combinations of axial forces in the two portions.

Euler buckling occurs for certain combinations of the axial forces for which a frequency parameter is zero. The first two critical combinations are tabulated for the example set of beam parameters.

The tables may be used to judge frequencies and buckling axial force combinations obtained by numerical methods like Rayleigh–Ritz, finite element, finite difference, etc. Although results are presented for the three types of beams, the method developed is applicable for any type of step changes in cross-section.

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