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Improvement on the iterated IRS method for structural eigensolutions

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Abstract

Iterated improved reduced system (IIRS) technique is a model reduction (or condensation) method by repeatedly updating a transformation matrix. An improvement on this dynamic condensation technique is proposed in this paper to modify the iterative transformation matrix and achieve faster convergence. Meanwhile, connection between the present algorithm and the subspace iteration method (SIM) is demonstrated. A proof of the convergence property is also presented. Applications of the method to two numerical examples have demonstrated that the proposed method can obtain the lowest eigensolutions of structures more accurately and efficiently, as compared with the current IIRS.

1. Introduction

In the structural analysis with the finite element (FE) method, a very large number of degrees of freedom (d.o.f.s) (several hundreds or thousands) are usually required to describe the structure accurately. In this situation, it is often necessary to reduce d.o.f.s for a variety of engineering and mechanical problems [1]. For example, although many eigensolution algorithms exist, model reduction method (or condensation, economization) is still an efficient technique to give fast computation of some lowest natural frequencies and corresponding mode shapes of large structures [2–7]. In recent years, it has also been used in the experimental modal analysis and related fields [8,9] since the number of measured points in experiments is much less than that of d.o.f.s in the FE analysis and thus it is necessary to reduce the complete system matrices to the size of the experimental model or expand the measured mode shapes to the full size of the FE model.

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The strategy of model reduction in solving eigenproblem is to remove some d.o.f.s (called slaves) of the original FE model and retain a much smaller set of d.o.f.s (called masters), then to solve the eigenfunction of the reduced model and approximate the eigensolutions of the original model. The pivot task of various reduction methods is to estimate the transformation matrix between the mode shapes corresponding to the masters and those to the slaves. Guyan [2] and Irons [3] firstly proposed the static condensation technique nearly 40 years ago, which neglects the inertia terms of the slavery d.o.f.s. Later some dynamic approaches were proposed to increase the accuracy of the condensation method. For example, Paz [4] studied Guyan's method with a shifted eigenvalue; O'Callahan [5] proposed the improved reduced system (IRS) by adding an extra term in the transformation matrix of Guyan's method.

Recently some iterative dynamic schemes have been developed which update a transformation matrix repeatedly until the eigenpairs meet the required precision. In particular, Friswell et al. [6] proposed an iterated IRS (IIRS) technique and the convergence was proved later [10]. Unfortunately, the convergence speed of this method cannot be comparable to that of the subspace iteration method (SIM), a commonly used algorithm in the structural community. Even some improvements have been made to increase the convergence [7], the convergence property has not been proved so far. Dependence on the master selection also prevents this kind of method from becoming a more popular eigensolver in engineering.

In this paper, an improvement on the IIRS is presented by modifying the iterative formula of the transformation matrix. We will demonstrate that this modification is equivalent to the standard SIM. The convergence proof is also given, in a similar way to that of Friswell et al. [10]. The effectiveness of this improvement is demonstrated by applications to two numerical examples.

2. Iterated improved reduced system (IIRS) method

The generalized eigenvalue problem of a system with N d.o.f.s is described as follows, in the block form, according to the chosen master d.o.f.s (retained) and slavery d.o.f.s (removed),

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{bmatrix} \Lambda_{mm}, \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the $N \times N$ symmetric stiffness and mass matrices; $\mathbf{\Phi}$ consists of the mass-normalized eigenvectors and Λ is a diagonal matrix containing corresponding eigenvalues, λ_i $(i = 1, 2, \ldots, m)$, on the diagonal. Only the first m modes are included in the above equation. The subscripts "m" and "s" represent the master and slave d.o.f.s, respectively, and the superscript "T" denotes the transpose of the matrix. The sizes of the master and slave d.o.f.s are assumed to be m and s with $m + s = N$. Without loss of generality, the eigenvalues are arranged in ascending order, i.e., $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$. From the second set of the above equation,

$$\mathbf{K}_{ms}^{\mathrm{T}} \mathbf{\Phi}_{mm} + \mathbf{K}_{ss} \mathbf{\Phi}_{sm} = \mathbf{M}_{ms}^{\mathrm{T}} \mathbf{\Phi}_{mm} \Lambda_{mm} + \mathbf{M}_{ss} \mathbf{\Phi}_{sm} \Lambda_{mm}. \tag{2}$$

$\mathbf{\Phi}_{sm}$ hence can be expressed as

$$\mathbf{\Phi}_{sm} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{ms}^{\mathrm{T}} \mathbf{\Phi}_{mm} + \mathbf{K}_{ss}^{-1} \left(\mathbf{M}_{ms}^{\mathrm{T}} \mathbf{\Phi}_{mm} \Lambda_{mm} + \mathbf{M}_{ss} \mathbf{\Phi}_{sm} \Lambda_{mm} \right). \tag{3}$$

Let

$$\mathbf{\Phi}_{sm} = \mathbf{t}\mathbf{\Phi}_{mm}, \tag{4}$$

where $\mathbf{t}$ is the transformation matrix between $\mathbf{\Phi}_{mm}$ and $\mathbf{\Phi}_{sm}$, which takes the form of

$$\mathbf{t} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{\Phi}_{mm}\mathbf{\Lambda}_{mm}\mathbf{\Phi}_{mm}^{-1}, \tag{5}$$

where $\mathbf{t}_G = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}$ and subscript "G" represents the item of Guyan technique thereafter. Then the transformation between the master d.o.f.s and the complete set of d.o.f.s becomes

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{mm} \\ \mathbf{t} \end{bmatrix} \mathbf{\Phi}_{mm} = \mathbf{T}\mathbf{\Phi}_{mm}, \tag{6}$$

where $\mathbf{I}_{mm}$ is the unit matrix of size $m \times m$.

Substituting Eq. (6) into Eq. (1) and pre-multiplying $\mathbf{T}^{\mathrm{T}}$, one can obtain a reduced eigenvalue problem of order m:

$$\mathbf{K}_R\mathbf{\Phi}_{mm} = \mathbf{M}_R\mathbf{\Phi}_{mm}\mathbf{\Lambda}_{mm}, \tag{7}$$

where $\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}$ and $\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}$ are the reduced stiffness and mass matrices. It shows that the eigenvalues of the reduced system are exactly the lowest m eigenvalues of the initial system and the associated eigenvectors are the lowest m eigenvectors of the initial system in the corresponding master d.o.f.s. The complete eigenvectors of the initial structure can be recovered by Eq. (6). From Eq. (7), one has

$$\mathbf{\Phi}_{mm}\mathbf{\Lambda}_{mm}\mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_R^{-1}\mathbf{K}_R. \tag{8}$$

Substituting Eq. (8) into Eq. (5), we can get

$$\mathbf{t} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{M}_R^{-1}\mathbf{K}_R. \tag{9}$$

Since $\mathbf{t}$ in Eq. (9) is implicit and cannot be directly solved, Friswell et al. [9] proposed the IIRS technique in which the iterative scheme is

$$\mathbf{t}^{(k)} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k-1)}\right)\left[\mathbf{M}_R^{(k-1)}\right]^{-1}\mathbf{K}_R^{(k-1)}, \tag{10a}$$

$$\mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{I}_{mm} \\ \mathbf{t}^{(k)} \end{bmatrix}, \tag{10b}$$

$$\mathbf{K}_R^{(k)} = \left[\mathbf{T}^{(k)}\right]^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(k)}, \tag{11a}$$

$$\mathbf{M}_R^{(k)} = \left[\mathbf{T}^{(k)}\right]^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k)}, \tag{11b}$$

where the superscript k denotes the kth ($k \geqslant 2$) iteration. When $k = 1$, $\mathbf{t}^{(1)} = \mathbf{t}_G$, which is right Guyan technique; and when $k = 2$, it is equivalent to the standard IRS method. The lowest m eigenvalues and the associated eigenvectors after kth iteration are estimated by solving the generalized eigenproblem of the reduced system $\left(\mathbf{K}_R^{(k)}, \mathbf{M}_R^{(k)}\right)$:

$$\mathbf{K}_R^{(k)}\mathbf{\Phi}_m^{(k)} = \mathbf{M}_R^{(k)}\mathbf{\Phi}_m^{(k)}\mathbf{\Lambda}_m^{(k)}. \tag{12}$$

3. The present improvement of IIRS

In this part, we present a modification of the current IIRS and demonstrate its connection with the standard SIM, one of the most powerful and popular technique in the structural community.

We rewrite Eq. (5) as

$$\mathbf{t} = \mathbf{t}_G + \mathbf{t}_d, \tag{13a}$$

$$\mathbf{t}_d = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm}\mathbf{\Phi}_{mm}^{-1} \tag{13b}$$

and

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{mm} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{mm} \\ \mathbf{t}_G + \mathbf{t}_d \end{bmatrix} = \mathbf{T}_G + \begin{bmatrix} \mathbf{0} \\ \mathbf{t}_d \end{bmatrix}. \tag{14}$$

Substituting Eq. (14) into $\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}$, one can get the reduced stiffness matrix

$$\begin{aligned}
\mathbf{K}_R &= \left(\mathbf{T}_G + \begin{bmatrix} \mathbf{0} \\ \mathbf{t}_d \end{bmatrix}\right)^{\mathrm{T}} \mathbf{K} \left(\mathbf{T}_G + \begin{bmatrix} \mathbf{0} \\ \mathbf{t}_d \end{bmatrix}\right) \\
&= \mathbf{T}_G^{\mathrm{T}}\mathbf{K}\mathbf{T}_G + \begin{bmatrix} \mathbf{0} & \mathbf{t}_d^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t}_G \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathbf{I}_m & \mathbf{t}_G^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{t}_d \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{t}_d^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{t}_d \end{bmatrix} \\
&= \mathbf{K}_G + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G\right) + \left(\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss}\right)\mathbf{t}_d + \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d.
\end{aligned} \tag{15}$$

Noting $\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G = \mathbf{K}_{ms}^{\mathrm{T}} - \mathbf{K}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}} = \mathbf{0}$ and similarly $\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss} = \mathbf{0}$, the above equation becomes

$$\mathbf{K}_R = \mathbf{K}_G + \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d. \tag{16}$$

Similarly we can obtain the reduced mass matrix as

$$\mathbf{M}_R = \mathbf{M}_G + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d + \mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d. \tag{17}$$

Therefore, Eq. (7) can be rewritten as

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}_R\mathbf{\Phi}_{mm} - \mathbf{M}_R\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm} = \left(\mathbf{K}_G + \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\right)\mathbf{\Phi}_{mm} \\
&\quad - \left[\mathbf{M}_G + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d + \mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d\right]\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm}.
\end{aligned} \tag{18}$$

On the right-hand side of the above equation, $\mathbf{t}_d$ of the second term is substituted by Eq. (13b),

$$\begin{aligned}
\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\mathbf{\Phi}_{mm} &= \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\left[\mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{t}_d\right]\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm} \\
&= \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right)\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm} + \mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm}.
\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (18) and removing the identical items, one can get

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}_G\mathbf{\Phi}_{mm} - \left[\mathbf{M}_G + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d\right]\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm} \\
&= \mathbf{K}_G\mathbf{\Phi}_{mm} - \mathbf{M}_d\mathbf{\Phi}_{mm}\mathit{\Lambda}_{mm},
\end{aligned} \tag{20}$$

where

$$\mathbf{M}_d = \mathbf{M}_G + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d = \mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}_G + \mathbf{T}_G^{\mathrm{T}}\mathbf{M}\begin{bmatrix}0\\ \mathbf{t}_d\end{bmatrix} = \mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}. \tag{21}$$

Therefore, from Eq. (20), we can get

$$\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_d^{-1}\mathbf{K}_G \tag{22}$$

and substitute it into Eq. (13),

$$\mathbf{t} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{M}_d^{-1}\mathbf{K}_G. \tag{23}$$

This leads to a modification of the iterative scheme, in which the formulae are

$$\mathbf{t}^{(k)} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left[\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k-1)}\right]\left[\mathbf{M}_d^{(k-1)}\right]^{-1}\mathbf{K}_G, \tag{24a}$$

$$\mathbf{T}^{(k)} = \begin{bmatrix}\mathbf{I}_{mm}\\ \mathbf{t}^{(k)}\end{bmatrix}, \tag{24b}$$

$$\begin{aligned}\mathbf{M}_d^{(k-1)} &= \mathbf{M}_G + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d^{(k-1)} = \mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k-1)}\\ &= \left[\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right] + \mathbf{t}_G^{\mathrm{T}}\left[\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k-1)}\right].\end{aligned} \tag{25}$$

Eqs. (11) and (12) are still used to estimate the lowest m eigenvalues and the associated eigenvectors:

$$\mathbf{K}_R^{(k)} = \left[\mathbf{T}^{(k)}\right]^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(k)}, \tag{11a}$$

$$\mathbf{M}_R^{(k)} = \left[\mathbf{T}^{(k)}\right]^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k)} = \left[\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k)}\right] + \left[\mathbf{t}^{(k)}\right]^{\mathrm{T}}\left[\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k)}\right], \tag{11b}$$

$$\mathbf{K}_R^{(k)}\mathbf{\Phi}_m^{(k)} = \mathbf{M}_R^{(k)}\mathbf{\Phi}_m^{(k)}\Lambda_m^{(k)}. \tag{12}$$

The convergence of the present method will be proved later. The advantage of this modification over the IIRS will be demonstrated with two numerical examples. In the computational cost point of view, the present method spends only a little more in one iteration since Eq. (24) consumes similar computation as Eqs. (10) and (25) can be simply obtained with the interim results that in forming Eq. (11b). However, as the present method converges faster than the IIRS and needs less iterations to achieve similar accuracy, the present method takes much less computation in total than the latter, which will be demonstrated in the later examples.

Friswell et al. [11] found that the IIRS is very closely related but not exactly equivalent to the standard SIM. Here we show that the present improvement is exactly identical to the SIM, which implies its advantage over IIRS.

First we briefly introduce the standard SIM [12] for comparison. Assume $\left(\mathbf{X}^{(k-1)}, \mathbf{P}^{(k-1)}\right)$ are the estimated first m eigenpairs of the initial system after $(k-1)$th $(k \geqslant 2)$ iteration, a new subspace $\bar{\mathbf{X}}^{(k)}$ is obtained by simultaneous inverse

$$\mathbf{K}\bar{\mathbf{X}}^{(k)} = \mathbf{M}\mathbf{X}^{(k-1)}\mathbf{P}^{(k-1)}. \tag{26}$$

Then the best approximation of eigenpairs in the basis of $\bar{\mathbf{X}}^{(k)}$ is achieved through Rayleigh–Ritz analysis [12]:

$$\left[\bar{\mathbf{X}}^{(k)}\right]^{\mathrm{T}}\mathbf{K}\bar{\mathbf{X}}^{(k)}\mathbf{Q}^{(k)} = \left[\bar{\mathbf{X}}^{(k)}\right]^{\mathrm{T}}\mathbf{M}\bar{\mathbf{X}}^{(k)}\mathbf{Q}^{(k)}\mathbf{P}^{(k)}, \tag{27}$$

where $\left(\mathbf{Q}^{(k)}, \mathbf{P}^{(k)}\right)$ are the eigenpairs of the reduced system spanned by the Ritz vector $\bar{\mathbf{X}}^{(k)}$, and the new eigenvectors, after kth iteration, are

$$\mathbf{X}^{(k)} = \bar{\mathbf{X}}^{(k)}\mathbf{Q}^{(k)}. \tag{28}$$

Back to the present method, assume $\left(\mathbf{\Phi}_{mm}^{(k-1)}, \Lambda_{mm}^{(k-1)}\right)$ are the calculated eigensolutions of the reduced system after $(k-1)$th iteration, an interim matrix $\mathbf{Y}^{(k)}$ satisfies

$$\mathbf{K}_G\mathbf{Y}^{(k)} = \mathbf{M}_d^{(k-1)}\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)}. \tag{29}$$

$\mathbf{Y}^{(k)}$ is non-singular as long as $\mathbf{\Phi}_{mm}^{(k-1)}$ and $\Lambda_{mm}^{(k-1)}$ are not. Right-multiplying $\mathbf{Y}^{(k)}$ in both sides of Eq. (24a) and substituting Eq. (29) into it, we can get

$$\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}^{(k)}\right)\mathbf{Y}^{(k)} = \left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)}, \tag{30}$$

or in compact form

$$\begin{bmatrix}\mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{Y}^{(k)} \\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)} \\ \mathbf{t}^{(k-1)}\mathbf{\Phi}_{mm}^{(k-1)}\end{bmatrix}\Lambda_{mm}^{(k-1)}. \tag{31}$$

On the other hand, substituting $\mathbf{M}_d^{(k-1)}$ in Eq. (25) into Eq. (29) and noting $\mathbf{K}_G = \mathbf{T}_G^{\mathrm{T}}\mathbf{K}\mathbf{T}_G = \mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G$ gives

$$\begin{aligned}(\mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G)\mathbf{Y}^{(k)} = {} & \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)} \\ & + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)}.\end{aligned} \tag{32}$$

Combining Eq. (30), the above one is

$$\begin{aligned}(\mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G)\mathbf{Y}^{(k)} &= \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)} + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}^{(k)}\right)\mathbf{Y}^{(k)} \\ &= \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}^{(k)}\right)\mathbf{Y}^{(k)} \\ &= \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)} + \left(\mathbf{K}_{ms}\mathbf{t}_G - \mathbf{K}_{ms}\mathbf{t}^{(k)}\right)\mathbf{Y}^{(k)}.\end{aligned} \tag{33}$$

Being removed the identical items from both sides, Eq. (33) is further simplified to

$$\left(\mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}^{(k)}\right)\mathbf{Y}^{(k)} = \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{(k-1)}\right)\mathbf{\Phi}_{mm}^{(k-1)}\Lambda_{mm}^{(k-1)}, \tag{34}$$

or in compact form

$$\begin{bmatrix}\mathbf{K}_{mm} & \mathbf{K}_{ms}\end{bmatrix}\begin{bmatrix}\mathbf{Y}^{(k)} \\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{mm} & \mathbf{M}_{ms}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)} \\ \mathbf{t}^{(k-1)}\mathbf{\Phi}_{mm}^{(k-1)}\end{bmatrix}\Lambda_{mm}^{(k-1)}. \tag{35}$$

Combining Eqs. (31) and (35), one has

$$\begin{bmatrix}\mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{Y}^{(k)} \\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)} \\ \mathbf{t}^{(k-1)}\mathbf{\Phi}_{mm}^{(k-1)}\end{bmatrix}\Lambda_{mm}^{(k-1)}, \tag{36}$$

or

$$\mathbf{K}\begin{bmatrix}\mathbf{Y}^{(k)}\\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix}=\mathbf{M}\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)}\\ \mathbf{t}^{(k-1)}\mathbf{\Phi}_{mm}^{(k-1)}\end{bmatrix}\Lambda_{mm}^{(k-1)}=\mathbf{M}\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)}\\ \mathbf{\Phi}_{sm}^{(k-1)}\end{bmatrix}\Lambda_{mm}^{(k-1)}. \tag{37}$$

Comparing Eq. (37) with Eq. (26), it is clear that they are exactly similar. In fact, if we define $\begin{bmatrix}\mathbf{\Phi}_{mm}^{(k-1)}\\ \mathbf{\Phi}_{sm}^{(k-1)}\end{bmatrix}=\mathbf{X}^{(k-1)}$ and $\Lambda_{mm}^{(k-1)}=\mathbf{P}^{(k-1)}$, the subspace $\begin{bmatrix}\mathbf{Y}^{(k)}\\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix}$ of the present method is equivalent to $\bar{\mathbf{X}}^{(k)}$ of the SIM. The only difference is that in SIM, $\bar{\mathbf{X}}^{(k)}$ is used as Ritz vector to perform Rayleigh–Ritz analysis (Eq. (27)) rather than $\mathbf{T}^{(k)}=\begin{bmatrix}\mathbf{I}_{mm}\\ \mathbf{t}^{(k)}\end{bmatrix}$ is used in the reduction methods (Eqs. (11) and (12)). However, this will not cause any difference since $\begin{bmatrix}\mathbf{Y}^{(k)}\\ \mathbf{t}^{(k)}\mathbf{Y}^{(k)}\end{bmatrix}=\begin{bmatrix}\mathbf{I}_{mm}\\ \mathbf{t}^{(k)}\end{bmatrix}\mathbf{Y}^{(k)}$ spans the same subspace as $\mathbf{T}^{(k)}=\begin{bmatrix}\mathbf{I}_{mm}\\ \mathbf{t}^{(k)}\end{bmatrix}$ does, when $\mathbf{Y}^{(k)}$ is not singular. Therefore, they give the same new eigenpairs after the next (kth) iteration.

4. Convergence of the present method

In this section, we will roughly prove that the present iterative formula for $\mathbf{T}^{(k)}$ in the form of Eq. (24a) converges to the actual $\mathbf{T}$, in a similar way to that of Friswell et al. [10] in proving the convergence of IIRS that takes form of Eqs. (10).

First we rewrite Eqs. (23) and (24) as

$$\mathbf{T}=\mathbf{T}_G+\mathbf{SMTM}_d^{-1}\mathbf{K}_G, \tag{38}$$

$$\mathbf{T}^{(k)}=\mathbf{T}_G+\mathbf{SMT}^{(k-1)}\left[\mathbf{M}_d^{(k-1)}\right]^{-1}\mathbf{K}_G, \tag{39}$$

where $\mathbf{S}$ is

$$\mathbf{S}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_{ss}^{-1}\end{bmatrix}. \tag{40}$$

Define an error matrix

$$\mathbf{E}^{(k)}=\mathbf{T}^{(k)}-\mathbf{T}=\mathbf{SM}\left\{\mathbf{T}^{(k-1)}\left[\mathbf{M}_d^{(k-1)}\right]^{-1}-\mathbf{TM}_d^{-1}\right\}\mathbf{K}_G. \tag{41}$$

Similarly let $\mathbf{E}^{(k-1)}=\mathbf{T}^{(k-1)}-\mathbf{T}$, $\left[\mathbf{M}_d^{(k-1)}\right]^{-1}$ is expanded as the first order Taylor series in terms of $\mathbf{E}^{(k-1)}$,

$$\begin{aligned}\left[\mathbf{M}_d^{(k-1)}\right]^{-1}&=\left[\mathbf{T}_G^{\mathrm{T}}\mathbf{MT}^{(k-1)}\right]^{-1}=\left\{\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\left[\mathbf{T}+\mathbf{E}^{(k-1)}\right]\right\}^{-1}\\ &=\left[\mathbf{M}_d+\mathbf{T}_G^{\mathrm{T}}\mathbf{ME}^{(k-1)}\right]^{-1}=\left\{\mathbf{M}_d\left[\mathbf{I}+\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{ME}^{(k-1)}\right]\right\}^{-1}\\ &\approx\left[\mathbf{I}-\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{ME}^{(k-1)}\right]\mathbf{M}_d^{-1}=\mathbf{M}_d^{-1}-\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{ME}^{(k-1)}\mathbf{M}_d^{-1}.\end{aligned} \tag{42}$$

Then by neglecting the high order of the error matrix,

$$\mathbf{T}^{(k-1)}\left[\mathbf{M}_d^{(k-1)}\right]^{-1} = \left[\mathbf{T}+\mathbf{E}^{(k-1)}\right]\left[\mathbf{M}_d^{-1}-\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{E}^{(k-1)}\mathbf{M}_d^{-1}\right]$$
$$\approx \mathbf{T}\mathbf{M}_d^{-1}+\mathbf{E}^{(k-1)}\mathbf{M}_d^{-1}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{E}^{(k-1)}\mathbf{M}_d^{-1}. \tag{43}$$

Substituting Eq. (43) into Eq. (41),

$$\mathbf{E}^{(k)} = \mathbf{S}\mathbf{M}\left(\mathbf{I}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\right)\mathbf{E}^{(k-1)}\mathbf{M}_d^{-1}\mathbf{K}_G. \tag{44}$$

Assume $\{\mathbf{\Phi}_{mj}\}$ ($j=1,2,\ldots,m$) is the jth eigenvector corresponding to the master d.o.f.s (or the jth eigenvector of the reduced system), from Eq. (20), we have

$$\mathbf{K}_G\left\{\mathbf{\Phi}_{mj}\right\} = \lambda_j\mathbf{M}_d\left\{\mathbf{\Phi}_{mj}\right\}, \tag{45}$$

Post-multiplying $\{\mathbf{\Phi}_{mj}\}$ on both sides of Eq. (44), it has

$$\mathbf{E}^{(k)}\left\{\mathbf{\Phi}_{mj}\right\} = \mathbf{S}\mathbf{M}\left(\mathbf{I}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\right)\mathbf{E}^{(k-1)}\left\{\mathbf{\Phi}_{mj}\right\}\lambda_j. \tag{46}$$

The error vector $\mathbf{E}^{(k-1)}\left\{\mathbf{\Phi}_{mj}\right\}$ can be written as a combination of the full eigenvectors, i.e.,

$$\mathbf{E}^{(k-1)}\left\{\mathbf{\Phi}_{mj}\right\} = \mathbf{\Phi}\{\mathbf{Z}\} = \begin{bmatrix}\mathbf{\Phi}_m & \mathbf{\Phi}_s\end{bmatrix}\begin{Bmatrix}\mathbf{Z}_m\\ \mathbf{Z}_s\end{Bmatrix}, \tag{47}$$

where $\mathbf{\Phi}$ is the full eigenvector matrix of the original system, $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_s$ store the first m and left s eigenvectors, respectively; $\{\mathbf{Z}\}$ is a coefficient vector with size of $N\times 1$ containing the contribution from each eigenvector, and $\mathbf{Z}_m$, $\mathbf{Z}_s$ are the contributions from $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_s$, respectively. Substituting Eq. (47) into Eq. (46) gives

$$\mathbf{E}^{(k)}\left\{\mathbf{\Phi}_{mj}\right\} = \mathbf{S}\mathbf{M}\left(\mathbf{I}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\right)\begin{bmatrix}\mathbf{\Phi}_m & \mathbf{\Phi}_s\end{bmatrix}\begin{Bmatrix}\mathbf{Z}_m\\ \mathbf{Z}_s\end{Bmatrix}\lambda_j. \tag{48}$$

In Eq. (48), it is reminded that $\mathbf{\Phi}_m=\mathbf{T}\mathbf{\Phi}_{mm}$, and $\mathbf{M}_d=\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}$, therefore

$$\left(\mathbf{I}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\right)\mathbf{\Phi}_m = \mathbf{\Phi}_m-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}\mathbf{\Phi}_{mm} = \mathbf{\Phi}_m-\mathbf{T}\mathbf{\Phi}_{mm} = \mathbf{0}. \tag{49}$$

This implies that in the $(k-1)$th iteration, the contribution from the first m eigenvectors $\mathbf{\Phi}_m$ to the error vector $\mathbf{E}^{(k-1)}\left\{\mathbf{\Phi}_{mj}\right\}$ will disappear after one more iteration (kth iteration). Hence Eq. (48) can be simplified as

$$\mathbf{E}^{(k)}\left\{\mathbf{\Phi}_{mj}\right\} = \mathbf{S}\mathbf{M}\left(\mathbf{I}-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\right)\mathbf{\Phi}_s\{\mathbf{Z}_s\}\lambda_j = \mathbf{S}\mathbf{M}\left(\mathbf{\Phi}_s-\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_s\right)\{\mathbf{Z}_s\}\lambda_j. \tag{50}$$

Again we suppose

$$\mathbf{T}\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_s = \mathbf{\Phi}\mathbf{C} = \begin{bmatrix}\mathbf{\Phi}_m & \mathbf{\Phi}_s\end{bmatrix}\begin{bmatrix}\mathbf{C}_m\\ \mathbf{C}_s\end{bmatrix}, \tag{51}$$

$\mathbf{C}$ is a coefficient matrix of size $N\times s$, $\mathbf{C}_m$ and $\mathbf{C}_s$ are the corresponding components of $\mathbf{C}$. Noting the orthogonality of the eigenvectors, the below equation satisfies

$$\mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}\mathbf{T} = \mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}\mathbf{T}\mathbf{\Phi}_{mm}\mathbf{\Phi}_{mm}^{-1} = \mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_m\mathbf{\Phi}_{mm}^{-1} = \mathbf{0}. \tag{52}$$

Then pre-multiply $\mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}$ in both sides of Eq. (51),

$$\begin{aligned}&\left(\mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}\mathbf{T}\right)\mathbf{M}_d^{-1}\mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_s=\mathbf{0}\\&\quad=\mathbf{\Phi}_s^{\mathrm{T}}\mathbf{M}\begin{bmatrix}\mathbf{\Phi}_m & \mathbf{\Phi}_s\end{bmatrix}\begin{bmatrix}\mathbf{C}_m\\ \mathbf{C}_s\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{I}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{C}_m\\ \mathbf{C}_s\end{bmatrix}=\mathbf{C}_s.\end{aligned}\tag{53}$$

Therefore, $\mathbf{C}_s$ is a zero matrix. Combining Eqs. (50) and (51) gives

$$\begin{aligned}\mathbf{E}^{(k)}\left\{\mathbf{\Phi}_{mj}\right\}&=\mathbf{S}(\mathbf{M}\mathbf{\Phi}_s-\mathbf{M}\mathbf{\Phi}_m\mathbf{C}_m)\{\mathbf{Z}_s\}\lambda_j\\&=\mathbf{S}\left(\mathbf{K}\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}-\mathbf{K}\mathbf{\Phi}_m\mathbf{\Lambda}_{mm}^{-1}\mathbf{C}_m\right)\{\mathbf{Z}_s\}\lambda_j\\&=\mathbf{S}\mathbf{K}\left(\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}-\mathbf{\Phi}_m\mathbf{\Lambda}_{mm}^{-1}\mathbf{C}_m\right)\{\mathbf{Z}_s\}\lambda_j\\&=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{t}_s & \mathbf{I}_{ss}\end{bmatrix}\left(\lambda_j\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}-\lambda_j\mathbf{\Phi}_m\mathbf{\Lambda}_{mm}^{-1}\mathbf{C}_m\right)\{\mathbf{Z}_s\}\\&=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{t}_G & \mathbf{I}_{ss}\end{bmatrix}\lambda_j\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}\{\mathbf{Z}_s\}+\lambda_j\begin{bmatrix}\mathbf{0}\\ \mathbf{t}_G\mathbf{\Phi}_{mm}-\mathbf{\Phi}_{sm}\end{bmatrix}\mathbf{\Lambda}_m^{-1}\mathbf{C}_m\{\mathbf{Z}_s\},\end{aligned}\tag{54}$$

where $\mathbf{\Lambda}_s$ is a diagonal matrix containing the largest s eigenvalues, i.e., λ_i $(i=m+1,m+2,\ldots,N)$. The right hand side of the above equation includes two terms. In the first term, all λ_j/λ_i are less than unity and thus $\lambda_j\mathbf{\Phi}_s\mathbf{\Lambda}_s^{-1}\{\mathbf{Z}_s\}<\mathbf{\Phi}_s\{\mathbf{Z}_s\}$. The second term will be small in norm bacause the expanded modes from static reduction should be close to the modes at the slave degrees of freedom and as a result, the norm of the term $(\mathbf{t}_G\mathbf{\Phi}_{mm}-\mathbf{\Phi}_{sm})$ will be small. Therefore, the norm of $\mathbf{E}^{(k)}\left\{\mathbf{\Phi}_{mj}\right\}$ will be much smaller than the norm $\mathbf{E}^{(k-1)}\left\{\mathbf{\Phi}_{mj}\right\}$, leading to convergence of the method.

5. Numerical examples

Two structures are applied to illustrate the effectiveness and accuracy of the proposed algorithm. It was found that the selection of master d.o.f.s certainly affects the convergence speed or accuracy of the reduction methods. Some strategies have been studied in the master d.o.f.s selection for condensation [13,14], sensor placement [15,16] or damage identification [17]. However, this is not the focus of the present paper. In this paper, the master d.o.f.s are selected as those with relatively higher ratios of $\mathbf{M}_{ii}/\mathbf{K}_{ii}$ $(i=1,2,\ldots,N)$, as usual. This is similar to the way in the SIM, in which the starting vectors are chosen as unit vectors with entries $+1$ at the d.o.f.s with largest ratios of $\mathbf{M}_{ii}/\mathbf{K}_{ii}$ [12]. When many d.o.f.s have identical values, which is common in the FE analysis, we select the master d.o.f.s uniformly distributed such that their corresponding mode shapes are linear independent as possible (or, it yields $\mathbf{\Phi}_{mm}$ well-conditioned and thus $\mathbf{t}=\mathbf{\Phi}_{sm}\mathbf{\Phi}_{mm}^{-1}$ can be iteratively approached in less iterations). It should be admitted that this is not the optimal way in the convergence speed point of view, but the numerical examples will show that this strategy can achieve a fast convergence for condensation methods.

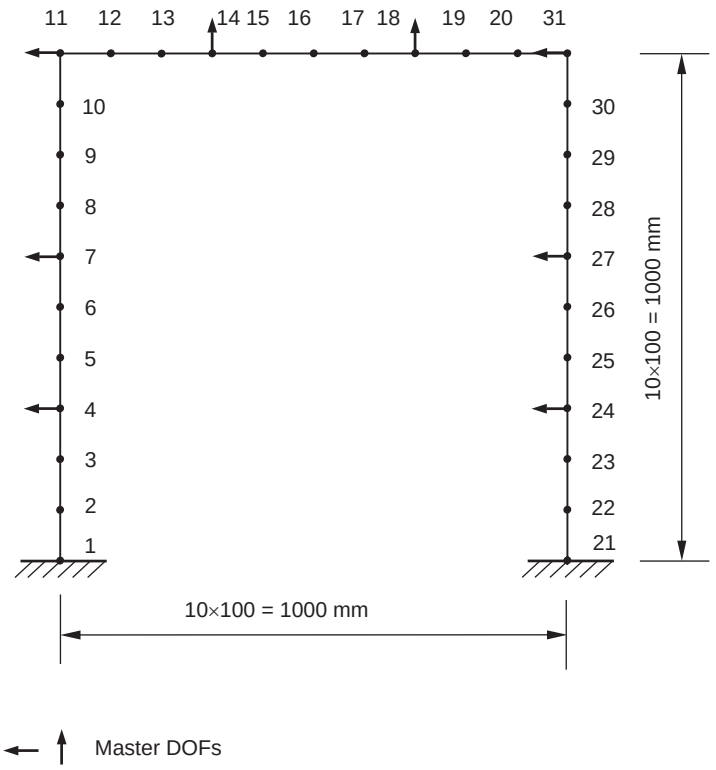

Fig. 1. Finite element model of the steel frame and the master degrees of freedom.

5.1. *A portal frame*

The first example is a one-span steel portal frame as shown in Fig. 1 [18]. The cross-section of the beam is $40.50 \times 6.0\,\text{mm}^2$, and the columns $50.50 \times 6.0\,\text{mm}^2$. The mass density is $7.67 \times 10^3\,\text{kg/m}^3$ and Young's modulus 200.0 GPa. The structure is modelled by 30 Euler–Bernoulli beam elements, as shown in Fig. 1. Each node has three d.o.f.s (horizontal, vertical displacements and rotation) and results in 87 d.o.f.s in total.

The lateral d.o.f.s of 8 points, as shown in Fig. 1 are chosen as master d.o.f.s. The full model with the order of 87 is reduced to that of 8. The first eight natural frequencies to 6-digit precision, which suffices most engineering purposes, obtained with the present and IIRS methods are listed in Tables 1 and 2, respectively. For clearness, the values upon convergence are not listed in the tables. The true values in the tables, which are treated as reference values for comparison purpose, are computed using MATLAB5.3 [19].

It is reminded that the results of the first iteration are exactly those of Guyan technique and those of the second iteration are standard IRS method, for both IIRS and present methods. The tables demonstrate that Guyan reduction does not reproduce the modal frequencies of the original system as expected. It also shows that IRS technique improves the results greatly especially for the first three modes. Using the present method, the first three frequencies have achieved 6-digit precision with one additional iteration. With a few more runs, the frequencies of modes 4–7

Table 1
The first eight natural frequencies (Hz) of the frame obtained with present technique

Iteration	Modes							
	1	2	3	4	5	6	7	8
1	4.693525	18.284524	28.997330	31.743911	66.929409	84.064750	93.537472	2055.324179
2	4.692759	18.220707	28.917160	31.465207	64.550780	76.822660	90.227493	164.111312
3		18.220703	28.917153	31.464890	64.503756	76.637334	90.085887	146.873634
4				31.464889	64.502556	76.629519	90.072785	141.868482
5					64.502525	76.629133	90.071486	139.509284
6					64.502524	76.629116	90.071356	138.404140
7						76.629115	90.071343	137.907607
8							90.071342	137.690083
9								137.595937
10								137.555410
True values	4.692759	18.220703	28.917153	31.464889	64.502524	76.629115	90.071342	137.524922

Table 2
The first eight natural frequencies (Hz) of the frame obtained with IIRS method

Iteration	Modes							
	1	2	3	4	5	6	7	8
1	4.693525	18.284524	28.997330	31.743911	66.929409	84.064750	93.537472	2055.324179
2	4.692759	18.220707	28.917160	31.465207	64.550780	76.822660	90.227493	164.111312
3		18.220703	28.917153	31.464892	64.505060	76.648132	90.091970	148.132151
4				31.464891	64.502684	76.639500	90.074388	143.102621
5				31.464891	64.502535	76.638719	90.071811	140.681521
6				31.464891	64.502525	76.638352	90.071422	139.514596
7				31.464891	64.502525	76.638025	90.071362	138.963704
8				31.464890	64.502525	76.637720	90.071353	138.699136
9				31.464890	64.502524	76.637435	90.071351	138.564220
10				31.464890		76.637167	90.071351	138.487919
True values	4.692759	18.220703	28.917153	31.464889	64.502524	76.629115	90.071342	137.524922

converge to the exact values consistently. In particular, modes 4,5,6 and 7 achieve 6-digit precision after 4,6,7 and 8 iterations in total, respectively. Therefore, the present method is very powerful in estimating the first a few modal frequencies. With the IIRS technique, the lower modes also converge fast but the higher modes do slower than the present one, as shown in Table 2. Actually modes 4,6,7 and 8 do not achieve the accuracy after 10 iterations. This example clearly demonstrates that the present method improves the convergence speed of IIRS and the computation cost is thus saved.

It is noted that the 8th natural frequency converges slowest. Detailed study finds that the ratio of the 8th natural frequency to the 9th is as high as 0.86. To get an accurate value of mode 8, some more iterations are needed. To avoid heavy computation, it is preferred to increase the number of

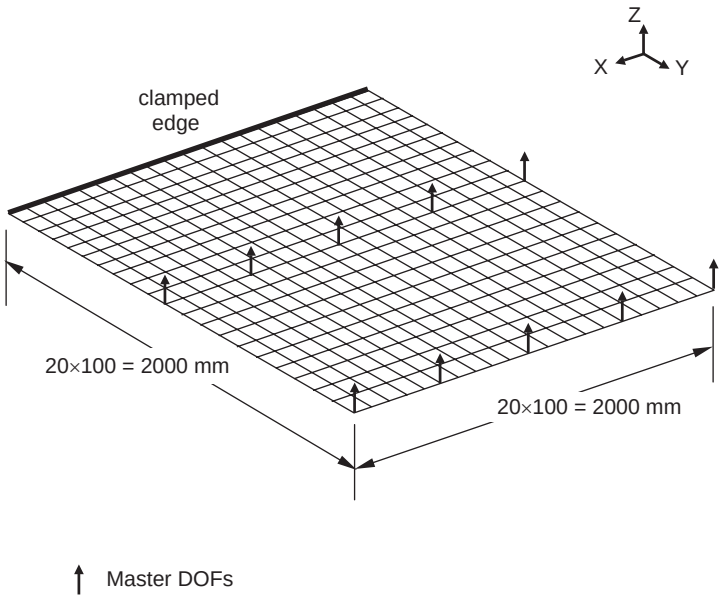

Fig. 2. Finite element model of the plate and the master degrees of freedom.

master d.o.f.s in the case rather than increase iteration number. Increasing the number of masters will normally reduce the iteration number involved (as λ_j/λ_{m+1} becomes smaller), but also increase the computational cost in each iteration. It is recommended that the size of masters is twice the frequencies (or eigenvalues) required. For instance, in this example, the first 4 natural frequencies converge very fast to the exact values with the chosen eight masters.

5.2. A cantilever plate

The second example is a cantilever plate as shown in Fig. 2. It is used here to verify the computational efficiency of present method for a relatively large structure. The nominal dimensions of the plate are $2000 \times 2000 \times 10$ mm. The Young's modulus is 206 GPa, the mass density 7.80×10^3 kg/m^3 and the Possion ratio is 0.3. The structure is discretized into $20 \times 20 = 400$ classical thin plate elements, resulting in 441 nodes and 1260 d.o.f.s in total. Each node has three d.o.f.s, namely, deflection in Z direction and rotations around X- and Y-axis. The first 5 natural frequencies and their associated eigenvectors are assumed to compute.

The master d.o.f.s are selected uniformly over the structure as shown in Fig. 2. The total number is 10, twice the number of frequencies required, to reduce iteration number. To illustrate the accuracy of the present method, 8-digit precision is used for frequencies. After 7 iterations, the present method converges and the first 5 natural frequencies are listed in Table 3. It can be seen that the present method can accurately predict the natural frequencies of the full structure with just a few iterations. With the IIRS, however, modes 3, 4 and 5 do not achieve such accuracy even after 20 iterations, which can be seen from Table 4.

Table 3
The first five natural frequencies (Hz) of the plate obtained with present method

Iteration	Modes				
	1	2	3	4	5
1	2.15180705	5.31056911	13.37957316	17.65512513	19.84993706
2	2.14781211	5.26386819	13.17819520	16.82681089	19.16743116
3	2.14781209	5.26386494	13.17743492	16.82476860	19.15604247
4			13.17742911	16.82474807	19.15565839
5			13.17742910	16.82474801	19.15564703
6					19.15564677
7					19.15564676
True values	2.14781209	5.26386494	13.17742910	16.82474801	19.15564676

Table 4
The first five natural frequencies (Hz) of the plate obtained with IIRS method

Iteration	Modes				
	1	2	3	4	5
1	2.15180705	5.31056911	13.37957316	17.65512513	19.84993706
2	2.14781211	5.26386819	13.17819520	16.82681089	19.16743116
3	2.14781209	5.26386496	13.17745076	16.82487491	19.15630419
4		5.26386494	13.17743775	16.82480114	19.15572298
5			13.17743486	16.82478341	19.15566618
6			13.17743338	16.82477448	19.15565425
7			13.17743248	16.82476902	19.15565042
10			13.17743112	16.82476070	19.15564765
15			13.17743026	16.82475536	19.15564699
20			13.17742989	16.82475304	19.15564685
True values	2.14781209	5.26386494	13.17742910	16.82474801	19.15564676

After the eigensolutions of the full model have been computed, an error estimation is performed via following criterion [12], for the ith approximated eigenpairs $(\bar{\mathbf{\Phi}}_i, \bar{\lambda}_i)$:

$$e_i = \frac{\left|\mathbf{K}\bar{\mathbf{\Phi}}_i - \bar{\lambda}_i \mathbf{M}\bar{\mathbf{\Phi}}_i\right|_2}{\left|\mathbf{K}\bar{\mathbf{\Phi}}_i\right|_2}, \tag{55}$$

where $\|\bullet\|_2$ is Euclidean norm of the vector.

The errors of IIRS and present methods, as defined in Eq. (55), are compared in Fig. 3 with respect to the iteration number used. Since the lower modes converge faster than higher ones, the figure only shows the convergence of 4th and 5th modes for clearness. It clearly shows that the present method converges much faster than IIRS does. After 10 iterations, the errors of the eigensolutions with the present method are less than 10^{-8}, but those of IIRS method are about 10^{-2}. As expected, the convergence speed of mode 4 is faster than that of mode 5, in the present method. But it is a bit abnormal that a reverse phenomenon happens for IIRS. This figure again

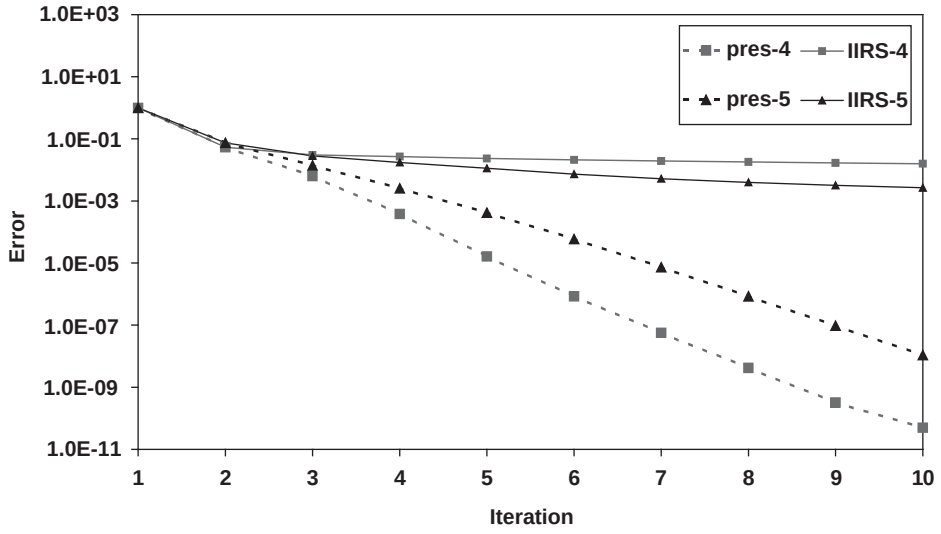

Fig. 3. Convergence of the eigensolutions of the plate with IIRS and present methods (IIRS-4: mode 4 with IIRS method, IIRS-5: mode 5 with IIRS method, pres-4: mode 4 with present method, pres-5: mode 5 with present method).

demonstrates that the present reduction algorithm can predict the eigensolutions of the structure effectively and accurately.

To compare the computational cost, the number of floating-point operations (flops) is counted in which addition, subtraction, multiplication and division count one flop for real number. After 10 iterations, the present method numbers 5.02×10^7 in total and IIRS 4.79×10^7. Apart from the computation consumed in the initialization and error estimation, the present method averagely costs 4.25×10^6 flops and IIRS 4.02×10^6 in one iteration. But the slight additional computation is worth spending since the convergence speed of the present method is improved significantly.

6. Conclusions and discussions

A new effective model reduction method has been developed for structural eigensolutions. This technique modifies the current iterated IRS technique and is found equivalent to the widely used SIM. The convergence of the method is mathematically verified.

The present algorithm has been applied to two practical examples. Numerical results have showed that the proposed technique can accurately predict the frequencies and the mode shapes of interest. As compared with the other commonly used condensation methods, such as the iterated IRS, the proposed method converges much faster to the exact values especially for higher modes and, therefore saves a great deal of computation. Due to mathematical complexity, the inherent advantage of the present method over the IIRS is not rigorously proven but instead, verified through extensive numerical simulations.

The convergence associated with the proposed method is mathematically verified. It is very difficult to prove the convergence in an elegant mathematical sense and the proof given in this

paper is not meant to be rigorous. Numerical examples show that the convergence speed of the present method is fast, when the master subset is selected in the usual way and the number of the master is twice that of the eigenvalues required. On the other hand, since the lowest eigenpairs converge faster than those higher ones, similar to other eigensolvers, some acceleration strategies such as shifting can be employed to improve computational efficiency. This will be further investigated in future.

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