Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 270 (2004) 833–845
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Energy and damping in an SDOF system subjected to a variable forcing frequency

A. Al Majid, R. Dufour*

Laboratoire de Dynamique des Machines et des Structures, UMR CNRS 5006, INSA de Lyon, Batiment Jean d' Alembert, 8, rue des Sciences, 69621 Villeurbanne, France

Received 29 July 2002; accepted 23 January 2003

Abstract

In order to model the dissipative effect due to variable forcing frequency, the authors presented in a previous article an original method based on an additional dimension and on the relativity concept. The associated metric of the space was identified experimentally, and the equation of motion was obtained from geodesic equations. In the present article the equation of motion is formulated by using the Newtonian approach. The total kinetic energy obtained has an original general expression that is reduced either to Einstein's energy in the presence of no variable forcing frequency, or to the classical expression in the case of no relativist effects, and which validates the assumptions made on the metric. An experiment carried out on a single-degree-of-freedom system is used to validate the approach.

1. Introduction

Investigation on damping is a topical subject of research, since influential parameters such as temperature, deflection, forcing frequency, and type of excitation make damping modelling delicate and complicated [1–3]. In particular it is well known that a forced transient response cannot be predicted with great accuracy if the evaluation of the damping is based only on steady state measurement [4–7].

In order to improve existing mechanical models and to account for the effect of damping in particular, the authors presented in a previous article [8] an original method based on an additional dimension that involved the experimental identification of the associated metric of the space, and the equation of motion obtained from geodesic equations, i.e., by minimizing the

*Corresponding author. Tel.: +33-472-43-82-02; fax: +33-472-43-89-30.
E-mail address: regis.dufour@insa-lyon.fr (R. Dufour).

0022-460X/$ - see front matter
doi:10.1016/S0022-460X(03)00634-5

universe (the interval separating two successive positions of the mechanical system). This method was applied to a single-degree-of-freedom (SDOF) mechanical system. The basic idea is that each mechanical system has its own time, that depends on external events.

In the present article, the equations of motion are established by using the Newtonian approach rather than by seeking the extrema of the functional universe.

Firstly, the metric of the space and the line element are presented. After classical considerations on impulse and energy in the classical frame of reference, the impulse and equation of motion are formulated in the proposed reference co-ordinate system (x^{-1}, x^0, x^1) [9–12]. Particular attention is paid to the original expression of the energy in order to confirm the metric evaluation and define its role. Finally, the transient forced response predicted with the proposed model is compared to the experimental results of Ref. [8].

2. Additional dimension for the forcing frequency

In order to predict the dynamic behaviour of an SDOF system subjected to a variable forcing frequency, let an orthogonal frame of reference be composed of three dimensions: a dimension of space, a dimension of time and an additional dimension of frequency. It is practical to use contravariant co-ordinates (x^{-1}, x^0, x^1) to define the position of a point in this space:

$$x^0 = \mathrm{i}at \quad \text{and} \quad x^1 = u, \tag{1}$$

with $\mathrm{i} = \sqrt{-1}$, a being a reference high speed, t the time of the laboratory and u the displacement. Let the first co-ordinate be an arc of a circle [13–15]:

$$x^{-1} = \mathrm{i}\rho\theta, \tag{2}$$

where the radius ρ is defined by

$$\rho = \frac{a}{\Omega}, \tag{3}$$

Ω being the natural frequency of the system.

In Eq. (2), variable θ is the angle of a circle devoted to a curvilinear axis and related by forcing pulsation ω (forcing frequency v) by the following relation:

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\pi \frac{\mathrm{d}n}{\mathrm{d}t} = 2\pi v. \tag{4}$$

According to the first Einstein axiom, see for example Ref. [16], the laws of physics are the same in all frames of reference. Consequently, any new law must be invariable even if the forcing frequency v is variable.

The metric tensor $\mathbf{g}$ of the three-dimension space is

$$\mathbf{g} = \begin{pmatrix} -q(\Xi) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

where function q which depends on $\boldsymbol{\Xi}$ defined by

$$\boldsymbol{\Xi} = \left(\int \xi \, \mathrm{d}t, \xi, \dot{\xi} \right)^{\mathrm{T}}, \tag{6}$$

expresses the effect of a non-dimensional forcing frequency $\xi = \omega/\Omega$ and of $\dot{\xi} = \mathrm{d}\xi/\mathrm{d}t$.

In order to define function q as well as possible, it is assumed that $q(\boldsymbol{\Xi}) = q(\xi)$ and that function $q(\xi)$ is nil in the three following cases:

- No forcing frequency, $\xi = 0$.
- Resonance phenomenon, $\xi = 1$.
- High forcing frequency $\xi \to \infty$.

Consequently, the metric $\mathrm{d}s$ can be expressed using the metric tensor (5) taking into account Einstein's notation for the repeated indices:

$$\mathrm{d}s^2 = -g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu, \tag{7}$$

or

$$\mathrm{d}s^2 = q(\xi)(\mathrm{d}x^{-1})^2 - (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2. \tag{8}$$

Substituting x^{-1}, x^0, x^1 by their expression (1) and (2), the line-element becomes

$$\mathrm{d}s = a \, \mathrm{d}t \sqrt{1 - \lambda^2 - \beta^2}, \tag{9}$$

where

$$\lambda = \xi \sqrt{q(\xi)} \quad \text{and} \quad \beta = \frac{\dot{u}}{a}, \tag{10}$$

$\dot{u} = \mathrm{d}u/\mathrm{d}t$ being the velocity of the system. The proper time τ of the system defined by

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{a}, \tag{11}$$

taking into account relation (9), can be related to the laboratory time t:

$$\mathrm{d}\tau = \mathrm{d}t \sqrt{1 - \lambda^2 - \beta^2}. \tag{12}$$

3. Impulse and energy in classical frame of reference

The equations of motion are derived taking into account the classical frame of reference containing the classical dimension x^1. Let Newton's second law be applied along axis x^1 to an SDOF system of mass m subjected to external force F^1:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv^1) = F^1, \tag{13}$$

or

$$\frac{dp^1}{dt} = F^1, \tag{14}$$

where $p^1 = mv^1$ is the impulse of the system. The multiplication of relation (13) by velocity $v^1 = \dfrac{dx^1}{dt}$ yields:

$$\frac{d}{dt}\left(\frac{m(v^1)^2}{2}\right) = F^1 v^1, \tag{15}$$

where the right-hand side is the power of the force while the left-hand side is the energy variation:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{m(v^1)^2}{2}\right). \tag{16}$$

Therefore the total energy of the system is defined with the arbitrary constant E_0.

$$E = \frac{m(v^1)^2}{2} + E_0. \tag{17}$$

Kinetic energy $T = m(v^1)^2/2$ depends on the movement of the system. If the system is at rest with no energy, E_0 is nil. E_0 stands for the total potential energy. In the presence of a potential field, the system is subjected to a force $\mathbf{F}$ expressed by

$$\mathbf{F} = -\text{grad } U(x^1) \tag{18}$$

with $U(x^1)$ being the potential energy which obeys the relationship:

$$d\left(\frac{mv^2}{2}\right) = -dU. \tag{19}$$

4. Impulse in the proposed frame of reference

Now let $\boldsymbol{\mathscr{P}}$ be the 3-impulse vector and $\boldsymbol{\mathscr{F}}$ be the 3-force vector in the reference co-ordinate system (x^{-1}, x^0, x^1). The 3-impulse vector is:

$$\boldsymbol{\mathscr{P}} = m\boldsymbol{\mathscr{U}}, \tag{20}$$

where $\boldsymbol{\mathscr{U}}$ is the speed vector comprising the three following components:

$$\mathscr{U}^i = \frac{dx^i}{d\tau}, \tag{21}$$

and m is the invariant mass of the system. The third equation of the vector equation of motion in (x^{-1}, x^0, x^1) follows Newton's second law and can be written as

$$\frac{d\mathscr{P}^1}{d\tau} = \mathscr{F}^1, \tag{22}$$

where $\mathscr{P}^1$ and $\mathscr{F}^1$ are the third components of the 3-impulse vector and of the 3-force vector respectively. However, it is necessary to write the equation of the motion in the 3-dimension vectorial form due to Einstein's first axiom [16]. By taking into account relation (20), Eq. (22) makes possible the generalization in the frame of reference (x^{-1}, x^0, x^1):

$$m\frac{\mathrm{d}\mathscr{U}^i}{\mathrm{d}\tau} = \mathscr{F}^i, \quad i = -1, 0, 1. \tag{23}$$

The third equation of relation (23) and relation (12) between proper time and laboratory time yields

$$m\frac{\mathrm{d}\mathscr{U}^1}{\sqrt{(1-\lambda^2-\beta^2)}\,\mathrm{d}t} = \mathscr{F}^1. \tag{24}$$

Let the third component of the 3-force vector in (x^{-1}, x^0, x^1) be

$$\mathscr{F}^1 = \frac{F^1}{\sqrt{1-\lambda^2-\beta^2}}, \tag{25}$$

where F^1, is the classical force along axis x^1. Consequently, Eq. (24) takes a form similar to the Newtonian equation. This can be verified by replacing $\mathscr{F}^1$ in Eq. (24) by its expression (25):

$$m\frac{\mathrm{d}\mathscr{U}^1}{\sqrt{(1-\lambda^2-\beta^2)}\,\mathrm{d}t} = \frac{F^1}{\sqrt{1-\lambda^2-\beta^2}}. \tag{26}$$

Simplification leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathscr{U}^1) = F^1, \tag{27}$$

which, taking into account Eqs. (12) and (21) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv^1}{\sqrt{1-\lambda^2-\beta^2}}\right) = F^1, \tag{28}$$

where the right-hand side contains the component of the classical force and in the left-hand side the quantity,

$$\mathscr{P}^1 = \frac{mv^1}{\sqrt{1-\lambda^2-\beta^2}}, \tag{29}$$

is logically called impulse, considering its first expression (20). In the absence of relativity effects, $\lambda = 0$, $\beta = 0$ and Eq. (28) is reduced to a classical Eq. (14).

5. Energy in the proposed frame of reference

The following energy considerations allow the identification of a general and original expression of the total energy of an SDOF system subjected to a variable forcing frequency, by validating the assumptions made on the metric and by defining the role of the energy.

Eq. (8) divided by $\mathrm{d}\tau^2$ and relations (11) and (21) gives the following equation:

$$a^2 = q(\xi)(\mathscr{U}^{-1})^2 - (\mathscr{U}^0)^2 - (\mathscr{U}^1)^2. \tag{30}$$

According to Eqs. (12) and (21) and taking into account the axes defined by Eqs. (1) and (2), Eq. (30) can be expressed as follows:

$$a^2 = -\left(\frac{a\lambda}{\sqrt{1-\lambda^2-\beta^2}}\right)^2 + \left(\frac{a}{\sqrt{1-\lambda^2-\beta^2}}\right)^2 - \left(\frac{v^1}{\sqrt{1-\lambda^2-\beta^2}}\right)^2 \tag{31}$$

which, after its own time derivative (or time derivative), its multiplication by constant mass m and by virtue of Eq. (27), Eq. (31) becomes

$$-m\lambda a\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda a}{\sqrt{1-\lambda^2-\beta^2}}\right) + ma\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\sqrt{1-\lambda^2-\beta^2}}\right) - v^1F^1 = 0, \tag{32}$$

where quantity $v'F^1$ is the time derivative of energy as expressed by Eq. (15). Consequently the derivative of energy E can be formulated:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = ma\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\sqrt{1-\lambda^2-\beta^2}}\right) - m\lambda a\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda a}{\sqrt{1-\lambda^2-\beta^2}}\right) \tag{33}$$

which yields

$$E = ma^2\left(\frac{1}{\sqrt{1-\beta^2-\lambda^2}} - \int\left(\lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{\sqrt{1-\beta^2-\lambda^2}}\right)\right)\mathrm{d}t\right). \tag{34}$$

During the steady state resonance phenomenon, function $q = 0$, and $\partial q/\partial\xi = 0$ and the only damping that remains is classical mechanical damping which is determined by carrying out a bandwidth measurement. It is reasonable to assume that the function q and also λ around the resonance phenomenon are very small and that this can be extended to any mode of operation. Assuming that $\lambda(t) = \varepsilon\Lambda(t)$ where ε is a small quantity, the Taylor series expansion of Eq. (34) around $\varepsilon = 0$ gives

$$E = \frac{ma^2}{\sqrt{1-\beta^2}}(1 + \psi(\beta, \lambda)) \tag{35}$$

with

$$\psi(\beta,\lambda)=\sum_{k=1}^{\infty}\left(\frac{\Gamma(1/2+k)}{\sqrt{\pi}\Gamma(k)}\left(\frac{1}{k}\left(\frac{\lambda}{\sqrt{1-\beta^2}}\right)^{2k}\right.\right.$$
$$\left.\left.-2\sqrt{1-\beta^2}\int\left(\lambda\left(\frac{\lambda}{\sqrt{1-\beta^2}}\right)^{2k-2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{\sqrt{1-\beta^2}}\right)\right)\mathrm{d}t\right)\right). \tag{36}$$

It is obvious that the function $\psi(\beta,\lambda)$ is a very small quantity, which however contributes an energy source of the dissipative effect in relation (35).

In the absence of variable forcing frequency, $\psi(\beta,0)=0$, the energy is reduced to Einstein's energy:

$$E_E=\frac{ma^2}{\sqrt{1-\beta^2}}. \tag{37}$$

Moreover in the case of no motion, relation (36) becomes:

$$\psi(0,\lambda)=0. \tag{38}$$

Therefore energy (35) is reduced to the energy at rest:

$$E_R=ma^2. \tag{39}$$

This section devoted to consideration on energy validates the assumptions made related to function q. Moreover it should be noted that in Eq. (35) an additional energy E_A expressed by

$$E_A=E-E_E=\psi(\beta,\lambda)E_E, \tag{40}$$

can become a dissipative energy when $\psi(\beta,\ \lambda)$ becomes negative.

6. Response of a low-speed SDOF system subjected to a variable forcing frequency

6.1. Equation of motion

The derivation of relation (28) gives the equation that governs the dynamic behaviour of the SDOF system:

$$\frac{m\ddot{u}}{\sqrt{1-\lambda^2-\beta^2}}+\frac{(\lambda\dot{\lambda}+\beta\dot{\beta})m\dot{u}}{(1-\lambda^2-\beta^2)^{3/2}}=F^1. \tag{41}$$

By using relations (10), the equation of motion (41) becomes

$$\frac{m\ddot{u}}{\sqrt{1-q\xi^2-\beta^2}}+\frac{\left(\left(q+\dfrac{\xi}{2}\dfrac{\partial q}{\partial \xi}\right)\xi\dot{\xi}+\beta\dot{\beta}\right)m\dot{u}}{(1-q\xi^2-\beta^2)^{3/2}}=F^1. \tag{42}$$

In the presence of forces applied on the mass such as viscous force ($-c\dot{u}$, c being the damping coefficient), restitution force ($-ku$, with k the stiffness), and excitation force $f(t)$, the resulting force applied is

$$F^1=-c\dot{u}-ku+f(t). \tag{43}$$

Assuming that the mechanical system studied is propelled by a low speed $\dot{u}$ regarding the speed of reference a, relations (10) cause β^2 to be neglected regarding the unity. Therefore, Eq. (42) can be simplified:

$$\frac{m}{\sqrt{1-q\xi^2}}\ddot{u}+\left(\frac{\xi\dot{\xi}\left(q+\dfrac{\xi}{2}\dfrac{\partial q}{\partial \xi}\right)m}{\sqrt{(1-q\xi^2)^3}}+c\right)\dot{u}+ku=f(t). \tag{44}$$

The inertia force depends in particular on ξ while the dissipative force also depends on $\dot{\xi}$, the variation of the forcing frequency. In the case where $q(\xi)$ is nil, Eq. (42) is reduced to the classical Newtonian equation of motion, the additional dimension vanishes and the space becomes flat.

In the case where $q(\xi)$ is not nil, solving Eq. (44) requires determining function $q(\xi)$. The application concerns an in-plane spring–pendulum system presented in Ref. [8]. Its parameters expressed in displacement u, correspond to

$$m=23.3\ \text{kg},\quad c=14.0\ \text{Ns/m},\quad k=528{,}840\ \text{N/m}, \tag{45}$$

that induce a 24 Hz undamped natural frequency.

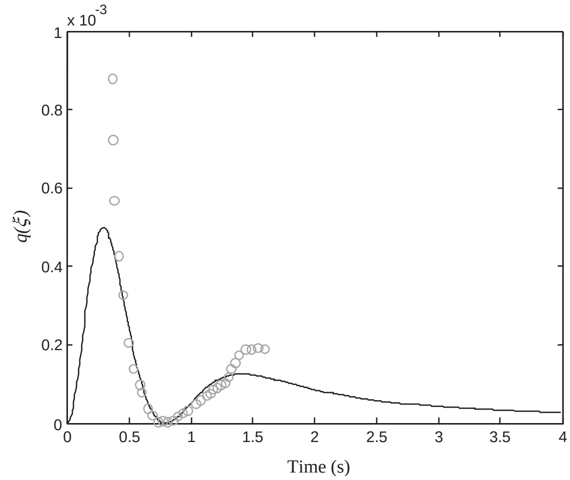

Fig. 1. Evolution of $q(\xi)$. Measured (dotted line), extrapolated (solid line).

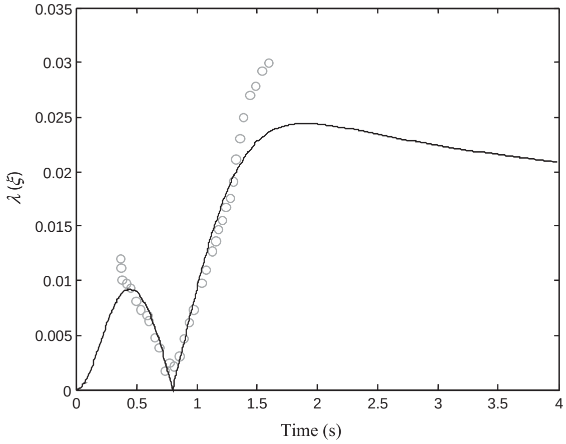

Fig. 2. Evolution of $\lambda(\xi)$. Measured (dotted line), extrapolated (solid line).

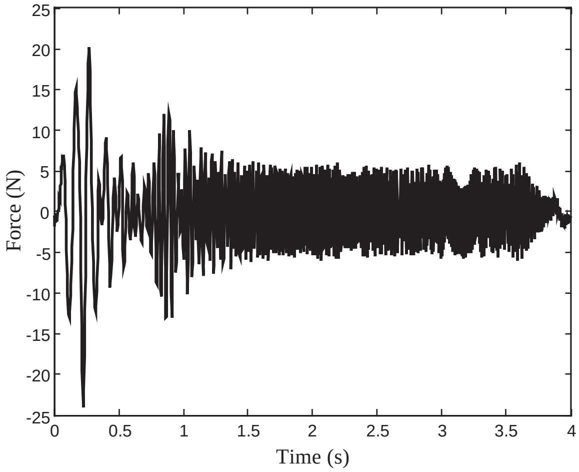

Fig. 3. Measured excitation force.

6.2. Determination of function $q(\xi)$

In Ref. [8] function $q(\xi)$ is determined by comparing the transfer function measured and predicted by the following steady state equation:

$$\frac{m}{\sqrt{1-q\xi^2}}\ddot{u} + c\dot{u} + ku = f_0 \sin(\omega_0 t), \quad (46)$$

where $f_0 \sin(\omega_0 t)$ is the constant harmonic force (with constant amplitude f_0 and constant frequency ω_0). The $q(\xi)$ function obtained by the measurement around the resonance

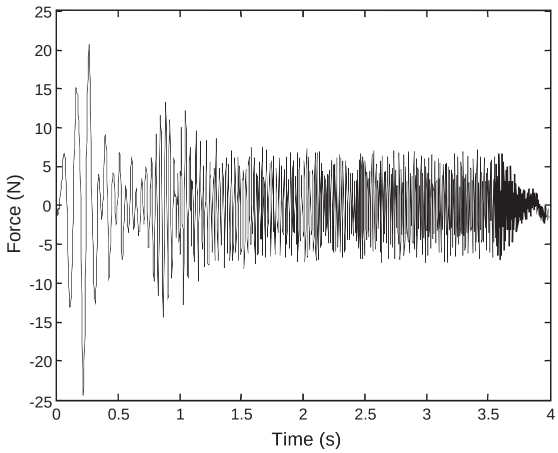

Fig. 4. Fourier expansion of the measured excitation force.

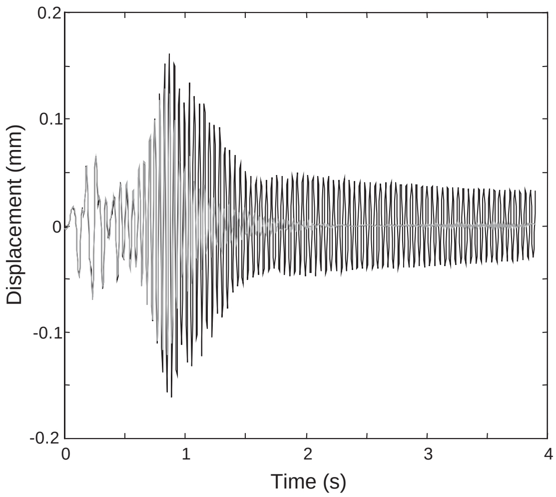

Fig. 5. Predicted with classical model (dark line) and measured (grey line) responses of the SDOF system [8].

phenomenon is extrapolated, Fig. 1, with the formula

$$q(\xi) = 0.0018\,\frac{(1 - \mathrm{e}^{-15\xi^2})(1 - \mathrm{e}^{-(\xi-1)^2})}{1 + \xi^3} \tag{47}$$

formulated by using the least square technique and the assumptions made in Section 2. Consequently it induces a change of function $\lambda(\xi)$ shown in Fig. 2. It should be noted that in the case of resonance phenomenon or free motion the additional dimension plays no role, the motion has no influence on the space and the classical approach is sufficient to model the problem.

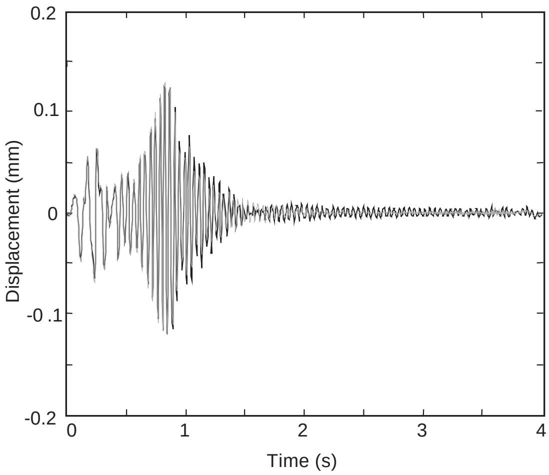

Fig. 6. Predicted with the proposed model (dark line) and measured (grey line) responses of the SDOF system.

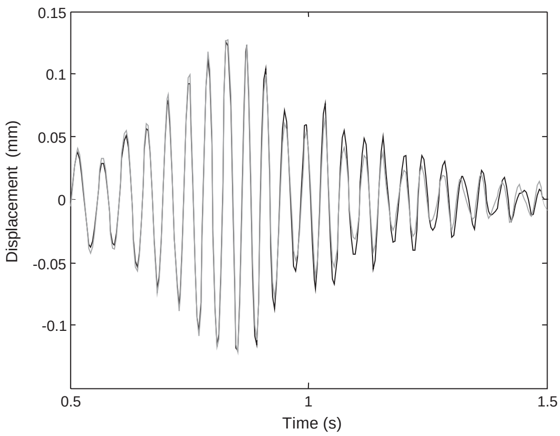

Fig. 7. Zoom of Fig. 6; key as Fig. 6.

6.3. Response to a variable forcing frequency

The SDOF system is subjected to the measured force, shown in Fig. 3, which has a variable forcing frequency estimated to be linear from 0 to 150 Hz within a 4-s time interval. The excitation force is introduced in the model by using its Fourier expansion.

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{500} \left(a_n \cos\left(\frac{2n\pi}{T}t\right) + b_n \sin\left(\frac{2n\pi}{T}t\right)\right) \tag{48}$$

with $T = 4$ s. The result is shown in Fig. 4.

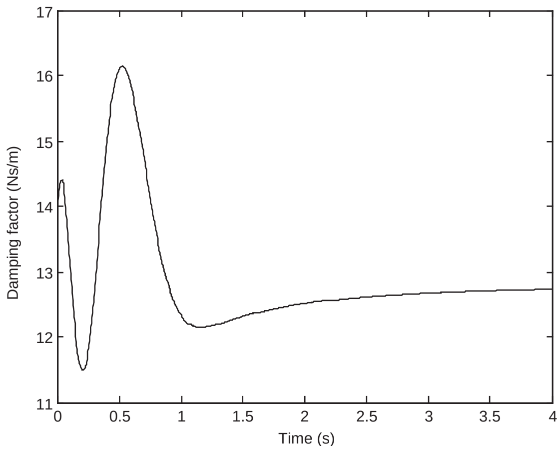

Fig. 8. Time history of the damping coefficient.

6.3.1. Classical model

The classical model corresponds to Eq. (44) with $q = 0$, for any value of ξ. The time integration is performed with the Newmark method and a 0.0007 s time step. Fig. 5 presents the predicted (classical model) and measured displacements. A considerable deviation is observed after the resonance phenomenon. A smaller time step, other integration methods and a non-linear model did not improve the predicted response. In Ref. [8] it was shown that the higher the variation of the forcing frequency, the higher the deviation is.

6.3.2. Proposed model

The proposed model corresponds to Eqs. (44) and (47). The time integration uses the Newmark method and a 0.0007 s time step. Fig. 6 presents the predicted (proposed model) and measured displacements. The zoom presented in Fig. 7 exhibits the quality of the prediction performed with the proposed model. Moreover, it is interesting to plot in Fig. 8 the time history of the damping factor to show its dependence on the change in the forcing frequency.

7. Conclusions

It has been shown that the variable forcing frequency introduces a damping effect in the dynamic behaviour of an SDOF system. The original model proposed uses the concept of relativity. A dimension corresponding to the forcing frequency is added to the classical co-ordinate frame of reference which by consequence becomes non-flat. The equation of the motion of an SDOF is formulated by analyzing the impulse, the Newtonian approach and by making assumptions on the metric of space confirmed by considerations on energy. The total kinetic energy obtained has an original general expression that is reduced either to Einstein's energy in the presence of no variable forcing frequency or to the energy at rest in the case of no relativist effects. The proposed method, contrary to the classical model, permits taking damping into account thus giving quite good prediction of the displacement of the SDOF system studied. The forced response predicted with the proposed model is validated experimentally.

It can be concluded that an external parameter that deforms the space is a source of force, which in the case of an excitation at variable frequency, corresponds to a damping. Rotordynamics is an applicative illustration of a possible relapse of the results. A rotor subjected to a high transient motion (start-up or shutdown) has a lateral deflection influenced by a damping force created by the time-varying speed of rotation.

Acknowledgements

The authors would like to thank the Région Rhône-Alpes Council for its support.

References

[1] A.D. Nashif, D.I.G. Jones, J.P. Henderson, *Vibration Damping*, Wiley, New York, 1985.
[2] M. Lalanne, Modeling damping in mechanical engineering structures, *Shock and Vibration Journal* 7 (1) (2000) 29–37.
[3] D.I.G. Jones, *Handbook of Viscoelastic Vibration Damping*, Wiley, New York, 2001.
[4] C.H. Lu, R. Mao, R.M. Evan-Iwanowski, Non-stationary parametric responses of shear deformable orthotropic plates, *Journal of Sound and Vibration* 184 (2) (1995) 329–342.
[5] B.N. Agrawal, R.M. Evan-Iwanowski, Resonances in nonstationary, nonlinear, multidegree-of-freedom systems, *American Institute of Aeronautics and Astronautics Journal* 11 (7) (1973) 907–912.
[6] R.M. Evan-Iwanowski, *Resonance Oscillations in Mechanical Systems*, Elsevier, Amsterdam, 1976.
[7] Yu. A. Mitropol'ski, *Problems of the Asymptotic Theory of Nonstationary Vibrations*, Nauka, Moscow, 1964.
[8] A. Al Majid, R. Dufour, An event dimension for modeling damping due to time-varying forcing frequency, *Nonlinear Dynamics* 23 (4) (2000) 303–318.
[9] S.M. Blinder, *Foundations of Quantum Dynamics*, Academic Press Inc., London, 1974.
[10] H. Lumbroso, *Ondes mécaniques et sonores*, Dunod, Paris, 1999.
[11] V. Ougarov, *Théorie de la relativité restreinte*, Mir, Moscow, 1974.
[12] M.J. Woodhouse, *Special Relativity*, Springer, Berlin, 1991.
[13] A. Einstein, P. Bergmann, On a generalization of Kaluza's theory of electricity, in: T. Appelquist, A. Chodos, P.D.O. Freund (Eds.), *Modern Kaluza–Klein Theories*, Addison-Wesley Publishing Company, Menlo Park, CA, 1987, pp. 107–689.
[14] Th. Kaluza, On the unity problem of physics, in: T. Appelquist, A. Chodos, P.G.O. Freund (Eds.), *Modern Kaluza–Klein Theories*, Addison-Wesley Publishing Company, Menlo Park, CA, 1987, pp. 61–68.
[15] O. Klein, Quantum theory and five dimensional theory of relativity, *Zeitschrift für Physik* 37 (1926) 895–906.
[16] W.D. McComb, *Dynamics and Relativity*, Oxford University Press Inc., New York, 1999.