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Verification of Kramers–Kronig relationship in porous materials having a rigid frame

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Abstract

The propagation of acoustic waves in porous materials having a rigid frame is well described by several models. A doubt about the causality of these models has been raised recently in the literature. A verification of the causality of these models is studied in this paper using the Kramers–Kronig dispersion relations adapted to the frequency power law dependence of the attenuation. It is shown that these models are causal in the high- and low-frequency range. A time domain wave equation and time-causal theory have been treated.

1. Introduction

Recently there have been suggestions in the acoustics literature [1] that models [2,3] describing the propagation of acoustic waves in the porous materials are not causal, especially at high frequencies and that Kramers–Kronig relations are not satisfied.

The ultrasonic characterization of porous materials saturated by air is of a great interest for a large class of industrial applications. These materials are frequently used in the automotive and aeronautics industries or in the building trade. The determination of the properties of a medium form waves that have been reflected by or transmitted through the medium is a classical inverse scattering problem. Such problems are often approached by taking a physical model of the scattering process, generating a synthetic response for some assumed values of the parameters,

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adjusting these parameters until reasonable agreement is obtained between the synthetic response and the observed data.

Ultrasonic characterization of materials is often achieved by measuring the attenuation coefficient and phase velocity in frequency domain or by measuring a loss temporal operator in time domain. In frequency domain, measurements of the attenuation coefficient may be more robust than measurements of phase velocity. In these situations, application of the Kramers–Kronig dispersion relations may allow the determination of the phase velocity from the measured attenuation coefficient.

At present most analyses of signal propagation are carried out in the frequency domain using the Fourier transform to translate the results in the time domain and vice versa. This however has several limitations. The first is that the transformation is difficult to compute numerically with sufficient accuracy for non-analytical functions. For example, using Fourier transform to obtain time domain results for a lossy material is a more complicated approach than using a true time domain analysis, and the numerical results are less accurate. The second disadvantage is that by working in the frequency domain some numerical information is lost or hard to recover. For example, in the case of noisy data it may be difficult to reconstruct the chronological events of a signal by phase unwrapping. Consequently, it is difficult to obtain a deep understanding of transient signal propagation using frequency domain method.

The time domain response of the material is described by an instantaneous response and a "susceptibility" kernel responsible for the memory effects. A time domain approach differs from the frequency analysis in that the susceptibility functions of the problem are convolution operators acting on the velocity and pressure fields, and therefore a different algebraic formalism has to be applied to solve the wave equation. The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behaviour of sound waves in these kinds of materials, including relaxation and frequency dependence.

The Kramers–Kronig relations can serve as a check on the causal consistency of a theoretical model. The derivation of dispersion relations based on causality had its beginning in 1926 with Kronig's work relating to the dispersion and the absorption of X-rays [4]. In 1927, Kramers showed that the existence of electromagnetic dispersion relations implies that no signal can propagate in a medium faster than the vacuum speed of light (relativistic causality) [5]. Eventually, the general causal basis for the dispersion relations was appreciated [6] using the physical restrictions on the behaviour of a stable linear system. In addition to their original application in electromagnetism, the Kramers–Kronig relations have been used in many other fields, especially nuclear physics and scattering [6–8] and electrical engineering [9–14]. Their application in acoustics is somewhat less developed. Ginzberg [15] proposed their use and, since then, these relations have been studied [16] and used in several areas like geophysics [17], underwater acoustics [18] and both medical and non-destructive evaluation ultrasound [19–22].

When a broadband acoustic pulse passes through a layer of medium, the waveform of the pulse changes as a result of the attenuation and dispersion in the medium. Many media, including porous materials, have been observed to have an attenuation function which increases with frequency. As a result, the higher frequency components of the pulse are attenuated more than the lower frequency components [23]. After passing through the layer, the transmitted pulse is not just a scaled down version of the incident pulse, but will have a different shape. Dispersion refers to the phenomenon that the phase velocity of a propagating wave changes with the frequency. Dispersion causes additional change in the waveform of the propagating pulse because the wave components with different frequencies travel at different speeds.

2. Mathematical background

Frequent use of Fourier transform analysis will be made so it is necessary to define the transform conventions employed. The time and space two-dimensional Fourier transforms, also used in Refs. [24–26], are

$$P(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z,t)\mathrm{e}^{-\mathrm{i}(kz-\omega t)}\,\mathrm{d}z\,\mathrm{d}t,$$
$$V(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(z,t)\mathrm{e}^{-\mathrm{i}(kz-\omega t)}\,\mathrm{d}z\,\mathrm{d}t. \quad (1)$$

Here the space z and wave number k variables have a conventional Fourier transform designated as FT_- and its inverse FT_-^{-1}. The time t and angular frequency ω transform are unconventional in the sense that the argument of the exponent in the transform has a "+i" rather than "−i" as in Eq. (1); therefore, the designations FT_+ and FT_+^{-1} are for the "+i" type and its inverse.

The corresponding derivative transforms [12] are

$$FT_-\left(\frac{\partial^n p}{\partial z^n}\right) = (\mathrm{i}k)^n P,$$
$$FT_+\left(\frac{\partial^n p}{\partial t^n}\right) = (-\mathrm{i}\omega)^n P. \quad (2)$$

Higher order derivatives will be expressed later in the notation $p_{z^n} = \partial^n p/\partial z^n$.

For example if $n = 2$, $p_{zz} = \partial^2 p/\partial z^2$.

Later it will become necessary to use the definition of fractional derivative [23,27–30] to derive wave equation in time domain. Therefore, we generalize the Fourier transform to fractional order

$$FT_+^{-1}(-\mathrm{i}\omega)^\nu P = D^\nu[P(t)], \quad (3)$$

where $D^\nu[x(t)]$ is the fractional derivative of order ν defined by

$$D^\nu[x(t)] = -\frac{1}{\Gamma(-\nu)}\int_0^t (t-u)^{-\nu-1}x(u)\,\mathrm{d}u, \quad (4)$$

where ν is real number and $\Gamma(x)$ is the gamma function [31]. A fractional derivative no longer represents the local variations of the function but on the contrary, it acts as a convolution integral operator. More details about the properties of fractional derivatives and about fractional calculus are given in Ref. [27].

In this paper generalized functions [32] and their transforms will be applied to derive a wave equation in the time domain and to write the causal relationships.

Some useful generalized function transform pair can be recast in the Fourier transform convention used for time (angular) frequency to obtain for odd integer value of y:

$$FT_{+}^{-1}[\omega^{y}\,\mathrm{sgn}(\omega)] = y!/[\pi(\mathrm{i}t)^{y+1}], \tag{5}$$

$$FT_{+}[\mathrm{sgn}(t)/t^{y+1}] = -2(i)^{y}\omega^{y}[\ln|\omega| + G]/y!, \tag{6}$$

where $\mathrm{sgn}(\omega)$ is the sign function and G is a constant set to zero.

Another set of cases of interest are those in which y is a non-integer. For non-integers, an appropriate generalized function transform pair is

$$FT_{+}^{-1}(|\omega|^{y}) = \Gamma(y+1)\cos[(y+1)\pi/2]/(\pi|t|^{y+1}), \tag{7}$$

$$FT_{+}[\mathrm{sgn}(t)/|t|^{y+1}] = \mathrm{i}\pi\,\mathrm{sgn}(\omega)|\omega|^{y}/y!\,\sin[(y+1)\pi/2]. \tag{8}$$

This case above is more general than the definition of fractional derivatives given by Samko et al. [27] in Eq. (3).

3. Theory

In a variety of media (e.g., porous media, liquids and tissue) over a finite bandwidth, the attenuation of acoustic waves appears to be adequately modelled by a power law dependence on frequency [33–39]:

$$\alpha = \alpha_0|\omega|^{y}, \tag{9}$$

where ω is angular frequency, and α_0 has units of Np/m and the loss is $\exp(-\alpha z)$. The absolute sign is a consequence of the real even properties of absorption as a function of frequency, y is a real positive finite number. For most materials, the power law exponent y has value from 0 to 2. Unlike the electrical network applications of the Kramers–Kronig relations, acoustics pose a unique problem as mentioned in Ref. [33]. Networks have frequency transfer functions expressed as algebraic ratio of pole-zero polynomial products which have a denominator which is at least one order higher than the numerator. In acoustics, the propagation problems of interest may occur in different forms, such as power law dependence on frequency. The Paley–Wiener theorem [40] states that for a transfer function of the form

$$H(\omega) = A(\omega)\mathrm{e}^{\mathrm{i}\theta(\omega)} = (\mathrm{e}^{-\alpha(\omega)z})\mathrm{e}^{\mathrm{i}\beta(\omega)z}; \tag{10}$$

the logarithm of the magnitude A must meet the requirement

$$\int_{-\infty}^{\infty}\frac{|\ln A(\omega)|}{1+\omega^2}\,\mathrm{d}\omega = \int_{-\infty}^{\infty}\frac{|\alpha(\omega)|}{1+\omega^2}\,\mathrm{d}\omega < \infty, \tag{11}$$

and A must be square-integrable in order for causality to hold [11]. These requirements restrict the values of power law of $\alpha(\omega)$ to that one.

An alternative time-causal theory that provided an inherently causal model [33] was developed for media with attenuation obeying a frequency power law because of the (initial) apparent failing of the Kramers–Kronig dispersion. A key to the development of the time-causal dispersion relations was the use of generalized functions, as opposed to ordinary point function, to represent physical quantities (e.g., attenuation coefficient and phase velocity). The generalized functions have been used for dispersion measurements in other areas of physics (e.g., particle physics) but they do not appear to have found as much use in ultrasonic measurements, with the exception of the afore-mentioned time-causal theory. By considering the complex wave number as generalized function, a generalized Paley–Wiener theorem [41] is available that permits the relaxation of the restriction on the high-frequency behavior of the attenuation coefficient. Specifically, any frequency power law attenuation represented as a generalized function satisfies the generalized Paley–Wiener theorem. Using this concept of generalized functions, Waters et al. [36] demonstrate the equivalence of the Kramers–Kronig frequency domain approach to that of the time-causal approach.

3.1. Porous materials having a rigid frame

In the acoustics of porous materials, one distinguishes two situations according to whether the frame is moving or not. In the first case, the dynamics of the waves due to the coupling between the solid skeleton and the fluid is well described by the Biot theory [42,43]. In air-saturated porous media, the structure is generally motionless and the waves propagate only in fluid. This case is described by the model of equivalent fluid which is a particular case of the Biot model.

3.2. Model of equivalent fluid

Let a homogeneous isotropic porous material with porosity ϕ be saturated with a compressible and viscous fluid of density ρ_f and viscosity η. It is assumed that the frame of this porous solid is not deformable when it is subjected to an acoustic wave. It is the case, for example, for a porous medium which has a large skeleton density or very large elastic modulus or weak fluid–structure coupling. To apply the results of continuum mechanics it is required that the wavelength of sound should be much larger than the sizes of the pores or grains in the medium.

In such a porous material, acoustic waves propagate only in the fluid, so it can be seen as an equivalent fluid, the density and the bulk modulus of which are "renormalized" by the fluid–structure interactions. A prediction of the acoustic behaviour of the porous material requires the determination of the dynamic tortuosity $\varepsilon(\omega)$ and the dynamic compressibility $\mu(\omega)$. These functions depend on the physical characteristics of the fluid in the pore space of the medium and are independent of the dynamic characteristics of the structure. The basic equations of the model of equivalent fluid are:

$$\rho_f \varepsilon(\omega) \frac{\partial v_i}{\partial t} = -\nabla_i p, \qquad \frac{\mu(\omega)}{K_a} \frac{\partial p}{\partial t} = -\nabla \cdot v. \tag{12}$$

In these relations, v and p are the particle velocity and the acoustic pressure, $K_a = \gamma P_0$ is the compressibility modulus of the fluid. The first equation is the Euler equation, and the second one is a constitutive equation obtained from the equation of mass conservation associated with the behaviour (or adiabatic) equation. $\varepsilon(\omega)$ and $\mu(\omega)$ are the dynamic tortuosity of the medium and the dynamic compressibility of the air included in the porous material.

These two factors are complex functions which heavily depend on the frequency $f = \omega/2\pi$. Their theoretical expressions are given by Johnson et al. [44], Allard [45] and Lafarge [46]:

$$\varepsilon(\omega) = \varepsilon_\infty \left(1 - \frac{\eta\phi}{\mathrm{i}\omega\varepsilon_\infty \rho_f k_0} \sqrt{1 - \mathrm{i}\frac{4\varepsilon_\infty^2 k_0^2 \rho_f \omega}{\eta \Lambda^2 \phi^2}} \right), \tag{13}$$

$$\mu(\omega) = \gamma - (\gamma - 1)\left(1 - \frac{\eta\phi}{\mathrm{i}\omega\rho_f k_0' P_r}\sqrt{1 - \mathrm{i}\frac{4k_0'^2\rho_f\omega P_r}{\eta\phi^2\Lambda'^2}}\right)^{-1}, \tag{14}$$

where $\mathrm{i}^2 = -1$, γ represents the adiabatic constant, P_r the Prandtl number, ε_∞ the tortuosity, k_0 the static permeability, k_0' the thermal permeability [47], Λ and Λ' are the viscous and thermal characteristic lengths, respectively [44,48]. This model was initially developed by Johnson [44], and completed by Allard [48] by adding the description of thermal effects. Later on, Lafarge [47] has introduced the parameter k_0' which describes the additional damping of sound waves due to the thermal exchanges between fluid and structure at the surface of the pores.

The functions $\varepsilon(\omega)$ and $\mu(\omega)$ express the viscous and thermal exchanges between the air and the structure which are responsible for the sound damping in acoustics materials. These exchanges are due on the one hand to the fluid–structure relative motion and on the other hand to the air compression–dilatations produced by the wave motion. The parts of the fluid affected by these exchanges can be estimated by the ratio of a microscopic characteristic length of the media, for example, the sizes of the pores, and the viscous and thermal skin depth thickness $\delta = (2\eta/\omega\rho_0)^{1/2}$ and $\delta' = (2\eta/\omega\rho_0 P_r)^{1/2}$, respectively. For the viscous effects this domain corresponds to the region of the fluid in which the velocity distribution is perturbed by the frictional forces at the interface between the viscous fluid and motionless structure. For the thermal effects, it corresponds to the fluid volume affected by the heat exchange between the two phases of the porous medium. In this model, the sound propagation is completely determined by the six following parameters: $\phi, \varepsilon_\infty, \sigma = \eta/k_0, k_0', \Lambda$ and Λ'. In the next section it will be shown that the values of these parameters are given by the low- and high-frequency wave equations for monochromatic waves. To restore their validity for transient signals, they need to be written in the time domain.

3.3. Viscous domain

In this domain, the viscous forces are important everywhere in the fluid, the compression–dilatation cycle in the porous material is slow enough to favour the thermal exchanges between fluid and structure. At the same time the temperature of the frame is practically unchanged by the passage of the sound wave because of the high value of its specific heat: the frame acts as a thermostat. In this case the isothermal compressibility is directly applicable. This domain corresponds to the range of frequencies such that viscous skin thickness $\delta = (2\eta/\omega\rho_0)^{1/2}$ is much larger than the radius of the pores r,

$$\frac{\delta}{r} \gg 1 \tag{15}$$

is called the low-frequency range. For these frequencies, the low-frequency approximations of the response factor $\varepsilon(\omega)$ and $\mu(\omega)$ are considered. When $\omega \to 0$, Eqs. (13) and (14), respectively, become

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\eta\phi}{\mathrm{i}\omega\varepsilon_0\rho_f k_0}\right), \tag{16}$$

$$\mu(\omega) = \gamma. \tag{17}$$

ε_0 is the low-frequency approximation of the tortuosity given by Lafarge [46] and Norris [49] from homogenation theory

$$\varepsilon_0 = \frac{\langle v(r)^2 \rangle}{\langle v(r) \rangle^2}, \tag{18}$$

where $\langle v(r) \rangle$ is the average velocity of the viscous fluid for direct current flow within a volume element small compared to the relevant wavelength, but large compared to the individual grains/pores of solid.

For a wave travelling along the direction oz, the generalized forms of the basic equations (12) in the frequency domain are now

$$\rho_f \varepsilon_0(-\mathrm{i}\omega)V + \sigma\phi V = -P_z \quad \text{and} \quad \frac{\gamma}{K_a}(-\mathrm{i}\omega)P = -V_z. \tag{19}$$

In this approximation the Euler equation expresses the balance between the driving force of the wave, the drag forces $\eta\phi v/k_0$ due to the flow resistance of the material and the inertial forces. The wave equation in time domain in viscous domain is given by

$$p_{zz} - \left(\frac{\varepsilon_0\gamma}{c_0^2}\right)p_{tt} - \left(\frac{\phi\sigma}{\rho_f\varepsilon_0 c_0^2}\right)p_t = 0. \tag{20}$$

The first coefficient of this equation $\varepsilon_0\gamma/c_0^2$ leads to the sound velocity in the air filling the structure of the material. This result shows that the viscous forces and the shape of the pores increase the fluid density by the factor $\varepsilon_0 > 1$. The second coefficient $\phi\sigma/\rho_f\varepsilon_0 c_0^2$ is the damping-distortion term due to viscous effects which take place in the porous material. From these equations it is possible to estimate ε_0 and the flow resistivity $\sigma = \eta/k_0$. By applying the double Fourier transform in z and t to Eq. (20), the dispersion relation

$$k^2 = (\omega/c_0)^2(\varepsilon_0\gamma) + \mathrm{i}\omega(\gamma\phi\sigma/\rho_f\varepsilon_0 c_0^2) \tag{21}$$

is obtained, where $k = \beta + \mathrm{i}\alpha$ and $\beta = \omega/c_p$. White [50] provides explicit equations for the real and imaginary parts of k.

Taking the square root of Eq. (21) gives the expression for the wave number $k(\omega)$:

$$k(\omega) = \frac{1}{c_0}\sqrt{\frac{\phi\gamma\sigma}{2\rho_f}}\sqrt{\omega}\left(1 + \frac{\omega\rho_f\varepsilon_0}{2\sigma\phi} + \mathrm{i}\left(1 - \frac{\omega\rho_f\varepsilon_0}{2\sigma\phi}\right)\right). \tag{22}$$

The imaginary part of the wave number $k(\omega)$ gives the attenuation $\alpha(\omega)$

$$\alpha(\omega) = \frac{1}{c_0}\sqrt{\frac{\phi\gamma\sigma}{2\rho_f}}\left(\sqrt{\omega} - \frac{\rho\varepsilon_0}{2\sigma\phi}\omega^{3/2}\right). \tag{23}$$

At very low frequency, the asymptotic expressions for $\varepsilon(\omega)$ and $\mu(\omega)$ are

$$\varepsilon(\omega) = -\frac{\sigma\phi}{\mathrm{i}\omega\rho_f}, \quad \mu(\omega) = \gamma. \tag{24}$$

In this range of frequencies, the basic equations (12) are

$$\sigma\phi V = -P_z \quad \text{and} \quad \frac{\gamma}{K_a}(-\mathrm{i}\omega)P = -V_z, \tag{25}$$

where the Euler equation is reduced to the Darcy's law which defines the static flow resistivity $\sigma = \eta/k_0$. The wave equation in time domain (20) in this case is reduced to

$$p_{zz} + \left(\frac{\sigma\phi\gamma}{K_a}\right)p_t = 0. \tag{26}$$

The fields which are varying in time, the pressure, the acoustic velocity, etc., follow a diffusion equation with the diffusion constant

$$D = \frac{K_a}{\sigma\phi\gamma}. \tag{27}$$

A quite similar result is given by Johnson [51]. However, the adiabatic constant γ does not appear in Johnson's model in which the thermal expansion is neglected.

The dispersion relation in this case is given by

$$k^2 = \mathrm{i}\omega(\phi\sigma/\rho_f\varepsilon_0 c_0^2) \tag{28}$$

and the expression of $k(\omega)$ is given by

$$k(\omega) = \frac{1}{c_0}\sqrt{\frac{\phi\sigma}{\rho_f\varepsilon_0}}(1+\mathrm{i})\sqrt{\omega}. \tag{29}$$

The attenuation $\alpha(\omega)$ is given by

$$\alpha(\omega) = \frac{1}{c_0}\sqrt{\frac{\phi\sigma}{\rho_f\varepsilon_0}}\sqrt{\omega}. \tag{30}$$

3.4. Asymptotic domain

In this domain, the viscous effects are concentrated in a small volume near the frame and the compression/dilatation cycle is faster than the heat transfer between the air and the structure, and it is a good approximation to consider that the compression is adiabatic.

This domain corresponds to the range of frequencies such that viscous skin thickness $\delta = (2\eta/\omega\rho_0)^{1/2}$ is smaller than the radius of the pores r, and corresponds to the high-frequency approximation of the responses factors $\varepsilon(\omega)$ and $\mu(\omega)$ when $\omega \to \infty$ are given by the relations

$$\varepsilon(\omega) = \varepsilon_\infty\left(1 + \frac{2}{\Lambda}\left(\frac{\eta}{-\mathrm{i}\omega\rho_f}\right)^{1/2}\right), \tag{31}$$

$$\mu(\omega) = 1 + \frac{2(\gamma-1)}{\Lambda'}\left(\frac{\eta}{-\mathrm{i}\omega P_r\rho_f}\right)^{1/2}. \tag{32}$$

Using Eqs. (3) and (4), the expressions of the responses ε and μ are given in time domain [23] by

$$\varepsilon(\omega) \xrightarrow{t} \tilde{\varepsilon}(t) = \varepsilon_\infty\left(\delta(t) + \frac{2}{\Lambda}\left(\frac{\eta}{\pi\rho_f}\right)^{1/2} t^{-1/2}\right)*, \tag{33}$$

$$\mu(\omega) \xrightarrow{t} \tilde{\mu}(t) = \left(\delta(t) + \frac{2(\gamma - 1)}{\Lambda'} \left(\frac{\eta}{\pi Pr \rho_f} \right)^{1/2} t^{-1/2} \right) *, \quad (34)$$

where $*$ denotes the time convolution and $\delta(t)$ is the Dirac function. In this model $t^{-1/2}*$ is interpreted as a semi-derivatives operator following the definition of the fractional derivative [23] given in Eq. (4).

When the wave propagates along the co-ordinate axis oz, the basic equations (12) will be written in the time domain as

$$\rho_f \varepsilon_\infty v_t + 2 \frac{\rho_f \varepsilon_\infty}{\Lambda} \left(\frac{\eta}{\pi \rho_f} \right)^{1/2} \int_{-\infty}^{t} \frac{v_{t'}}{\sqrt{t - t'}} \, \mathrm{d}t' = -p_z, \quad (35)$$

$$\frac{1}{K_a} p_t + 2 \frac{\gamma - 1}{K_a \Lambda'} \left(\frac{\eta}{\pi Pr \rho_f} \right)^{1/2} \int_{-\infty}^{t} \frac{p_{t'}}{\sqrt{t - t'}} \, \mathrm{d}t' = -v_z. \quad (36)$$

In these equations the convolutions express the dispersive nature of the porous material. They take into account the memory effects due to the fact that the response of the medium to the wave excitation is not instantaneous but needs more time to become effective. The retarding force is no longer proportional to the time derivative of the acoustic velocity but is found to be proportional to the fractional derivative of order 1/2 of this quantity. This occurs because the volume of fluid participating to the motion is not the same during the whole length of the signal as it is in the case of a fully developed steady flow. The phenomena may be understood by considering such a volume of fluid in a pore to be in harmonic motion. At high frequencies, only a thin layer of fluid is excited: the average shear stress is high. At a lower frequency, the same amplitude of fluid motion allows a thicker layer of fluid to participate in the motion and consequently the shear stress is less. The penetration distance of the viscous forces and therefore the excitation of the fluid depends on frequency. In the time domain, such a dependence is associated with a fractional derivative.

The propagation equation in time domain is given by

$$p_{zz} - \left(\frac{\varepsilon_\infty}{c_0^2} \right) p_{tt} - \frac{2\varepsilon_\infty}{c_0^2} \sqrt{\frac{\eta}{\pi \rho_f}} \left(\frac{1}{\Lambda} + \frac{(\gamma - 1)}{\sqrt{P_r} \Lambda'} \right) p_{t^{3/2}} = 0. \quad (37)$$

The solution of the propagation equation (37) in time domain is given in Ref. [29]. Application of the transform, Eqs. (1)–(3), to this wave leads to a characteristic dispersion relation:

$$k^2(\omega) = \left(\frac{\varepsilon_\infty}{c_0^2} \right) \omega^2 + \frac{\varepsilon_\infty}{c_0^2} \sqrt{\frac{2\eta}{\rho_f}} \left(\frac{1}{\Lambda} + \frac{(\gamma - 1)}{\sqrt{P_r} \Lambda'} \right) \sqrt{\omega} (1 + \mathrm{i}); \quad (38)$$

the complex wave number is given in this time domain by

$$k(\omega) = \left(\frac{\sqrt{\varepsilon_\infty}}{c_0} \right) \omega + \frac{\sqrt{\varepsilon_\infty}}{c_0} \sqrt{\frac{\eta}{2\rho_f}} \left(\frac{1}{\Lambda} + \frac{(\gamma - 1)}{\sqrt{P_r} \Lambda'} \right) \sqrt{\omega} (1 + \mathrm{i}); \quad (39)$$

the attenuation $\alpha(\omega)$ is the imaginary part of the wave number $k(\omega)$ given by

$$\alpha(\omega) = \frac{\sqrt{\varepsilon_\infty}}{c_0}\sqrt{\frac{\eta}{2\rho_f}}\left(\frac{1}{\Lambda} + \frac{(\gamma-1)}{\sqrt{P_r}\Lambda'}\right)\sqrt{\omega}. \tag{40}$$

4. Time domain wave equation

It is possible to write a general dispersion relation for the propagation of ultrasonic waves in a wide variety of media [24] as

$$k^2(\omega) = (\omega/c)^2 + \mathrm{i}2(\omega/c)(\alpha_0|\omega|^y), \tag{41}$$

where $k(\omega) = \beta(\omega) + \mathrm{i}\alpha(\omega)$ and c is the velocity of the medium. This relation is valid if $(\alpha(\omega)/\beta(\omega))^2 \ll 1$ and this inequality defines a finite frequency range in which the general lossy wave equation developed is valid. This approximation is used widely for the linear case [20,25,52] and non-linear cases such as the Burger's equations and KZK equations [53]. In the case of porous materials, the above approximation is well satisfied when $\delta/r \ll l$, for high-frequency range. Multiplying Eq. (41) by i^2P, results in a generalized frequency domain lossy wave equation:

$$(\mathrm{i}k)^2P - (-\mathrm{i}\omega/c)^2P - \mathrm{i}^3 2(\omega/c)(\alpha_0|\omega|^y)P = 0. \tag{42}$$

In the time domain, all coefficients of the differential terms of the wave equations are real constants. This characteristic ensures real results for real excitation signals. Even when a consistent, valid, frequency domain plane wave solution approach is used through either a complex compressibility [20] or complex elastic constant [54], it cannot apply directly to the solution of pulsed case, as also pointed out by Nachman et al. [55].

A problem arises when an attempt is made to transform the general frequency domain lossy wave equation above (Eq. (42)) back to the space and time domain, as discussed in Refs. [23,24]. Under conventional Fourier transform (1), inversion by the k and ω transform is defined only when y is an even integer n:

$$p_{zz} - 1/c^2 p_{tt} - (-1)^{n/2} 2\alpha_0/c p_{t^{n+1}} = 0. \tag{43}$$

For other powers of y, the Fourier transform derivative relations (1), fail to recover differential terms with real constants. It is the case for the propagation in porous material at high-frequency range in which the introduction of the fractional derivative definition is needed to write the wave equation in time domain.

The approach that has been taken by Szabo [24] is to apply generalized functions and their transforms to the problem [32]. If y is an odd integer, it is helpful to rewrite Eq. (41) in an equivalent form

$$k^2(\omega) = (\omega/c)^2 + \mathrm{i}\omega(2\alpha_0/c)\,\mathrm{sgn}(\omega)\omega^y \tag{44}$$

for this odd integer case, a useful generalized function pair can be recast in the Fourier transform convention used for time (angular) frequency Eq. (5), this pair permits one to write equivalent time domain lossy wave equation

$$p_{zz} - 1/c^2 p_{tt} + (2/\pi c)\alpha_0(y+1)!(-1)^{(y+1)/2}/p*1/t^{(y+2)} = 0. \tag{45}$$

Another set of cases of interest are those in which y is a non-integer. An appropriate generalized function transform pair from Lighthill and Szabo is given in Eq. (7); this non-integer case includes the case of fractional derivative defined previously in Eq. (4); the general propagation equation in time domain is that given by

$$p_{zz} - (1/c^2)p_{tt} + (2/\pi c)\alpha_0 \Gamma(y+2)\cos[(y+1)\pi/2]p*1/|t|^{(y+2)} = 0. \tag{46}$$

In summary this wave equation can be expressed in the compact form as in Ref. [24]:

$$p_{zz} - 1/c^2 p_{tt} - (2/c)\partial/\partial t[L_{\gamma,y,t}*p] = 0, \tag{47}$$

where $L_{\gamma,y,t}$ is a time domain convolution loss operator that is a function of time t, loss α and y, and it differs for y as an even or odd integer or as a non-integer.

5. Causal theory

Causal means that an effect cannot precede its cause. For a time waveform initiated at $t = 0$, its spectral characteristics must meet certain requirements so that complete time cancellation occurs for $t<0$. Concise reviews of causality considerations can be found in Refs. [7,56] and in more detail in Ref. [6]. The real and imaginary parts of causal complex transfer function are related by Hilbert transforms as shown by Tithmarsh's theorem [57].

To take advantage of these Hilbert transform relations, we define as in Refs. [24,36] a propagation factor,

$$\gamma(\omega) = -\alpha(\omega) + i\beta'(\omega) \tag{48}$$

with

$$\beta(\omega) = \beta_0 + \beta'(\omega), \tag{49}$$

where $\beta'(\omega)$ is the extra dispersion term needed for causal propagation, $\beta_0 = \omega/c$, $c = c_0/\sqrt{\varepsilon_\infty}$ in high-frequency range and $c = c_0/\sqrt{\varepsilon_0}$ at low-frequency range. At very-low-frequency range, we have no propagation mode and then $\beta_0 = 0$. It has been recognized that the causal Hilbert transform relationships have more general applicability and that specifically they also relate the real and imaginary parts of complex propagation constant [6–10,13,15]. Both $\alpha(\omega)$ and $\beta'(\omega)$ are also related through their Hilbert transforms in the present sign convention:

$$\beta'(\omega) = [-1/(\pi\omega)]*[-\alpha(\omega)], \tag{50}$$

$$-\alpha(\omega) = [1/(\pi\omega)]*\beta'(\omega). \tag{51}$$

By defining

$$L_{\alpha,y,t} = FT_+^{-1}[-\alpha(\omega)],$$

$$L_{\beta',y,t} = FT_+^{-1}[\beta'(\omega)] \tag{52}$$

it is easy to write the above Hilbert transform in time domain, given

$$L_{\beta',y,t} = -i\,\mathrm{sgn(t)} L_{\alpha,y,t}, \tag{53}$$

$$L_{\alpha,y,t} = i\,\mathrm{sgn(t)} L_{\beta',y,t}. \tag{54}$$

Eqs. (53) and (54) are the time-causal relations. These relations can be shown to be the Fourier transform equivalents of the Hilbert transforms [8]. Because generalized function time domain operators satisfying Eqs. (53) and (54) have no restriction on the value of y (assumed to be finite, real), they have more general validity (for $y > 1$) than the Kramers–Kronig relations expressed in the frequency domain. In the frequency domain, the Kramers–Kronig relations require knowledge of either α or β' at all frequencies, however, in time domain, each convolution of causal operator is naturally limited by a finite length of an imput pressure of total propagation operator.

The temporal propagation operator $L_{\gamma,y,t} = FT_+^{-1}[\gamma(\omega)]$ is given by

$$\begin{aligned} L_{\gamma,y,t} &= L_{\alpha,y,t} + \mathrm{i}L_{\beta',y,t} \\ &= [1 + \mathrm{sgn}(t)]L_{\alpha,y,t} \\ &= 2\mathrm{H}(t)L_{\alpha,y,t} \end{aligned} \tag{55}$$

Here H(t) is the step function [12] defined by

$$\mathrm{H}(t) = \begin{cases} 0, & t<0, \\ 1/2, & t = 0, \\ 1, & t > 0. \end{cases} \tag{56}$$

The final causal lossy wave equation in time domain is given by

$$p_{zz} - 1/c^2 p_{tt} - (4/c)\partial/\partial t[\mathrm{H}(t)L_{\alpha,y,t} * p] = 0. \tag{57}$$

Because of the step function, the propagation time operator $L_{\gamma,y,t}$ is causal.

It is easy to find causal dispersion relations [33,38] for power law attenuation. The relative dispersion term is given by

$$\beta'(\omega) = FT_+[L_{\beta',y,t}], \tag{58}$$

Loss operators, found from Eq. (52) with the help of generalized functions, came in three flavors, according to whether y is an even or odd integer or a non-integer. These results, in combination with Eq. (53) and generalized functions, can be used to obtain the dispersion results below. For $y = 0$ or an even integer

$$L_{\beta',y,t} = 0. \tag{59}$$

Therefore,

$$\beta'(\omega) = 0. \tag{60}$$

For y odd integer, from the results of Szabo [24,33] and Eqs. (53) and (58), the following transform is sought:

$$\beta'(\omega) = FT_+([-\mathrm{i}\,\mathrm{sgn}(t)][-\alpha_0(-1)^{(y+1)/2}y!/(\pi t^{y+1})]). \tag{61}$$

Using a generalized function transform given in Eq. (6) helps in obtaining

$$\beta'(\omega) = -2\alpha_0\omega^y[\ln|\omega| + G]/\pi. \tag{62}$$

For y non-integer, one seeks the transform of

$$\beta'(\omega) = FT_+([-\mathrm{i}\,\mathrm{sgn}(t)](-\alpha_0 y(y+1)\cos[(y+1)\pi/2]/(\pi|t|^{y+1}))). \tag{63}$$

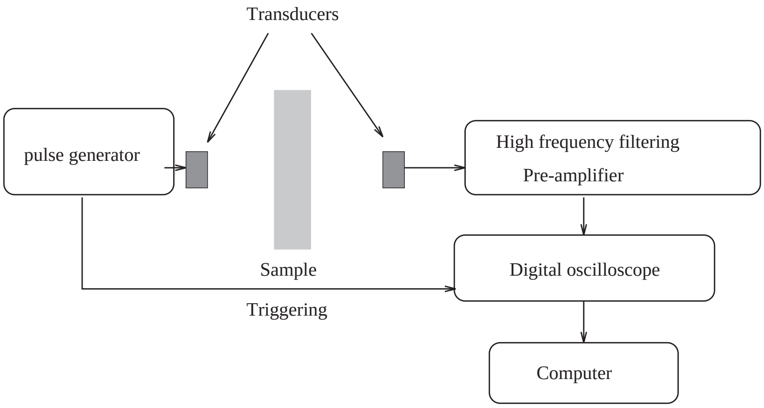

Fig. 1. Experimental set-up of the ultrasonic measurements.

Again with the help of the transform of a generalized function Eq. (8), it is found that

$$\beta'(\omega) = -\alpha_0 \cot[(y+1)\pi/2]\omega|\omega|^{y-1}. \tag{64}$$

These results were confirmed by Waters et al. [36,38].

6. Application to porous materials

Consider the case of porous material having a rigid frame. As shown previously the attenuation $\alpha(\omega)$ is in the form $\alpha_0\omega^y$, in the high-frequency range with $y = 0.5$ and

$$\alpha_0 = \frac{\sqrt{\varepsilon_\infty}}{c_0}\sqrt{\frac{\eta}{2\rho_f}}\left(\frac{1}{\Lambda} + \frac{\gamma - 1}{\sqrt{P_r}\Lambda'}\right),$$

one then has the case of non-integer of y. From causal theory (Eq. (64)) the extra dispersion term $\beta'(\omega)$ is given in this case by

$$\beta'(\omega) = -\alpha_0 \cot(3\pi/4)\sqrt{\omega}. \tag{65}$$

This extra dispersion term coincide with the one given by Johnson–Allard model (Eq. (39)):

$$k(\omega) = \left(\frac{\sqrt{\varepsilon_\infty}}{c_0}\right)\omega + \alpha_0\omega^{0.5}(1+\mathrm{i}). \tag{66}$$

The Kramers–Kronig relations are then well satisfied. The Johnson et al. [44] and Allard [45] models are then causal at high-frequency range. Some numerical simulations are compared to experimental results in the high-frequency domain approximation of Johnson and Allard model. Experiments are carried out in air with two broadband Ultran NCT202 transducers having a 190 kHz central frequency in air and a bandwith at 6 dB extending from 150 to 230 kHz. Pulses of 400 V are provided by a 5052PR Parametrics pulser/receiver (Fig. 1). Received signals are amplified up to 90 dB and filtred above 1 MHz to avoid high-frequency noise. Fig. 2 shows the comparison between experimental data of the attenuation of a plastic foams M1 having the

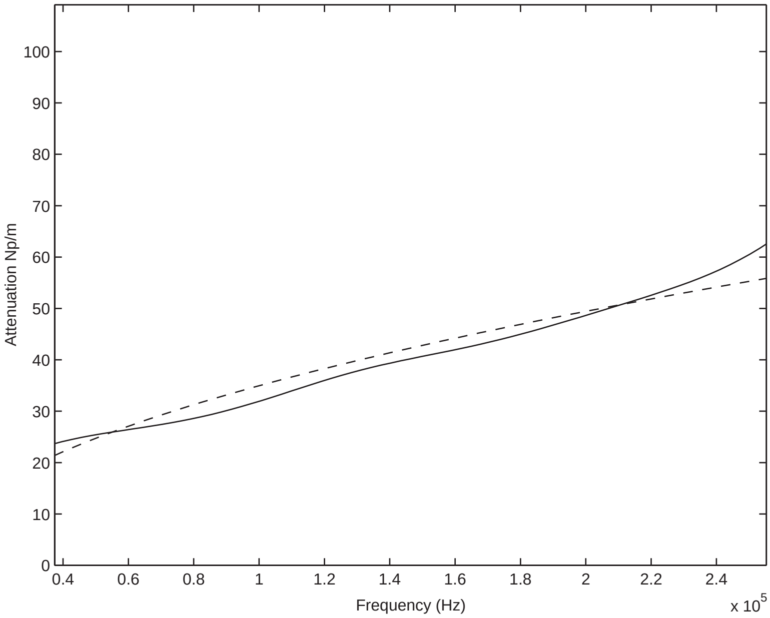

Fig. 2. Comparison between the Johnson and Allard model of attenuation (dashed line) and experimental data of attenuation (solid line).

following parameters: $\varepsilon_\infty = 1.07$, $\phi = 0.96$, $\Lambda = 220$ μm and $\Lambda' = 660$ μm and the simulated data of the attenuation using the Johnson and Allard model (Eq. (40)).

The wave velocity in frequency domain in the porous material $c(\omega)$ is given by

$$c(\omega) = \frac{1}{\sqrt{\varepsilon_\infty}/c_0 + \beta'(\omega)/\omega} \tag{67}$$

for non-integer value of y; the expression of $\beta'(\omega)$ is given by the Kramers–Kronig relations and causal theory Eq. (64) and the expression of the wave velocity in this case will be given by

$$c(\omega) = \frac{1}{\sqrt{\varepsilon_\infty}/c_0 - \alpha_0 \cot[(y+1)\pi/2]|\omega|^{y-1}}. \tag{68}$$

Fig. 3 shows the comparison between the causal wave velocity Eq. (68) for the plastic foam M1 (Johnson and Allard Model of the attenuation) and experimental data of the wave velocity. These results confirm the causality of the Johnson and Allard model in the high-frequency range. This is in contradiction with analysis given by Berthelot [1] in which the author concludes by using the Kramers–Kronig relations that the Johnson model is not causal in this range of frequency. In the very-low-frequency approximation $y = 0.5$, there is no propagation mode and $\beta_0 = 0$. One is in the same situation as the high-frequency range with Kramers–Kronig relations adapted for a non-integer value of y.

To test the models of porous material at all frequencies, one must use the general relation of the wave number $k(\omega)$ given by

$$k(\omega) = \frac{\omega}{c_0}\sqrt{\varepsilon(\omega)\mu(\omega)}. \tag{69}$$

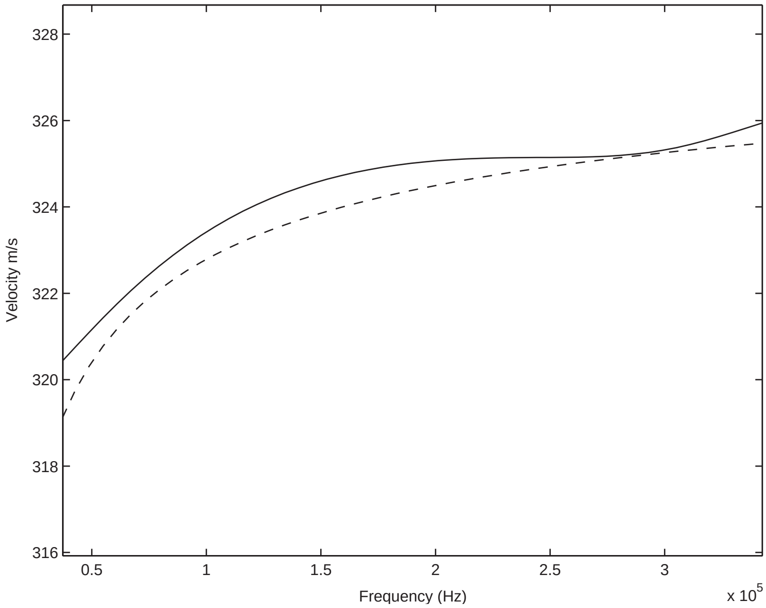

Fig. 3. Comparison between the wave velocity calculated by Kramers–Kronig relations from the Johnson and Allard model of attenuation (dashed line) and experimental data of the wave velocity (solid line).

$\varepsilon(\omega)$ and $\mu(\omega)$ are given by the models of Johnson [44], Allard [48] and Lafarge [47]. The attenuation which is the imaginary part of the wave number is not a power law of frequency dependence in this general case and it is not possible to use the analysis developed before. We must then use the general relation of Kramers–Kronig given by Eqs. (50) and (51). Fig. 4 shows the comparison between the wave velocity simulated by the Kramers–Kronig relations and the wave velocity simulated using the real part of $k(\omega)$:

$$c(\omega) = \frac{\omega}{\beta(\omega)}; \tag{70}$$

the weak difference between the two curves is essentially due to the numerical simulations.

In Fig. 5, a comparison between the numerical dynamic response of the Johnson and Allard model and experimental data is given in terms of transmitted wave. Two plastic foams are considered (M2 and M3). The theoretical expression of the dynamic response is given in Ref. [29]. The simulated signals are computed from the convolution of the Green function of the medium with the incident signal generated by the transducer (Fig. 6). For porous media having a high porosity like plastic foams, the reflected signal can be neglected. These materials have such a small amount of rigid frame that the incident wave does not feel its effects. The experimental data are deduced from the transmitted field scattered by a slab of two different plastic foams of finite depth $0 \leqslant x \leqslant L$ and having different flow resistivities. The parameters of the foam M2 are: thickness 5 cm, $\alpha_\infty = 1.055$, $\Lambda = 234$ μm, $\Lambda' = 702$ μm, flow resistivity $\sigma = 9000$ N m^{-4} s and porosity $\phi = 0.97$, those of the foam M3 are: thickness 1.1 cm, $\alpha_\infty = 1.26$, $\Lambda = 60$ μm, $\Lambda' = 180$ μm, $\sigma = 38000$ N m^{-4} s and $\phi = 0.98$. The parameters of the foams are given by the Leclaire et al.

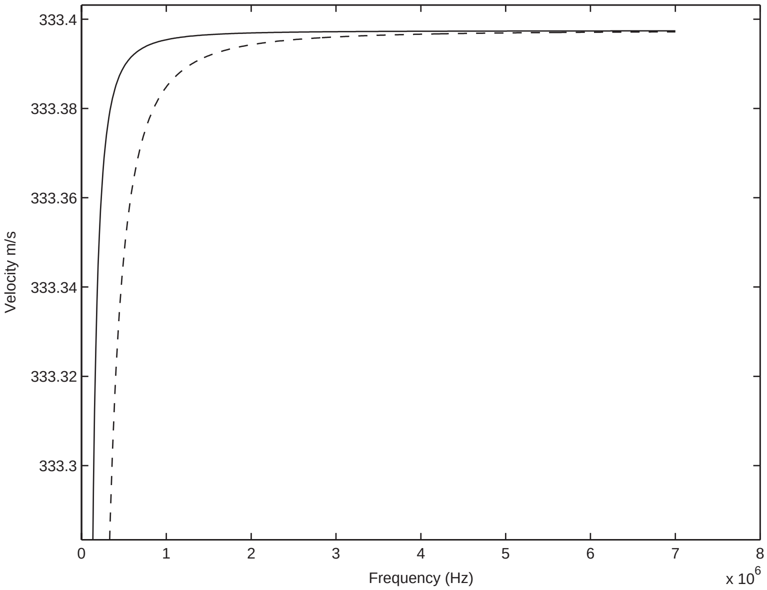

Fig. 4. Comparison between the wave velocity simulated from the Kramers–Kroning relations in general case and the wave velocity obtained by the real part of the wave number at all frequencies.

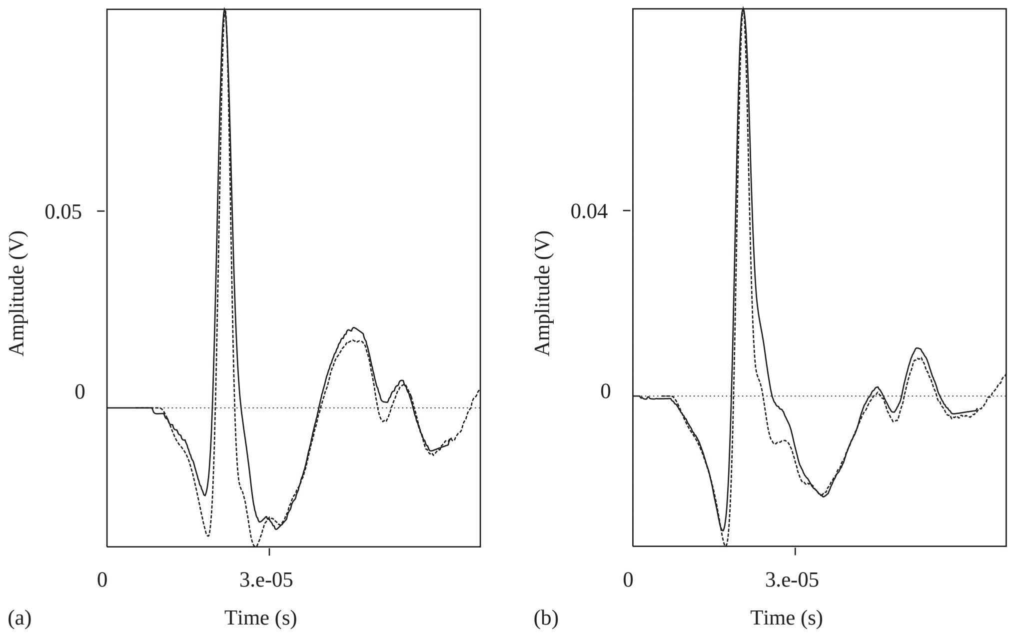

Fig. 5. (a) Experimental (solid line) and simulated signals (dashed line) for the foam M2. (b) Experimental (solid line) and simulated signals (dashed line) for the foam M3.

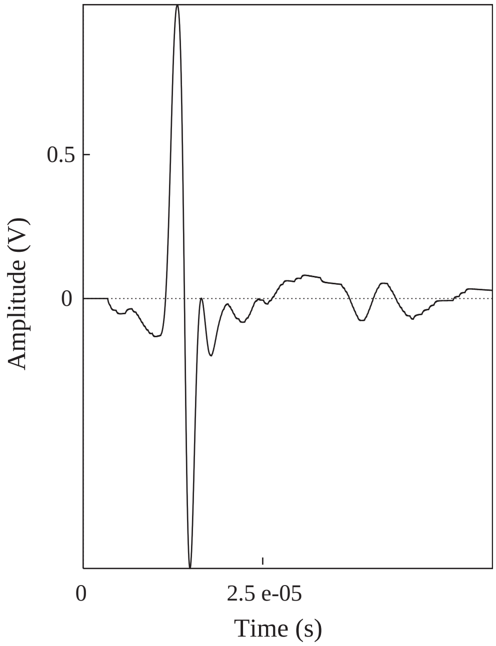

Fig. 6. Incident signal generated by the transducer.

method [58]. The expression of the Green function in time domain is given by

$$G(x,t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant x/c, \\ \dfrac{x}{c}\dfrac{b'}{4\sqrt{\pi}}\dfrac{1}{(t-x/c)^{3/2}}\exp\left(-\dfrac{b'^2x^2}{16c^2(t-x/c)}\right) & \\ \quad + \Delta\int_0^{t-x/c} h(t,\xi)\mathrm{d}\xi & \text{if } t \geqslant x/c, \end{cases} \tag{71}$$

where $h(\tau,\xi)$ is of the form

$$h(\tau,\xi) = -\frac{1}{4\pi^{3/2}}\frac{1}{\sqrt{(\tau-\xi)^2 - x^2/c^2}}\frac{1}{\xi^{3/2}}\int_{-1}^{1}\exp\left(-\frac{\chi(\mu,\tau,\xi)}{2}\right)(\chi(\mu,\tau,\xi)-1)\frac{\mu\mathrm{d}\mu}{\sqrt{1-\mu^2}} \tag{72}$$

with the following notations: $\chi(\mu,\tau,\xi) = (\Delta\mu\sqrt{(\tau-\xi)^2 - x^2/c^2} + b'(\tau-\xi))^2/8\xi$, $b' = Bc^2\sqrt{\pi}$, and $\Delta = b'^2$. The boundary and initial conditions are given by

$$p(0,t) = g^i(t) \quad \text{and} \quad \lim_{\substack{t\to 0\\ t>0}} p(x,t) = \lim_{\substack{t\to 0\\ t>0}} \frac{\partial p}{\partial t}(x,t) = 0, \tag{73}$$

where $g^i(t)$ is the incident signal generated by the transducer. The difference between experimental data and simulated data in Fig. 5 for the response of the medium is slight and shows the causality of the Johnson and Allard model.

It is easy to proof the causality of the Johnson model at all frequencies by simple analysis without Kramers–Kronig relations; the next section is devoted to this aim, the same analysis can be made for Allard and Lafarge model.

6.1. Causality of the Johnson model

Consider the dynamic tortuosity or permeability. The frequency ω can be considered as a complex variable. These functions belong to the general class of functions termed "generalized susceptibilities", a very useful description of their general properties is given in Ref. [59].

The causality is equivalent to the statement that the singularities in the complex ω are located in the lower half plane ($\mathrm{Im}(\omega)<0$), the long wavelength condition which is specific to our problem requires in addition that they are on the imaginary axis (Appendix A of Ref. [44]). The model should satisfy the following conditions: all zeros, poles and branch points of the dynamic tortuosity and compressibility must be in the lower half-plane. The dynamic tortuosity $\varepsilon(\omega)$ and the dynamic permeability $k(\omega)$ are given by

$$\varepsilon(\omega) = \varepsilon_\infty\left(1 + \frac{F(-\mathrm{i}\tilde{\omega})}{-\mathrm{i}\tilde{\omega}}\right), \quad k(\omega) = \frac{k_0}{F(-\mathrm{i}\tilde{\omega}) - \mathrm{i}\tilde{\omega}}$$

with

$$F(-\mathrm{i}\tilde{\omega}) = \left(1 - \frac{M}{2}\mathrm{i}\tilde{\omega}\right)^{1/2},$$

where

$$\tilde{\omega} = \omega\left(\frac{\varepsilon_\infty \rho_f k_0}{\eta\phi}\right) \quad \text{and} \quad M = \frac{8\varepsilon_\infty}{\Lambda^2\phi k_0}.$$

For the model to be acceptable, it must satisfy the following conditions:

- Causality: for every pole, branch point or zero $\tilde{\omega} = \tilde{\omega}_s$, of $\varepsilon(\omega)$ or $k(\omega)$, we must have $\mathrm{Im}(\tilde{\omega}_s)\leqslant 0$.
- Long wavelength: $Re(\tilde{\omega}_s) = 0$.

It is sufficient to consider the singularities of $k(\omega)$:

1. Branch point at $\tilde{\omega} = \tilde{\omega}_s$ such that $1 - (M/2)\mathrm{i}\tilde{\omega}_s = 0$, i.e the frequency $\tilde{\omega} = \tilde{\omega}_s$ is purely imaginary negative. In this case causality and long wavelength condition is respected.
2. Pole at $\tilde{\omega} = \tilde{\omega}_p$, such that

$$F(-\mathrm{i}\tilde{\omega}_p) - \mathrm{i}\tilde{\omega}_p = 0. \tag{74}$$

Setting $z = \mathrm{i}\tilde{\omega}_p$, one has a pole of the permeability (zero of the tortuosity) when z satisfies

$$\sqrt{1 - \frac{M}{2}z} = z. \tag{75}$$

From Eq. (75) z is real and positive. There are no other singularities of the permeability. So, to establish causality, it would suffice that all solutions $\tilde{\omega}_p$ of Eq. (74) have $\mathrm{Im}\,\tilde{\omega}_p\leqslant 0$, i.e., the solutions z of the above equation (75) have $\mathrm{Re}(z)\geqslant 0$.

The long wavelength condition would require, in addition, $\mathrm{Im}(z) = 0$. These conditions are well satisfied and the Johnson et al. model respects the causality and long wavelength condition.

7. Conclusion

In this paper a verification of Kramers–Kronig relations in the Johnson and Allard and Lafarge model is given. High- and low-frequency ranges are considered and the attenuation has been shown to be a power law of frequency dependence in the two ranges of frequencies. Simplified relations of Kramers–Kronig adapted to the case of media having a power law of frequency dependence have been used given a proof that these models are causal at high- and low-frequency range. An experimental validation of the causality of these models at high-frequency range is given. A time domain wave equation and time domain causal theory have been treated also. The attraction of a time domain based approach is that analysis is naturally bounded by the finite duration of ultrasonic pressures and is consequently the most appropriate approach for the transient signal.

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