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Effect of smoothing piecewise-linear oscillators on their stability predictions

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Abstract

The effects of smoothing non-smooth function on estimation of dynamical stability of the periodic response determined in the frequency domain are considered in this paper. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force–displacement relationship subjected to a harmonic force excitation is analyzed. Stability of the periodic response obtained in the frequency domain by the incremental harmonic balance method is determined by using the Floquet–Liapounov theorem. Results obtained are verified in the time domain by the method of piecing the exact solutions and Runge–Kutta integration routine.

1. Introduction

Among the great number of various types of non-linear dynamic systems, a very specific group constitutes non-linear systems described by differential equations which contain discontinuous, i.e., non-smooth restoring or damping force characteristic. Such non-linear systems (for example systems with clearance, Coulomb damper, impacting oscillators, etc.) can be easily described in piecewise manner, i.e., by using piecewise-linear or piecewise non-linear functions. Due to such piecewise representation of non-linear (non-smooth) functions, the governing equations cannot be expressed in closed form and many analytical algorithms based on a local power-series representation of a non-linear function cannot be applied to these equations. Responses (both periodic and aperiodic) of the systems with non-smooth non-linearities can be relatively easily

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determined in the time domain by using digital simulation. But procedures of that kind can be exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions (where many combinations of initial conditions have to be examined for obtaining all possible steady-state solutions) and for lightly damped systems, since a great number of excitation periods must be simulated to obtain a steady-state response. Very efficient methods for solving this type of non-linear differential equations in the frequency domain are multi-harmonic balance methods [1–8]. These methods become exceptionally efficient in combination with path following techniques [9–11] and can be successfully applied to a wide range of non-linear problems. They are very well suited for parametric analysis and bifurcation analysis. Nevertheless, the time and effort needed for analyzing smooth non-linear systems are very often considerably shorter and smaller than for analyzing corresponding non-smooth ones [12,13]. Because of that it could be advantageous to approximate original non-smooth function by a corresponding smoothed one. This could be especially important in bifurcation analysis [9], and for analysis of systems which exhibit chaotic responses [12]. Systems with clearance and Coulomb damper described in piecewise-linear manner and corresponding systems described by smoothed functions (by hyperbolic tangent and sigmoid functions) are considered by Lok and Wiercigroch [12], Wiercigroch and Sin [14], and Wiercigroch [13]. A very good agreement of the time-domain responses obtained for discontinuous and smoothed systems was achieved, even for the cases of chaotic responses (for certain ranges of parameters). Narayanan and Sekar [9] presented a path following an algorithm based on a predictor-corrector method which enables the bifurcation analysis. It is known that applying that method to discontinuous systems can cause considerable difficulties, and in that case the authors recommend replacing the non-smooth function by corresponding smoothed one. Moreover it is also observed [15,16] that even neglecting the very small harmonic terms of a response, which do not significantly influence the root mean square (r.m.s.) and are small in comparison to other terms of the spectrum, can lead to incorrect prediction of dynamical stability of the solution, i.e., that estimation of dynamical stability could be an extremely sensitive procedure. Accordingly, it could seem reasonable to examine the effects of smoothing non-smooth function on estimation of dynamical stability of the periodic solution. For that purpose, a simple single-degree-of-freedom piecewise-linear system subjected to a harmonic force excitation is analyzed in this paper. In the frequency domain, periodic solutions are obtained by the incremental harmonic balance method (IHBM). Their stability is estimated by the Floquet–Liapounov theorem [17,18]. The results obtained are verified in the time domain by the method of piecing the exact solutions (MPES) and by Runge–Kutta fourth and fifth order numerical integration routines.

2. Model of a mechanical system with a clearance

Model of a simple mechanical system with clearance is shown in Fig. 1. It consists of an inertia element m, a linear viscous damper c, and a non-linear elastic element $kg(x)$. The non-linear elastic element is defined by a piecewise-linear function $g(x)$ and a coefficient k. The piecewise-linear function $g(x)$ and its derivative are shown in Fig. 2. b denotes one-half of the clearance space. When the system is excited by a periodic harmonic force $F(t)$, the motion of the system can be

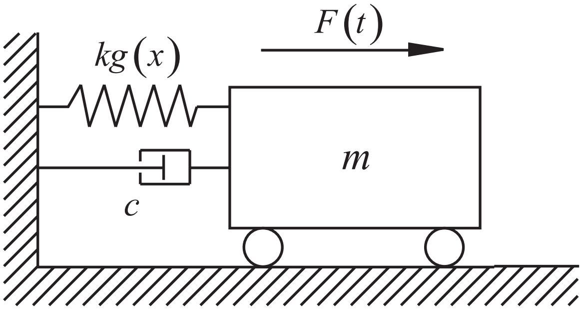


Fig. 1. Model of vibration system.

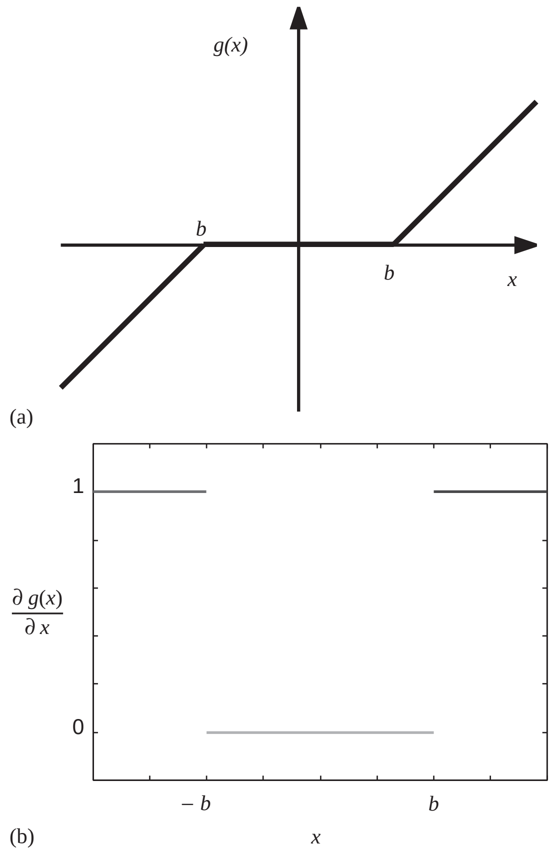


Fig. 2. Non-linear function $g(x)$ (a) and its derivative (b).

described by the non-linear differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kg(x) = F(t) = f_0 + f_C\cos(\Omega t) + f_S\sin(\Omega t), \tag{1}$$

where f_0 represents mean transmitted force, f_C and f_S are force component amplitudes of the corresponding harmonic terms, and Ω is the excitation frequency. Since the procedure of prediction of the dynamical stability is based on derivative of a non-linear function, expressions for non-linear functions and their derivatives are given. For original piecewise-linear function

$$g_1(x) = h^*(x - b^*), \quad (2)$$

$$\frac{\partial g_1(x)}{\partial x} = h^*, \quad (3)$$

where

$$h^* = \left\{ \begin{matrix} 1, & b < x \\ 0, & -b \leqslant x \leqslant b \\ 1, & x < -b \end{matrix} \right\}, \quad b^* = \left\{ \begin{matrix} b, & b < x \\ 0, & -b \leqslant x \leqslant b \\ -b, & x < -b \end{matrix} \right\}. \quad (4)$$

For hyperbolic tangent smoothing function

$$g_2(x) = \tfrac{1}{2}[2x + (x + b)\tanh(-h(x + b)) + (x - b)\tanh(h(x - b))], \quad (5)$$

$$\frac{\partial g_2(x)}{\partial x} = \frac{1}{2}\left[2 + \tanh(-h(x + b)) + \tanh(h(x - b)) - \frac{h(x + b)}{\cosh^2(-h(x + b))} + \frac{h(x - b)}{\cosh^2(h(x - b))}\right] \quad (6)$$

and for sigmoid smoothing function

$$g_3(x) = \frac{(x + b)}{1 + e^{a(b+x)}} + \frac{(x - b)}{1 + e^{a(b-x)}}, \quad (7)$$

$$\frac{\partial g_3(x)}{\partial x} = \frac{1 + e^{a(b+x)}(1 - a(x + b))}{(1 + e^{a(b+x)})^2} + \frac{1 + e^{a(b-x)}(1 + a(x - b))}{(1 + e^{a(b-x)})^2}, \quad (8)$$

where h and a are control parameters in Eqs. (5)–(8). The accuracy of the approximation increases with increasing h and a. The hyperbolic tangent smoothing approximation $g_2(x)$ and its derivative $\partial g_2(x)/\partial x$ is shown in Fig. 3 for certain values of h. From Figs. 2a and 3a, one can see that piecewise-linear and hyperbolic tangent functions exhibit a very good agreement for large values of the control parameter h. Figs. 2b and 3b show that the maximum value of disagreement between $\partial g_1(x)/\partial x$ (i.e. $\partial g(x)/\partial x$) and $\partial g_2(x)/\partial x$ is unchanged irrespective of the increase of h. Increasing of h causes only decreasing of the width of their significant disagreement region. The same situation holds for sigmoid smoothing function $g_3(x)$ also. But, in the case of sigmoid smoothing function, magnitudes of numerator and denominator of (8) can be extremely large for large values of a. For example, values $a \geqslant 240$ and $x \geqslant 2$ cause overflow. Therefore, only the approximation of piecewise-linear function $g_1(x)$ by hyperbolic tangent smoothing function $g_2(x)$ is used in this paper. Extensive analysis of similar systems is performed by Lok and Wiercigroch [12], Wiercigroch and Sin [14] and Wiercigroch [13].

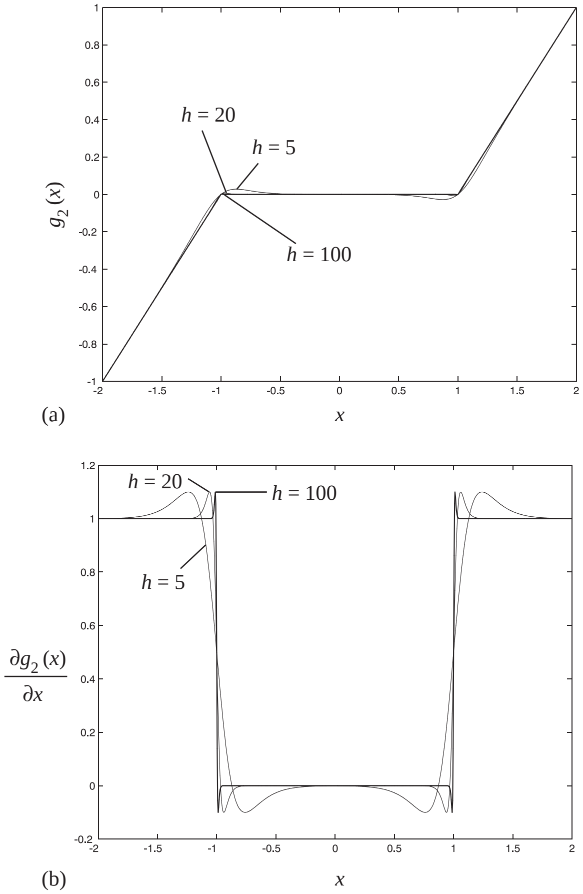


Fig. 3. Hyperbolic tangent smoothing function (a) and its derivative (b) for $b = 1$ and different values of h.

3. Short description of the applied methods

3.1. Incremental harmonic balance method (IHBM)

By introducing a non-dimensional time θ as a new independent variable, the differential equation (1) can be rewritten in the non-dimensional form

$$\frac{\eta^2}{\nu^2}\frac{d^2\bar{x}}{d\theta^2} + \frac{2\zeta\eta}{\nu}\frac{d\bar{x}}{d\theta} + g(\bar{x}) = \bar{f}_0 + \bar{f}_C\cos(\nu\theta) + \bar{f}_S\sin(\nu\theta), \tag{9}$$

where

$$\bar{x} = \frac{x}{l}, \quad \bar{b} = \frac{b}{l}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_0}, \quad \bar{f}_C = \frac{f_C}{ml\omega_0^2}, \quad \bar{f}_S = \frac{f_S}{ml\omega_0^2},$$

$$\eta = \frac{\Omega}{\omega_0}, \quad \tau = \omega_0 t, \quad \Omega t = \eta\tau = \nu\theta.$$

In this way, the period of the response (with ν subharmonics taken in consideration) is always 2π, making it possible (by using the IHBM) to consider any number of superharmonics and subharmonics included in the supposed approximate solution. Any characteristic dimension of the system is denoted by l here.

Supposed approximate solution is given by

$$\bar{x} = \sum_{i=0}^{N} a_i \cos i\theta + b_i \sin i\theta = [\mathbf{T}]\{\mathbf{a}\}, \tag{10}$$

where

$$[\mathbf{T}] = [1,\ \cos\theta,\ \cos 2\theta, ..., \cos N\theta,\ \sin\theta,\ \sin 2\theta, ..., \sin N\theta],$$

$$\{\mathbf{a}\} = [a_0, a_1, ..., a_N,\ b_1, b_2, ..., b_N]^{\mathrm{T}}.$$

$N = \nu K$ represents the number of all harmonics included in the supposed solution, ν is the number of subharmonics and K is the number of superharmonics. By applying this method, which consists of two basic steps: incrementation and Galerkin's procedure, the non-linear differential equation (9) is transformed into the system of $2N + 1$ linearized incremental algebraic equations

$$[\mathbf{k}]^j\ \{\Delta\mathbf{a}\}^{j+1} = \{\mathbf{r}\}^j, \tag{11}$$

$$\{\mathbf{a}\}^{j+1} = \{\mathbf{a}^j\} + \{\Delta\mathbf{a}\}^{j+1}, \tag{12}$$

with Fourier coefficients ($a_0,\ a_i,\ b_i,\ i = 1, \ldots, N$) as unknowns. In Eqs. (11) and (12) j is number of iterations. In each incremental step, only linear (i.e., linearized) algebraic equations have to be formed and solved. The comprehensive description of the method, its application to piecewise-linear systems and the way of determining elements of Jacobian matrix $[\mathbf{k}]$ and the corrector $\{\mathbf{r}\}$ in explicit form is given by Wong et al. [4]. In the case of hyperbolic tangent or sigmoid smoothing approximation of a piecewise-linear function, some elements of $[\mathbf{k}]$ and $\{\mathbf{r}\}$ can be determined only by numerical integration.

3.2. The method of piecing exact solutions (MPES)

The force–displacement relationship $g(x)$ (Fig. 2a) is piecewise linear. Local solutions of the differential equations (1) are known explicitly inside each of the stage stiffness, and can be repeatedly matched at $x = b$ and $-b$, to obtain a global solution of (1). Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in a closed form. But the matching of local solutions can be numerically done very easily. The only approximation made by applying this procedure is in the precision of numerical determination of the times in which the system changes stiffness region ($x = b$, $-b$). The comprehensive description of the method is given in Refs. [19,16].

4. The stability of the steady-state solution

When the periodic solution is obtained, the stability of the given solution can be determined by examining the perturbed solution $\bar{x}^*$:

$$\bar{x}^* = \bar{x} + \Delta\bar{x}^*, \tag{13}$$

where $\Delta\bar{x}^*$ is a small perturbation of a periodic solution $\bar{x}$. By substitution of Eq. (13) in (9), and after expanding non-linear function $g(\bar{x})$ in a Taylor's series about the periodic solution with neglecting non-linear incremental terms, one obtains linear homogeneous differential equation with time changing periodic coefficients $\partial g(\bar{x})/\partial\bar{x}$:

$$\frac{\eta^2}{\nu^2}\frac{\mathrm{d}^2\Delta\bar{x}^*}{\mathrm{d}\theta^2} + \frac{2\zeta\eta}{\nu}\frac{\mathrm{d}\Delta\bar{x}^*}{\mathrm{d}\theta} + \frac{\partial g(\bar{x})}{\partial\bar{x}}\Delta\bar{x}^* = 0. \tag{14}$$

When the steady-state solution $\bar{x}(\theta)$ is determined, the values of $\partial g(\bar{x})/\partial\bar{x}$ are known inside a period of the response. A very efficient and very often used method for determining the stability of the periodic solution is based on the Floquet–Liapounov theorem [17,18]. For that purpose Eq. (14) can be rewritten in the state variable form as

$$\left\{\frac{\mathrm{d}\bar{\mathbf{X}}^*}{\mathrm{d}\theta}\right\} = [\mathbf{A}(\theta)]\{\bar{\mathbf{X}}^*\}, \tag{15}$$

where

$$\{\bar{\mathbf{X}}^*\} = \left\{\begin{matrix}\Delta\bar{x}^* \\ \mathrm{d}\Delta\bar{x}^*/\mathrm{d}\theta\end{matrix}\right\}, \quad \left\{\frac{\mathrm{d}\bar{\mathbf{X}}^*}{\mathrm{d}\theta}\right\} = \left\{\begin{matrix}\mathrm{d}\Delta\bar{x}^*/\mathrm{d}\theta \\ \mathrm{d}^2\Delta\bar{x}^*/\mathrm{d}\theta^2\end{matrix}\right\}, \quad [\mathbf{A}(\theta)] = \begin{bmatrix} 0 & 1 \\ -\dfrac{\nu^2}{\eta^2}\left(\dfrac{\partial g(\bar{x})}{\partial\bar{x}}\right) & -\dfrac{2\nu\zeta}{\eta}\end{bmatrix}. \tag{16}$$

Since the matrix $[\mathbf{A}(\theta)]$ is a periodic function of θ with the period 2π, the stability criteria are related to the eigenvalues of the monodromy matrix which is defined as the state transition matrix at the end of one period. According to the Floquet–Liapounov theorem, the solution is stable if all the moduli of the eigenvalues of the monodromy matrix are less than unity. Otherwise the solution is unstable. Generally, it is not possible to derive an analytic expression for the transition matrix. But, if the non-linear force–displacement relationship is piecewise linear, its derivative $(\partial g_1(x)/\partial x = h^*)$ is, according to (4), constant inside each of the intervals $[\theta_i, \theta_{i+1}]$ (Fig. 4).

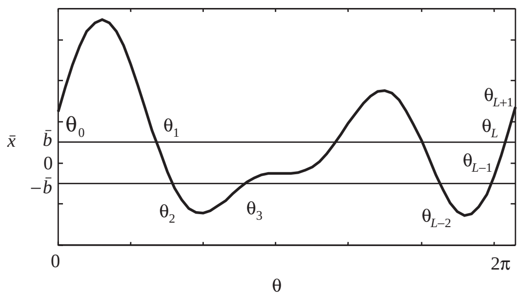


Fig. 4. Solutions of the equations $\bar{x} = \bar{b}$ and $-\bar{b}$.

Consequently, $[\mathbf{A}(\theta_i, \theta_{i+1})]$ is also a constant matrix inside that interval. According to D'Souza and Garg [20], for the constant $[\mathbf{A}(\theta_i, \theta_{i+1})]$ (inside the interval $[\theta_i, \theta_{i+1}]$), transition matrix $[\mathbf{\Phi}(\theta_{i+1}, \theta_i)]$ can be expressed as

$$[\mathbf{\Phi}(\theta_{i+1}, \theta_i)] = \mathrm{e}^{[\mathbf{A}(\theta_i, \theta_{i+1})]\ (\theta_{i+1}-\theta_i)} \tag{17}$$

and for the whole interval $[0, 2\pi]$ according to Wong et al. [4] one obtains

$$[\mathbf{\Phi}(2\pi, 0)] = \prod_{i=0}^{L} \mathrm{e}^{[\mathbf{A}(\theta_i, \theta_{i+1})](\theta_{i+1}-\theta_i)}. \tag{18}$$

Besides the accuracy of numerical determination of times θ_i in which the system changes stage stiffness region ($\bar{x} = \bar{b}$, $-\bar{b}$), the only approximation occurring in this procedure is the accuracy of computation of the matrix exponential $\mathrm{e}^{[\mathbf{A}(\theta)](\theta_{i+1}-\theta_i)}$ and the product of matrix exponentials $\prod_{i=0}^{L} \mathrm{e}^{[\mathbf{A}(\theta_i, \theta_{i+1})]\ (\theta_{i+1}-\theta_i)}$. The accuracy of determination of θ_i is arbitrary, i.e., it depends only on numerical precision of the computer used. To evaluate the matrix exponential and the product of matrix exponentials as accurately as possible, the algorithms recommended by Cardona et al. [10] are used in this paper. If a non-linear force–displacement relationship $g(x)$ is approximated by a continuous non-linear function, its derivative is a time changing function, and, consequently, $[\mathbf{A}(\theta)]$ is then a time-changing matrix. So, the monodromy matrix cannot be obtained in the previously described way, i.e., by using (17) and (18). Among various methods of approximating monodromy matrix, Friedman et al. [21] concluded that the most efficient procedure is one proposed by Hsu and Cheng [22], i.e., to approximate the periodic matrix $[\mathbf{A}(\theta)]$ by a series of step functions. For that purpose a period of the response (2π) is divided into M equal intervals $\Delta\theta = 2\pi/M$. Inside each of the intervals, the time changing matrix $[\mathbf{A}(\theta)]$ is replaced by its average value, i.e., by a constant matrix $[\mathbf{A}_j]$, $j = 1, 2, \ldots, M$.

For the jth interval, the transition matrix can be expressed as

$$[\mathbf{\Phi}_j] = \mathrm{e}^{[\mathbf{A}_j]\Delta\theta} \tag{19}$$

and for the whole period of the response $[0, 2\pi]$ as

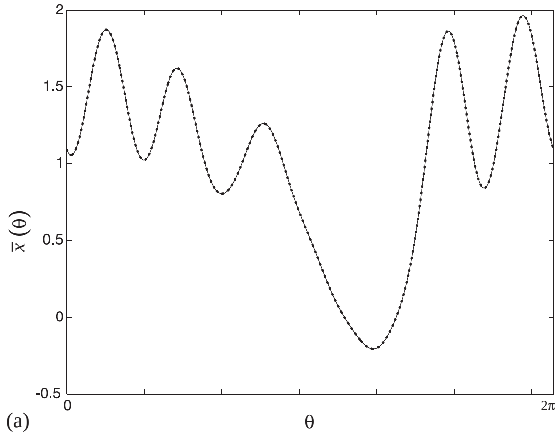
$$[\mathbf{\Phi}(2\pi, 0)] = \prod_{j=1}^{M} \mathrm{e}^{[\mathbf{A}_j]\Delta\theta}. \tag{20}$$

For numerical evaluation of (19) and (20), the algorithms recommended by Cardona et al. [10] are used, i.e., the same ones as for the evaluation of (17) and (18).

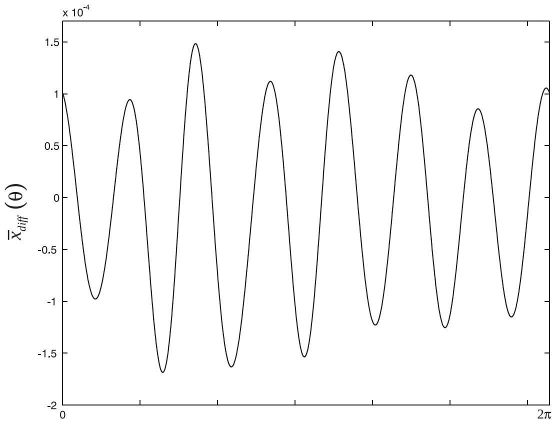
5. Results

Comparison of the responses of vibration system described by $\bar{b} = 1$, $\zeta = 0.03$, with excitation parameters $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$ at $\eta = 0.147$, obtained for the original piecewise-linear function $g_1(\bar{x})$ and its hyperbolic tangent smoothing approximation $g_2(\bar{x})$ for the control parameter $h = 50$ is shown in Fig. 5a. Fig. 5b shows difference of this two responses

$$\bar{x}_{diff} = \bar{x}_{g1} - \bar{x}_{g2}, \tag{21}$$



(a)



(b)

Fig. 5. Comparison of responses obtained for the original piecewise-linear (——) and hyperbolic tangent smoothing function (●) (a), and their difference (b), for $h = 50$, $\bar{b} = 1$, $\zeta = 0.03$, $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$ at $\eta = 0.147$.

where $\bar{x}_{g1}$ denotes the response obtained by original piecewise-linear function $g_1(\bar{x})$ and $\bar{x}_{g2}$ denotes the response obtained by hyperbolic tangent smoothing function $g_2(\bar{x})$ for $h = 50$. One can see an expected very good agreement of the obtained responses. Fig. 6 shows results of dynamical stability estimation depending on control parameter h and the number M of equal intervals $\Delta\theta$ (inside which the time changing matrix $[\mathbf{A}(\theta)]$ is replaced by a constant matrix $[\mathbf{A}_j]$ ($j = 1, 2, \ldots, M$)). "×" denotes solutions that are estimated as stable, and blank space denotes solutions that are estimated as unstable. For the system considered, the frequency-domain response should be estimated as stable because the corresponding periodic solution is obtained also in the time-domain (for piecewise-linear function by MPES and for hyperbolic tangent smoothing function by Runge–Kutta fourth and fifth order numerical integration routine). Fig. 6 shows that estimation of dynamical stability depends both on control parameter h and the number M of intervals $\Delta\theta$. "*of*" denotes results of stability estimation depending on M when stability of the response of original piecewise-linear system is estimated by using (19) and (20) instead (17)

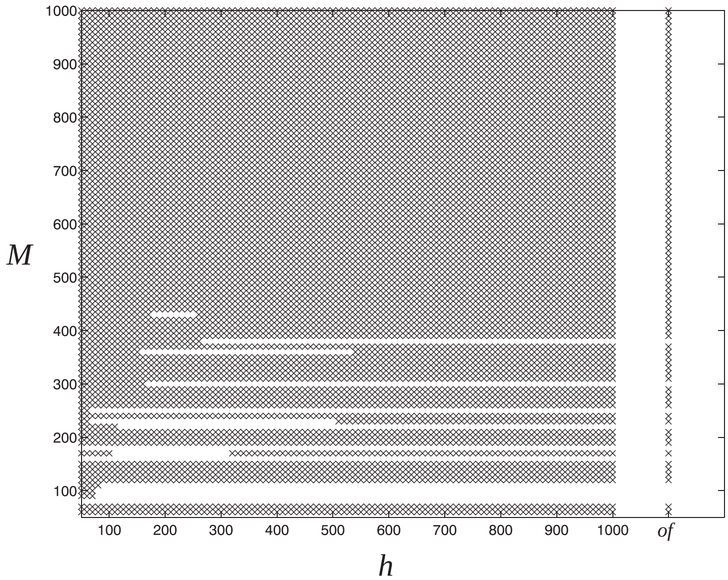


Fig. 6. The stability of the response for $\bar{b} = 1$, $\zeta = 0.03$, $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$ at $\eta = 0.147$.

and (18), i.e., when the procedure for continuous function is applied to original piecewise-linear function. One can see excellent agreement of these results and the results obtained by using hyperbolic tangent smoothing function $g_2(\bar{x})$ for $h = 1000$. This implies that if h is large enough, estimation of dynamical stability depends more upon the number of intervals $\Delta\theta$ than upon the control parameter h. The next example refers to a response of the same vibration system, but at $\eta = 0.50086$. One of eigenvalues of monodromy matrix obtained for the original piecewise-linear function (obtained by using (17) and (18)) is very close to unity ($|\lambda_{max}| = 0.9998$), i.e., the system is near a bifurcation. Fig. 7 shows dependence of stability estimation on the control parameter h and the number M of intervals $\Delta\theta$. It differs from the results obtained for $\eta = 0.147$ ($|\lambda_{max}| = 0.2774$) shown in Fig. 6, for $\eta = 0.50086$ increasing of both h and M does not increase frequency of appearing stable solutions. Fig. 7c shows the dependence of dynamical stability inside the region $10 \leqslant h \leqslant 1\,000\,000$, $500 \leqslant M \leqslant 30\,000$. It is very interesting that incorrect estimations occur even for cases $h = 1\,000\,000$ and $M \geqslant 25\,000$, as well as in the case when the procedure for continuous function is applied to original piecewise-linear function (results denoted by "*of*"). One could suspect that incorrect estimations of dynamical stability are not caused by insufficient number of intervals $\Delta\theta$ but by numerical inaccuracy of evaluation of monodromy matrix (Section 4). To examine what is the source of such incorrect stability estimations (the algorithm of evaluating monodromy matrix or insufficient number of intervals $\Delta\theta$), the stability of the system with original piecewise-linear function is analyzed by (17) and (18). For that purpose, the precision of determining times θ_i in which the system changes stage stiffness region is varied. This procedure causes similar effect as varying the number M of intervals $\Delta\theta$. Here the "equivalent number of intervals M_{ekv}" corresponds to the number of intervals $\Delta\theta$ needed for making the width of the interval $\Delta\theta$ to be equal as the specified largest error ε, permitted in the procedure of determining times θ_i, i.e., $M_{ekv} = 2\pi/\varepsilon = 2\pi/\Delta\theta$. Fig. 8 shows stability estimation results inside regions

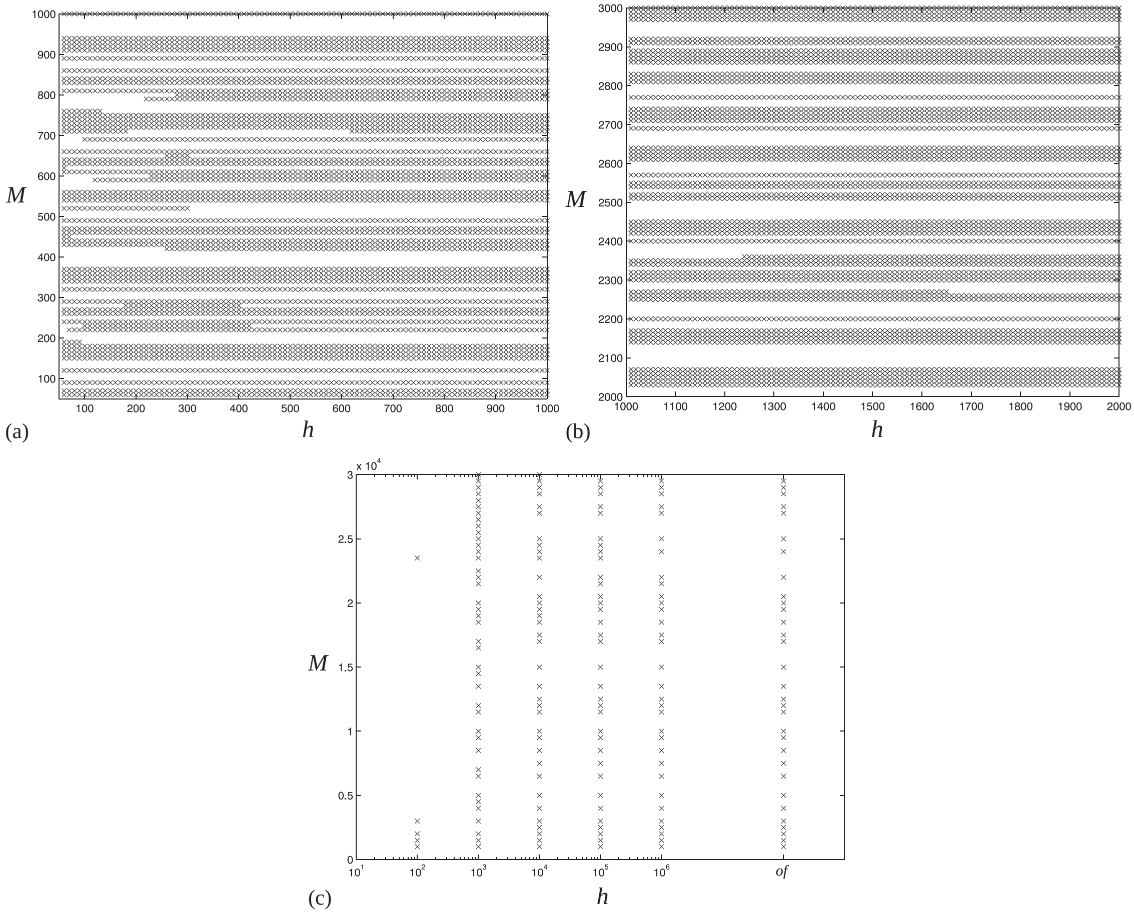


Fig. 7. The stability of the response for $\bar{b} = 1$, $\zeta = 0.03$, $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$ at $\eta = 0.50086$.

$2\pi \times 10^3 \leqslant M_{ekv} \leqslant 2\pi \times 10^8$ ($\varepsilon_{max} = 10^{-3}$, $\varepsilon_{min} = 10^{-8}$), with an increment $\Delta M_{ekv} = 2\pi \times 10^5$ and $2\pi \times 10^2 \leqslant M_{ekv} \leqslant 2\pi \times 10^5$ ($\varepsilon_{max} = 10^{-2}$, $\varepsilon_{min} = 10^{-5}$) with an increment $\Delta M_{ekv} = 2\pi \times 10^2$. These results show that incorrect stability estimation (Fig. 7c) is caused by insufficient number of intervals $\Delta\theta$, i.e., that near a bifurcation the procedure of determining monodromy matrix can be very sensitive to numerical accuracy of determining θ_i, and in this way, on the number of intervals $\Delta\theta$. One could also conclude that incorrect prediction of the dynamical stability of the solution is caused not as much by smoothing itself (in the case of sufficiently large values of smoothing control parameter), but by sensitivity of the procedure of evaluation of monodromy matrix of non-linear systems with discontinuous (or rapidly changing) first derivative of the force–displacement relationship. It is worth emphasizing that these effects could be especially important in bifurcation analysis. On the other hand, one's attention should be directed to another effect [15,16], which in contrast to the previously described ones, very significantly influences determination of dynamical stability, regardless whether the system is near a bifurcation or not. Fig. 9 shows a spectrum of the time-domain response of the considered system with

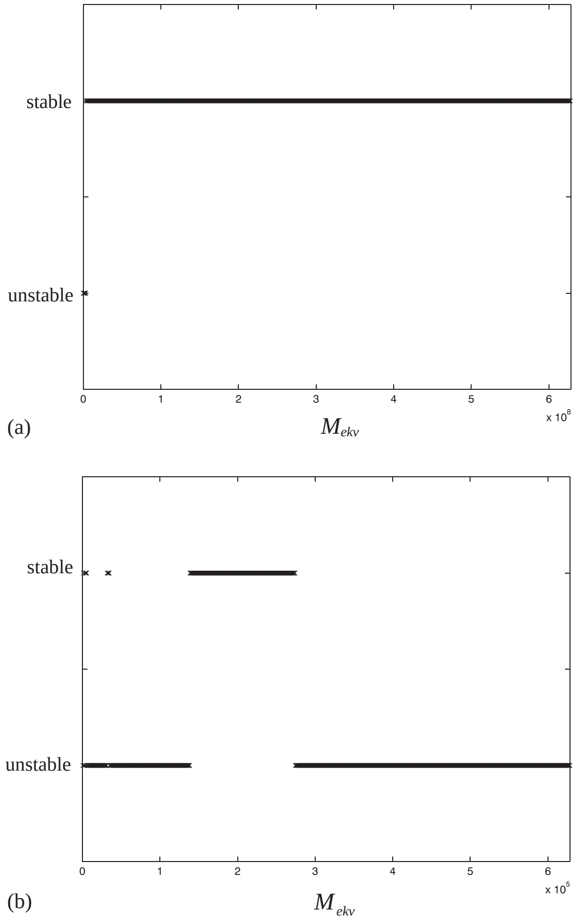


Fig. 8. The stability of the response for the original piecewise-linear system in dependence on numerical precision of determining θ_i ($\bar{b} = 1$, $\zeta = 0.03$, $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$ at $\eta = 0.50086$).

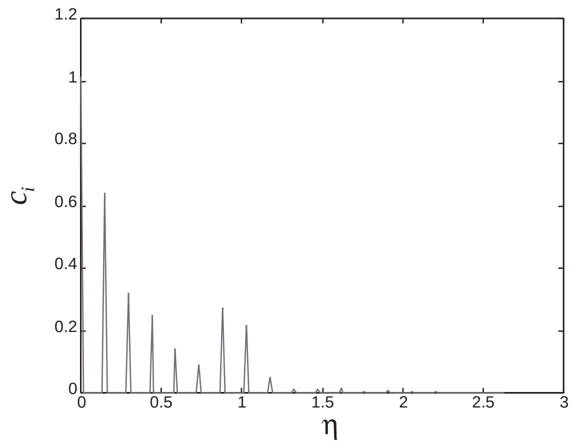


Fig. 9. The spectrum of the time-domain response at $\eta = 0.147$ ($c_i = \sqrt{a_i^2 + b_i^2}$).

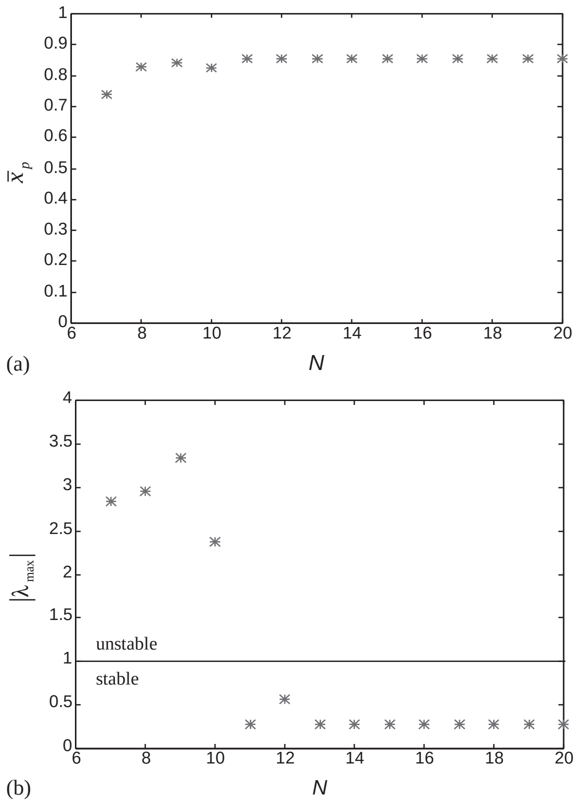


Fig. 10. The effective amplitude $\bar{x}_p$ (obtained by IHBM) (a) and maximum modulus of the eigenvalues of the corresponding monodromy matrix (b) in dependence on the number of harmonics N included in the supposed solution at $\eta = 0.147$.

piecewise-linear force–displacement relationship (obtained by MPES) for $\eta = 0.147$. Fig. 10 shows the dependence of effective amplitude $\bar{x}_p$ (obtained by IHBM) and maximum modulus of the eigenvalues of the corresponding monodromy matrix $|\lambda_{max}|$, on the number of harmonics N included in the supposed approximate solution. One can see (Fig. 10) that even neglecting very small harmonic terms in the actual time-domain response (11th harmonic in the spectrum whose amplitude is 2.8% of the amplitude of the first harmonic) causes significant error in determination of the eigenvalues of the monodromy matrix. Similar situation is shown in Figs. 11 and 12 for $\eta = 0.5243$. In this case, neglecting the 20th harmonic in the spectrum, whose amplitude is only 0.015% of the amplitude of the largest (second) harmonic, causes an incorrect prediction of the dynamical stability of the solution.

6. Conclusion

The influence of smoothing non-smooth functions on estimation of dynamical stability of the periodic response determined in the frequency domain is considered in this paper. For that

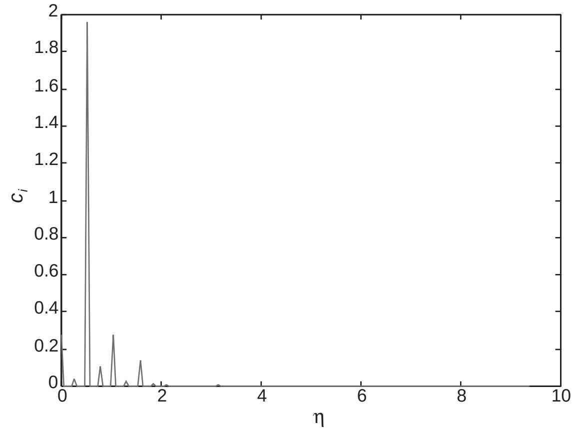


Fig. 11. The spectrum of the time-domain response at $\eta = 0.5243$ ($c_i = \sqrt{a_i^2 + b_i^2}$).

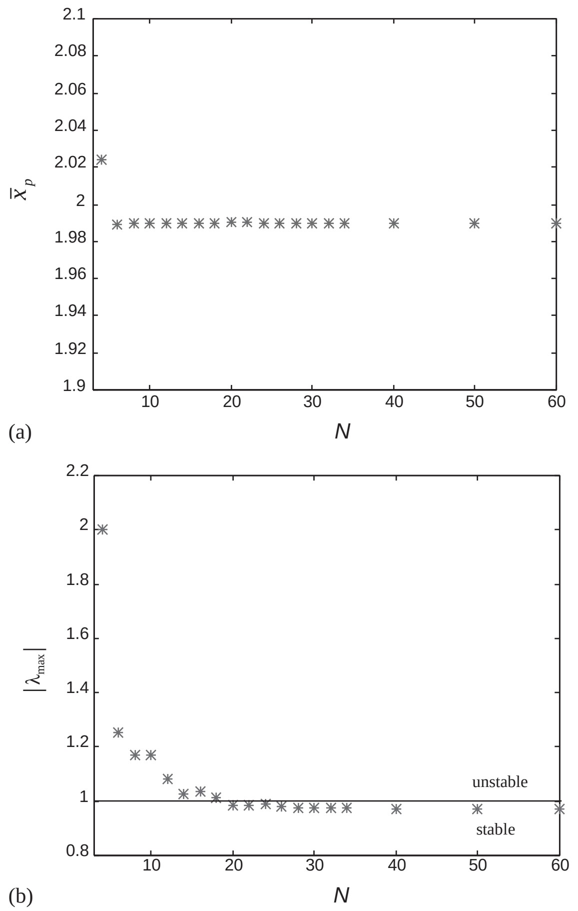


Fig. 12. The effective amplitude $\bar{x}_p$ (obtained by IHBM) (a) and maximum modulus of the eigenvalues of the corresponding monodromy matrix (b) in dependence on the number of harmonics N included in the supposed solution at $\eta = 0.5243$.

purpose, a simple single-degree-of-freedom system with piecewise-linear force–displacement relationship subjected to a harmonic force excitation is analyzed. The considerable advantage of using this piecewise-linear model is in the possibility of expressing monodromy matrix exactly as a product of matrix exponentials, which is not possible for a general non-linear function. In this way, the inaccuracy of evaluating monodromy matrix can be caused only by insufficient precision of numerical determination of the times in which the system changes stage stiffness region, and by numerical procedures of evaluation matrix exponential and product of matrix exponentials. Based upon the obtained results one can conclude that incorrect prediction of dynamical stability of the response of the systems with smoothed function is caused not as much by smoothing itself (if a smoothing control parameter is sufficiently large) but by sensitiveness of the procedure of evaluation of the monodromy matrix. If the system is close to a bifurcation, the stability estimation can be an extremely sensitive procedure and large values of smoothing functions control parameter are required, as well as a very large number of step functions used in approximate determining of the monodromy matrix by Hsu's procedure. In that case, the required large smoothing function control parameter can lead to overflow if sigmoid smoothing function is used, and hyperbolic tangent smoothing function is estimated as a better solution than sigmoid smoothing function. It is also shown that even neglecting very small harmonic terms of actual time-domain response (which in-significantly influence the r.m.s values of the response and are small in comparison to other terms of the spectrum) can cause a very large error in evaluation of the eigenvalues of the monodromy matrix, and can lead to incorrect prediction of the dynamical stability of the solution, regardless of whether the system is close to a bifurcation or not. These results are interesting not only in the case of smoothing a non-smooth function, but also whenever the stability of the periodic response of systems with discontinuous (or with rapidly changing) first derivative of the force–displacement relationship is estimated by Floquet–Liapounov theorem.

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