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Analysis of non-linear dynamics of a two-degree-of-freedom vibration system with non-linear damping and non-linear spring

S.J. Zhu[a], Y.F. Zheng[b,*], Y.M. Fu[b]

[a] Institute of Noise and Vibration, Naval University of Engineering, Wuhan, Hubei, People's Republic of China
[b] Department of Engineering Mechanics, Hunan University, Changsha, Hunan, People's Republic of China

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Abstract

In this paper, the non-linear dynamics of a two-degree-of-freedom vibration system with non-linear damping and non-linear spring is studied. The analytic results show that the purposes of reducing amplitude and oscillation can be realized by adjusting properly the system parameters and considering the value of exciting frequency. The conclusions can provide some available evidences for the design and improvement of the non-linear absorber. Numerical simulations show the system exhibits periodic motions, quasiperiodic motions and chaotic motions.

1. Introduction

In the domain of mechanical vibration research, dynamic absorbers have extensive application in reducing vibrations of machinery. Whiston [1] studied the non-linear response of a one-dimensional oscillator system excited by harmonic excitation. Natsiavas [2] discussed the steady state response for a class of strongly non-linear multiple-degree-of-freedom oscillators, where the vibration absorber is modelled as a mass with linear damping and restoring force. Vakakis and Paipetic [3] investigated the effect of a viscously damped dynamic absorber on an undamped multi-degree-of-freedom vibrating system. Many researchers have also used the non-linear absorbers. Soom [4] and Jordanov [5] studied the optimal parameter design of linear and non-linear dynamic vibration absorbers for damped primary systems. They examined optimization criteria other than the traditional one and obtained small improvements in steady state response

*Corresponding author.
E-mail address: yfzheng2000@163.net (Y.F. Zheng).

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by using non-linear springs. However, the presence of the non-linearities introduces dangerous instabilities, which in some cases may result in amplification rather than reduction of the vibration amplitudes [6,7]. Natsiavas [8] applied the method of averaging to investigate the steady state oscillations and stability of non-linear dynamic vibration absorbers. He pointed out that proper selection of the system parameters would result in substantial improvements of non-linear absorbers and avoid dangerous effects that are likely to occur due to the presence of the non-linearities. Oueini et al. [9,10] exploited the saturation phenomenon in devising an active vibration suppression technique. They introduced a second order absorber and coupled it with the plant through a user-defined quadratic and cubic feedback control law. All the above studies which are based on the absorber are composed of linear damper, linear or non-linear spring and mass, not considering the effect of the non-linear damper. The present investigation is to study the non-linear dynamic behaviour of a two-degree-of-freedom vibration system with non-linear damping and non-linear spring. And by numerical integration, periodic motions, quasiperiodic motions and chaotic motions of the system are discussed by tracing the bifurcation diagram.

2. Basic equations

A model of a two-degree-of-freedom oscillator under consideration is shown in Fig. 1. The co-ordinate x_1 represents the displacement of the main mass m_1 with respect to its foundation, while x_2 stands for the relative displacement of the absorber mass m_2 with respect to the mass m_1. The absorber is excited by a harmonic force p and $p = p_0 \cos \omega t$, where ω is the external exciting frequency. The system is attached by means of non-linear springs and non-linear dampers. Then, the non-linear spring restoring forces and damping forces can be expressed through

$$\begin{aligned} f_i(x_i) &= k_i x_i + k_i' x_i^3, \quad i = 1,2, \\ R_i(x_i) &= c_i \dot{x}_i + c_i' x_i^2 \dot{x}_i, \quad i = 1,2. \end{aligned} \tag{1}$$

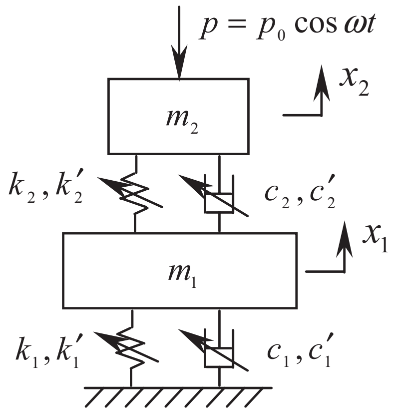

Fig. 1. Basic model.

By the principle of D'Alembert, the governing equations of motion for the system are

$$
\begin{aligned}
&(m_1 + m_2)\ddot{x}_1 + m_2\ddot{x}_2 + f_1(x_1) + R_1(x_1, \dot{x}_1) = -p_0 \cos \omega t, \\
&m_2\ddot{x}_1 + m_2\ddot{x}_2 + f_2(x_1) + R_2(x_2, \dot{x}_2) = -p_0 \cos \omega t.
\end{aligned} \tag{2}
$$

Suppose that $x_c = p_0/k_1$ and introduce the following dimensionless parameters:

$$
\begin{aligned}
&\tau = \omega t, \quad \mu = m_2/m_1, \quad \bar{\omega}_i^2 = k_i/m_i, \quad \omega_i = \bar{\omega}_i/\omega, \quad \eta_i = \omega_i^2, \quad \zeta_i = c_i/2\sqrt{k_i m_i}, \\
&\xi_i = 2\zeta_i\omega_i, \quad y_i = x_i/x_c, \quad \delta = \omega_2/\omega_1, \quad \Omega = \omega/\bar{\omega}_1, \quad \bar{\alpha}_i = k_i' x_c^2/k_i, \quad \alpha_i = \bar{\alpha}_i\omega_i^2, \\
&\bar{\beta}_i = c_i' x_c^2/\sqrt{k_i m_i}, \quad \beta_i = \bar{\beta}_i\omega_i, \quad \bar{\lambda} = p_0/m_1\bar{\omega}_1^2 x_c, \quad \lambda = \bar{\lambda}\omega_1^2,
\end{aligned}
$$

where ω_i is the non-dimensional frequency and $\bar{\omega}_i$ $(i = 1, 2)$ are the natural frequencies of the main mass and the absorber, respectively. Now the dimensionless governing equations of motion of the system are obtained and can be written as

$$
\begin{aligned}
&(1 + \mu)\ddot{y}_1 + \mu\ddot{y}_2 + \xi_1\dot{y}_1 + \beta_1 y_1^2\dot{y}_1 + \omega_1^2 y_1 + \alpha_1 y_1^3 = -\lambda \cos \tau, \\
&\mu\ddot{y}_1 + \mu\ddot{y}_2 + \mu\xi_2\dot{y}_2 + \mu\beta_2 y_2^2\dot{y}_2 + \mu\omega_2^2 y_1 + \mu\alpha_2 y_2^3 = -\lambda \cos \tau.
\end{aligned} \tag{3}
$$

The above formula can be put in matrix form:

$$
[M][\ddot{y}] + [C][\dot{y}] + [K][y] = [f], \tag{4}
$$

where

$$
[y] = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad [M] = \begin{bmatrix} 1 + \mu & \mu \\ \mu & \mu \end{bmatrix}, \quad [C] = \begin{bmatrix} \xi_1 & 0 \\ 0 & \mu\xi_2 \end{bmatrix},
$$

$$
[K] = \begin{bmatrix} \eta_1 & 0 \\ 0 & \mu\eta_2 \end{bmatrix}, \quad [f] = \begin{bmatrix} -\lambda \cos \tau - \beta_1 y_1^2 \dot{y}_1 - \alpha_1 y_1^3 \\ -\lambda \cos \tau - \mu\beta_2 y_2^2 \dot{y}_2 - \mu\alpha_2 y_2^3 \end{bmatrix}.
$$

Applying the method of averaging, the steady state responses are assumed as

$$
\begin{aligned}
y(\tau) &= u(\tau)\cos \tau + v(\tau)\sin \tau, \\
\dot{y}(\tau) &= -u(\tau)\sin \tau + v(\tau)\cos \tau,
\end{aligned} \tag{5}
$$

where $u(\tau) = [u_1(\tau), u_2(\tau)]^{\mathrm{T}}$, $v(\tau) = [v_1(\tau), v_2(\tau)]^{\mathrm{T}}$ are assumed to be slow functions about the time τ. Differentiating the first formula of Eq. (5) with respect to the time τ, we obtain

$$
\dot{y}(\tau) = \dot{u}(\tau)\cos \tau - u(\tau)\sin \tau + \dot{v}(\tau)\sin \tau + v(\tau)\cos \tau. \tag{6}
$$

Substituting the second formula in Eq. (5) into Eq. (6), the resulting equation is

$$
\dot{u}(\tau)\cos \tau + \dot{v}(\tau)\sin \tau = 0. \tag{7}
$$

Also differentiating the second formula of Eq. (5), we obtain

$$
\ddot{y}(\tau) = -\dot{u}(\tau)\sin \tau - u(\tau)\cos \tau + \dot{v}(\tau)\cos \tau - v(\tau)\sin \tau.
$$

Substituting the expressions about $\ddot{y}$, $\dot{y}$ and y into Eq. (4), the following equation is found:

$$(M\dot{v} - Mu + Cv + Ku)\cos\tau - (M\dot{u} + Mv + Cu - Kv)\sin\tau = f(u, v, \tau). \tag{8}$$

Then, Eq. (7) is multiplied by $M\cos\tau$, Eq. (8) is multiplied by $-\sin\tau$ and the two equations are added. The resulting equation is then integrated from 0 to 2π by assuming that u and v remain constant. The final result is

$$M\dot{u} = \frac{1}{2}(K - M)v - \frac{1}{2}\begin{pmatrix} \xi_1 u_1 + Q_1 \\ \mu\xi_2 u_2 + Q_2 \end{pmatrix}. \tag{9}$$

Similarly, Eq. (7) is multiplied by $M\sin\tau$, Eq. (8) is multiplied by $\cos\tau$ and the two equations are added. Then, integrating the resulting equation from 0 to 2π, we obtain

$$M\dot{v} = \frac{1}{2}(M - K)u - \frac{1}{2}\begin{pmatrix} \xi_1 v_1 + Q_3 + \lambda \\ \mu\xi_2 v_2 + Q_4 + \lambda \end{pmatrix}, \tag{10}$$

where

$$Q_1 = \tfrac{1}{4}(\beta_1 u_1 - 3\alpha_1 v_1)(u_1^2 + v_1^2), \quad Q_2 = \tfrac{1}{4}(\mu\beta_2 u_2 - 3\mu\alpha_2 v_2)(u_2^2 + v_2^2),$$
$$Q_3 = \tfrac{1}{4}(3\alpha_1 u_1 + \beta_1 v_1)(u_1^2 + v_1^2), \quad Q_4 = \tfrac{1}{4}(3\mu\alpha_2 u_2 + \mu\beta_2 v_2)(u_2^2 + v_2^2).$$

Eqs. (9) and (10) represent a set of first order, ordinary differential equations. For the periodic steady state vibration, the conditions are given as

$$\dot{u} = \dot{v} = 0. \tag{11}$$

Substituting conditions (11) into Eqs. (9) and (10), a set of four coupled non-linear algebraic equations for u_1, v_1, u_2 and v_2 is obtained:

$$\begin{aligned}
&\xi_1 u_1 - (\eta_1 - \mu - 1)v_1 + \mu v_2 + Q_1 = 0, \\
&\mu\xi_2 u_2 + \mu v_1 - (\mu\eta_2 - \mu)v_2 + Q_2 = 0, \\
&(1 + \mu - \eta_1)u_1 + \mu u_2 - \xi_1 v_1 - Q_3 - \lambda = 0, \\
&\mu u_1 + (\mu - \mu\eta_2)u_1 - \mu\xi_2 v_2 - Q_4 - \lambda = 0.
\end{aligned} \tag{12}$$

3. Stability analysis

In order to determine the stability of a periodic solution, a small perturbation of the solutions of Eq. (12) are introduced and they written as

$$\begin{aligned}
&u_1(\tau) = u_{10} + u_{11}(\tau), \quad v_1(\tau) = v_{10} + v_{11}(\tau), \\
&u_2(\tau) = u_{20} + u_{21}(\tau), \quad v_2(\tau) = v_{20} + v_{21}(\tau),
\end{aligned} \tag{13}$$

where $u_{11}(\tau)$, $v_{11}(\tau)$, $u_{21}(\tau)$ and $v_{21}(\tau)$ are small perturbations , u_{10}, v_{10}, u_{20} and v_{20} are the steady state solutions of Eq. (12). Substituting Eq. (13) into Eqs. (9) and (10), expanding the resulting equations in Fourier series with respect to u_{11}, v_{11}, u_{21} and v_{21}, using conditions (11) and keeping the linear parts, the perturbed equations are obtained:

$$[M]\begin{bmatrix}\dot{u}_{11}\\ \dot{u}_{21}\end{bmatrix}=\frac{1}{2}[K-M]\begin{bmatrix}v_{11}\\ v_{21}\end{bmatrix}-\frac{1}{2}\begin{bmatrix}\xi_1 u_{11}+\dfrac{\partial Q_1}{\partial u_{11}}u_{11}+\dfrac{\partial Q_1}{\partial v_{11}}v_{11}\\ \mu\xi_2 u_{21}+\dfrac{\partial Q_2}{\partial u_{21}}u_{21}+\dfrac{\partial Q_2}{\partial v_{21}}v_{21}\end{bmatrix},$$

$$[M]\begin{bmatrix}\dot{v}_{11}\\ \dot{v}_{21}\end{bmatrix}=\frac{1}{2}[M-K]\begin{bmatrix}u_{11}\\ u_{21}\end{bmatrix}-\frac{1}{2}\begin{bmatrix}\xi_1 v_{11}+\dfrac{\partial Q_3}{\partial u_{11}}u_{11}+\dfrac{\partial Q_3}{\partial v_{11}}v_{11}\\ \mu\xi_2 v_{21}+\dfrac{\partial Q_4}{\partial u_{21}}u_{21}+\dfrac{\partial Q_4}{\partial v_{21}}v_{21}\end{bmatrix}. \quad (14)$$

By judging the eigenvalues of the coefficient determinant of Eq. (14), the stability of periodic solutions can be determined. If the real part of all the eigenvalues is negative, then the periodic solution is stable; otherwise, it is unstable. If a real eigenvalue changes sign, it is a saddle-node-type bifurcation and may result in jump phenomena. If there exists a pair of complex conjugate eigenvalues whose real part changes sign, it is termed Hopf bifurcation and results in quasiperiodic vibrations.

4. Numerical results and discussion

In all the numerical calculations, the non-dimensionless exciting frequency Ω is taken as a bifurcation parameter. r_1 and r_2 denote the response amplitudes of the main mass and absorber, respectively, namely

$$r_1=\sqrt{u_1^2+v_1^2},\quad r_2=\sqrt{u_2^2+v_2^2}.$$

In order to validate the result right in this paper, the parameters taken are $\mu=0.05$, $\delta=1$, $\alpha_1=0$, $\alpha_2=0.01$, $\beta_1=\beta_2=0$, $\bar{\lambda}=0.1$, $\zeta_1=\zeta_2=0.01$, and we let $\lambda=0$ in the second equation of Eq. (10). The solid curves denote the stable solutions and the dotted curves denote the unstable solutions in Figs. 2–6. Fig. 2 shows the steady state response–frequency curve of the main mass. The two unstable branches shown in Fig. 2 are generated through saddle-node bifurcations. The results are consistent with those in Ref. [8]. Meanwhile, in order to evaluate the effectiveness of the cubic non-linear damping on the amplitude reduction of the system, we take $\beta_2=0.003$ and the other parameters are taken as above. From the curves, it can be seen that the non-linear damping has an obvious effect on the amplitude reduction.

Then, several groups of various parameters are tried out to discuss the vibration property of the system and the selections of the relative parameters are taken by virtue of Ref. [5]. The steady state responses of the main mass are shown in Figs. 3 and 4 considering different non-linear spring coefficients. In Fig. 3, with increase in the non-linear spring stiffness $\bar{\alpha}_2$, the vibration amplitude reduces and the critical bifurcation parameter increases. It can be concluded that increasing the

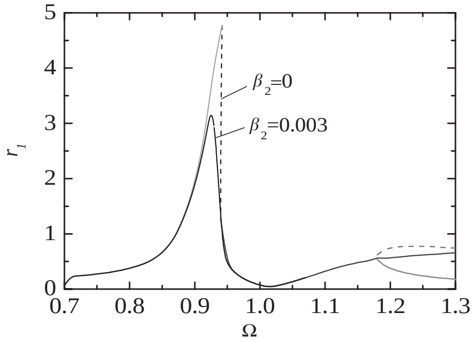

Fig. 2. Steady state response of the main mass ($\mu = 0.05$, $\delta = 1$, $\alpha_1 = 0$, $\alpha_2 = 0.01$, $\beta_1 = 0$, $\bar{\lambda} = 0.1$, $\zeta_1 = \zeta_2 = 0.01$).

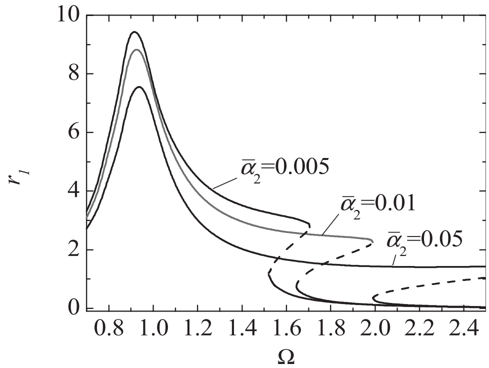

Fig. 3. Steady state response of the main mass ($\mu = 0.1$, $\delta = 0.95$, $\bar{\alpha}_1 = 0$, $\bar{\beta}_1 = \bar{\beta}_2 = 0$, $\bar{\lambda} = 1$, $\zeta_1 = \zeta_2 = 0.1$).

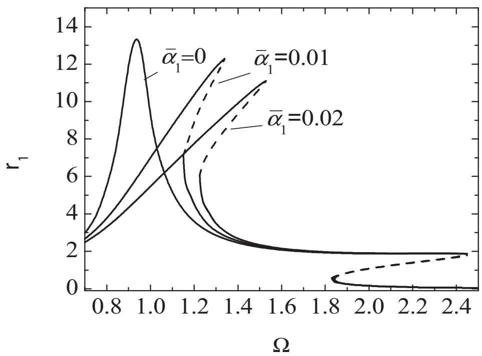

Fig. 4. Steady state response of the main mass ($\mu = 0.1$, $\delta = 1$, $\bar{\alpha}_2 = 0.02$, $\bar{\beta}_1 = \bar{\beta}_2 = 0$, $\bar{\lambda} = 1$, $\zeta_1 = 0.05$, $\zeta_2 = 0.1$).

non-linear spring stiffness $\bar{\alpha}_2$ can reduce the amplitude of the main mass and obtain the effect of reduction of the vibration amplitude. But from Fig. 4, the amplification of the non-linear spring stiffness $\bar{\alpha}_1$ makes the hardening degree of the system increase. When $\Omega < 1$, the increase of $\bar{\alpha}_1$ can

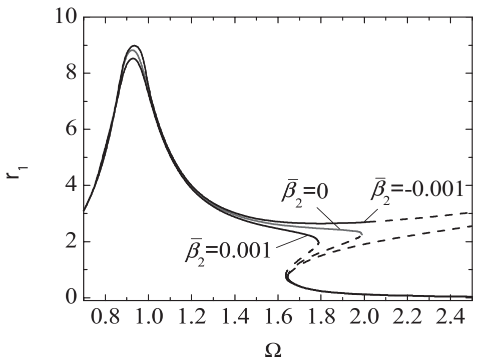

Fig. 5. Steady state response of the main mass ($\mu = 0.1$, $\delta = 0.95$, $\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 0.01$, $\bar{\beta}_1 = 0$, $\bar{\lambda} = 1$, $\zeta_1 = \zeta_2 = 0.1$).

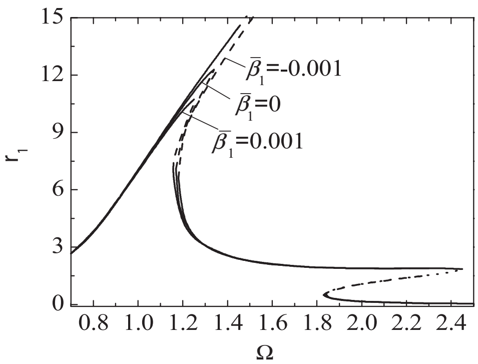

Fig. 6. Steady state response of the main mass ($\mu = 0.1$, $\delta = 1$, $\bar{\alpha}_1 = 0.01$, $\bar{\alpha}_2 = 0.02$, $\bar{\beta}_2 = 0$, $\bar{\lambda} = 1$, $\zeta_1 = 0.05$, $\zeta_2 = 0.1$).

reduce the vibration amplitude. However, when $1 < \Omega < 1.6$, the increase of $\bar{\alpha}_1$ does not obtain reduction of the amplitude. When $\Omega > 1.6$, the value of $\bar{\alpha}_1$ has little effect on the vibration of the system. It is observed that increase of $\bar{\alpha}_1$ can result in reduction of the amplitude only at $\Omega < 1$. The unstable branches are also generated through saddle-node bifurcations.

Figs. 5 and 6 show the steady state response of the main mass considering different non-linear damper. From Fig. 5, increase in non-linear damper $\bar{\beta}_2$ results in the reduction of the vibration amplitude in the range of $\Omega < 1.4$, while its effect is relatively evident when $\Omega > 1.4$. The vibration amplitude reduces and the range of the bifurcation parameter becomes smaller both with increase in $\bar{\beta}_2$. When $\bar{\beta}_2 < 0$, another unstable branch is generated through saddle-node bifurcations. It is obvious that the increasing of $\bar{\beta}_2$ obtains a significant effect in the range $\Omega > 1.4$. However, from Fig. 6, the increasing of non-linear damper $\bar{\beta}_1$ has little influence on the vibration amplitude of the system. Therefore, it can be concluded that the purpose of reduction of the vibration amplitude is arrived at with difficulty by increasing $\bar{\beta}_1$.

In the following section, numerical simulation is applied for Eq. (3). The non-dimensional frequency Ω is also taken as the bifurcation parameter. The other parameters of the system

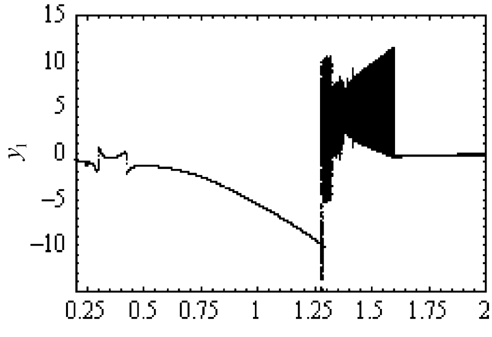

Fig. 7. Bifurcation diagram of the main mass by using Ω as the control parameter:

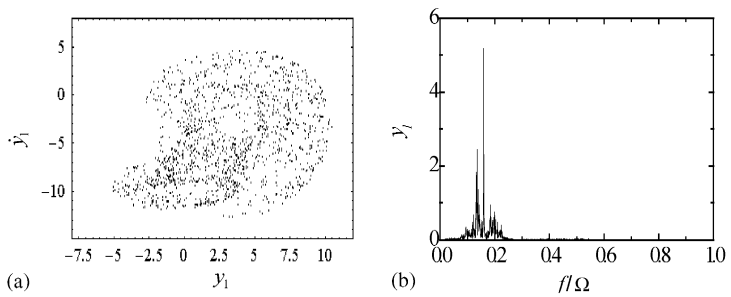

Fig. 8. $\Omega = 1.2941$: (a) Poincaré map; (b) amplitude–frequency spectrum.

are taken as

$$\mu = 0.1, \quad \delta = 0.95, \quad \bar{\alpha}_1 = 0.01, \quad \bar{\alpha}_2 = 0.002, \quad \bar{\beta}_1 = 0, \quad \bar{\beta}_2 = 0.0001, \quad \bar{\lambda} = 1, \quad \zeta_1 = \zeta_2 = 0.01.$$

Fig. 7 shows the bifurcation diagram of the main mass by using Ω as the control parameter. In this paper, we are mainly concerned with tracing the bifurcation diagram and identifying the periodic motions, quasiperiodic motions or chaotic motions.

From Fig. 7, the motion of the system is a periodic motion before about $\Omega < 1.3$. When $\Omega = 1.2941$, the system enters the chaotic motions. The relative Poincaré map and amplitude–frequency spectrum are shown in Fig. 8. It shows that the Poincaré map has the appearance of a strange attractor typical of chaos and the spectrum is a continuous curve. In Fig. 9, the Poincaré maps for the system are shown for the different exciting frequencies. At $\Omega = 1.322$ (Fig. 9a), there occurs a closed curve among a crowd of points, which shows the system will move to quasiperiodic motion. At $\Omega = 1.325$ (Fig. 9b), the system exhibits a quasiperiodic response. With the increase of the exciting frequency, the torus breaks up into six small sections and the system turns into a period 6 response at $\Omega = 1.3294$ (Fig. 9c). The Poincaré map has six points. Then, the six points generally expand and they are finally united into a closed curve at $\Omega = 1.335$ (Fig. 9d). A single closed curve bifurcates into two nested closed curves at $\Omega = 1.34$ (Fig. 9e). At $\Omega = 1.59$ (Fig. 9f), the system moves into period 1 response by an inverse Hopf bifurcation, then by a Hopf bifurcation resulting in a quasiperiodic response (Fig. 9g). After $\Omega = 1.605$ (Fig. 9h), the system enters into a periodic motion.

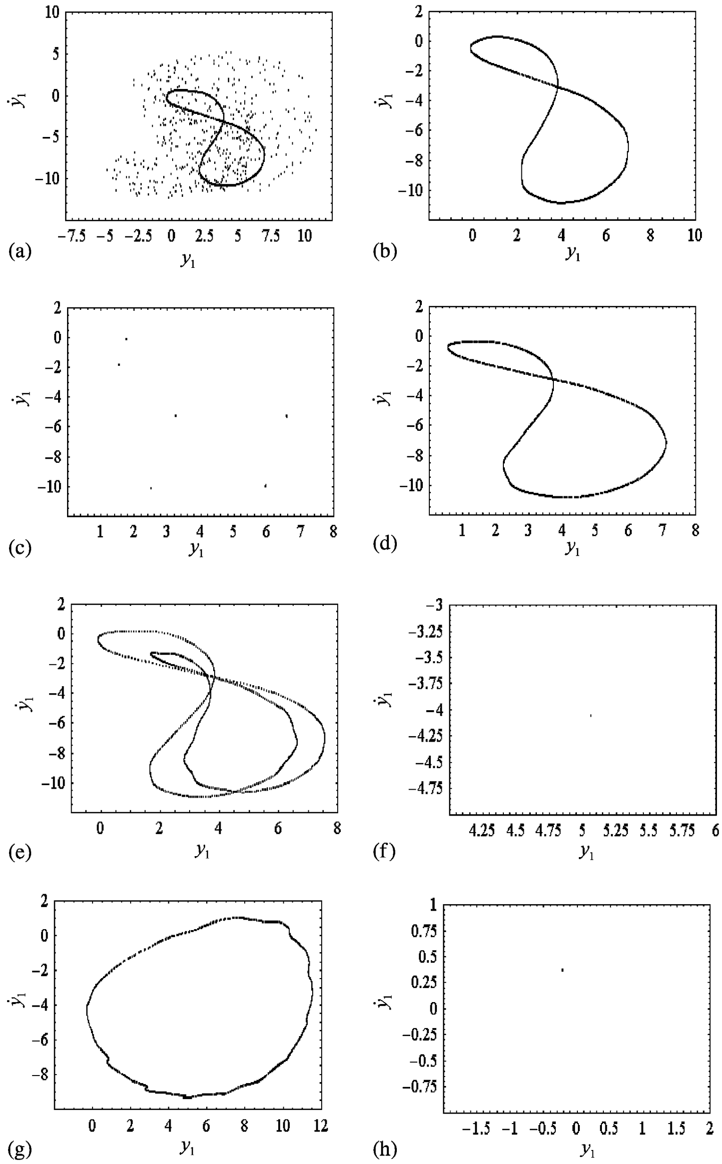

Fig. 9. Poincaré maps of the main mass for different exciting frequencies: (a) $\Omega = 1.322$, (b) $\Omega = 1.325$, (c) $\Omega = 1.3294$, (d) $\Omega = 1.335$, (e) $\Omega = 1.34$, (f) $\Omega = 1.59$, (g) $\Omega = 1.595$, and (h) $\Omega = 1.605$.

5. Conclusion

The model, in this paper, is a two-degree-of-freedom vibration system with non-linear damping and non-linear spring. The study presents firstly the stability and bifurcation of the system. Results show the effect of reduction of the vibration amplitude can be obtained by properly selecting the values of non-linear dampers, non-linear spring stiffness and the range of exciting frequency. When $\Omega < 1$, increasing non-linear spring stiffness and non-linear damper both result in reduction of the vibration amplitude. When $\Omega > 1$, the changes of non-linear spring stiffness and damper of the main mass do not reduce the vibration amplitude, but it can be realized adjusting the non-linear spring stiffness and damper of the absorber. Finally, numerical simulation is applied to investigate the periodic motion, quasiperiodic motion and chaotic motion of the system. The motion state of the system is transformed in a range of different exciting frequencies. The system exhibits complex motions.

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