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Letter to the Editor

Non-proportionally damped systems subjected to damping modifications by several viscous dampers

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1. Introduction

Ref. [1] dealt with a system consisting of a proportionally damped, linear, discrete mechanical system with n degrees of freedom (d.o.f.) to which m additional viscous dampers are attached. Making use of the approach in Refs. [2,3], the $n \times n$ characteristic determinant of the combined system was reduced to a determinant of order $m \times m$, where it was assumed that $m \ll n$, which is a more frequently encountered case in practice. As a result, an alternative formulation was presented for the characteristic equation of the mentioned mechanical system which can be very convenient for numerical calculations at higher n d.o.f. values.

The basis of the approach in the cited references is a formula with a rather lengthy proof, which is on the determinant of a diagonal matrix modified by a total of m rank-one matrices. In the meantime, the present author has given the proof of a more general but simple formula for the determinant of the sum of a regular square matrix (not necessarily diagonal) and several dyadic products, i.e., rank-one matrices [4]. Based on this development, in the present study, one is able to replace the original proportionally damped mechanical system in Ref. [1] by a non-proportionally damped, i.e., a more general system. Hence, the $n \times n$ characteristic determinant of a non-proportionally damped linear mechanical system with n d.o.f. modified by m additional viscous dampers, is reduced to a much smaller determinant of order $m \times m$. Further, an alternative form of the characteristic equation and an explicit analytical expression for the eigenvectors of the modified system are given. Then the eigenvalue, eigenvector and receptance matrix sensitivities with respect to a damping-related parameter are derived.

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2. Theory

As is known, the motion of a linear discrete mechanical system with n d.o.f. is governed in the physical space by the following matrix differential equation:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{D}}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\tilde{\mathbf{D}}$ and $\mathbf{K}$ are the $(n \times n)$ mass, damping and stiffness matrices, respectively and $\mathbf{q}$ is the $(n \times 1)$ vector of the generalized co-ordinates. It is assumed that the damping matrix $\tilde{\mathbf{D}}$ is non-proportional.

Suppose that m new viscous dampers are added to the mechanical system, such that the damping matrix of the modified system can be written as

$$\mathbf{D} = \tilde{\mathbf{D}} + \sum_{i=1}^{m} \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}}, \tag{2}$$

where the vectors $\mathbf{d}_i$ include both damping constant and the orientation information in the physical space [5].

The main aim of this study is to obtain the characteristic equation, the eigencharacteristics and the receptance matrix of the modified system and then to calculate sensitivities with respect to a damping-related parameter.

2.1. Determination of the characteristic equation

The transformation

$$\mathbf{q}(t) = \mathbf{\Phi}\boldsymbol{\eta}(t), \tag{3}$$

where $\mathbf{\Phi}$ is the modal matrix of the undamped system, results in the following equation of motion in the modal space:

$$\ddot{\boldsymbol{\eta}} + \left(\mathbf{D}^* + \sum_{i=1}^{m} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}}\right)\dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2\boldsymbol{\eta} = \mathbf{0}. \tag{4}$$

Here, the relations

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}, \quad \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2), \quad i = 1, \ldots, n \tag{5}$$

are used which are due to the mass orthonormalization of the mode vectors of the undamped system. $\mathbf{I}$ denotes the $(n \times n)$ unit matrix and ω_i is the ith eigenfrequency of the undamped system.

Additionally, the definitions

$$\mathbf{D}^* = \mathbf{\Phi}^{\mathrm{T}}\tilde{\mathbf{D}}\mathbf{\Phi}, \quad \mathbf{d}_i^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{d}_i \tag{6}$$

are introduced. It is worth noting that the first part of the transformed damping matrix, i.e., $\mathbf{D}^*$ is not a diagonal matrix, which is due to the non-proportional character of the original damping matrix $\tilde{\mathbf{D}}$.

If a solution of the equation of motion in the modal space, Eq. (4) is assumed in the form of

$$\boldsymbol{\eta}(t) = \tilde{\boldsymbol{\eta}}\mathrm{e}^{\lambda t}, \tag{7}$$

where λ and $\tilde{\boldsymbol{\eta}}$ represent an eigenvalue and the corresponding eigenvector, respectively, the eigenvalue problem

$$\left[(\lambda^2\mathbf{I} + \lambda\mathbf{D}^* + \boldsymbol{\Omega}^2) + \lambda \sum_{i=1}^{m} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}}\right]\tilde{\boldsymbol{\eta}} = \mathbf{0} \tag{8}$$

is obtained, which means that the eigenvalues λ are obtained from the characteristic equation

$$\det[(\lambda^2\mathbf{I} + \lambda\mathbf{D}^* + \boldsymbol{\Omega}^2) + \lambda \sum_{i=1}^{m} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}}] = 0. \tag{9}$$

In the above equation, unlike in Ref. [1], the sum of the first three matrices is non-diagonal, because the matrix $\mathbf{D}^*$ is non-diagonal, as mentioned previously. Hence, the matrix determinant of which is to be equated to zero, consists of a non-diagonal matrix modified by m dyadic products, i.e., m rank-one matrices.

In Ref. [4], the present author established the following formula:

$$\det\left(\mathbf{A} + \sum_{i=1}^{m} \mathbf{x}_i \mathbf{y}_i^{\mathrm{T}}\right) = \det\mathbf{A}\det\mathbf{G}, \tag{10}$$

for the calculation of the determinant of a general regular matrix $\mathbf{A}$ modified by the sum of m dyadic products. Here the matrix $\mathbf{G}$ is of the form

$$\mathbf{G} = [g_{ij}] = [\delta_i^j + \mathbf{y}_i^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_j] \quad (i,j = 1, \ldots, m), \tag{11}$$

δ_i^j-being the Kronecker delta.

Now, making use of formula (10) with Eq. (11), the equation in (9) leads to the following characteristic equation for the modified system:

$$\det(\mathbf{G}) = 0, \tag{12}$$

where

$$\mathbf{G} = [g_{ij}] = [\delta_i^j + \mathbf{d}_i^{*\mathrm{T}}(\lambda^2\mathbf{I} + \lambda\mathbf{D}^* + \boldsymbol{\Omega}^2)^{-1}\lambda\mathbf{d}_j^*] \quad (i,j = 1, \ldots, m). \tag{13}$$

It is worth noting that each element of the matrix $\mathbf{G}$ consists of a sum of n terms, for $i \neq j$. In comparison to Eq. (9), here, one has to find the roots of a determinant of size $(m \times m)$, where in practice it is usually $m \ll n$. This means that the eigenvalues λ can be obtained as the roots of a determinant of a highly reduced order.

2.2. Alternative form of the characteristic equation. Eigenvectors and frequency response matrix of the modified system

This section is devoted to the derivation of an alternative form of the characteristic equation which enables one to obtain explicit expressions of the eigenvectors on one side, and to calculate the sensitivities of the eigencharacteristics of the system with respect to a damping related parameter on the other side.

Begin with rewriting the damping matrix of the modified system in Eq. (2) in the form

$$\mathbf{D} = \left(\tilde{\mathbf{D}} + \sum_{i=1}^{m-1} \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}}\right) + \mathbf{d}_m \mathbf{d}_m^{\mathrm{T}}, \tag{14}$$

where it is insignificant which term is taken out of the summation.

The transformation in Eq. (3) yields now

$$\ddot{\boldsymbol{\eta}} + (\mathbf{D}^* + \mathbf{D}_1^* + \mathbf{d}_m^* \mathbf{d}_m^{*\mathrm{T}})\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2 \boldsymbol{\eta} = \mathbf{0}, \tag{15}$$

where in addition to Eqs. (5) and (6)

$$\mathbf{D}_1^* = \sum_{i=1}^{m-1} \mathbf{d}_i^* \mathbf{d}_i^{*\mathrm{T}}, \quad \mathbf{d}_m^* = \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{d}_m \tag{16}$$

are introduced.

Assumption of a solution of the form (7) leads now to the following eigenvalue problem:

$$(\mathbf{A} + \lambda \mathbf{d}_m^* \mathbf{d}_m^{*\mathrm{T}})\tilde{\boldsymbol{\eta}} = \mathbf{0} \tag{17}$$

with

$$\mathbf{A} = \lambda^2 \mathbf{I} + \lambda(\mathbf{D}^* + \mathbf{D}_1^*) + \boldsymbol{\Omega}^2. \tag{18}$$

This means that the eigenvalues λ are obtained from the characteristic equation

$$\det(\mathbf{A} + \lambda \mathbf{d}_m^* \mathbf{d}_m^{*\mathrm{T}}) = 0. \tag{19}$$

By using the well-known formula

$$\det(\mathbf{A} + \alpha \mathbf{b}\mathbf{b}^{\mathrm{T}}) = (\det \mathbf{A})\,(1 + \alpha \mathbf{b}^{\mathrm{T}} \mathbf{A}^{-1} \mathbf{b}) \tag{20}$$

for the determinant of the sum of a regular square matrix and a dyadic [6], one obtains the characteristic equation of the modified system in the following form:

$$1 + \lambda \mathbf{d}_m^{*\mathrm{T}} \mathbf{A}^{-1} \mathbf{d}_m^* = 0, \tag{21}$$

where $\mathbf{A}$ is given in Eq. (18).

Eigenvectors of the modified system, $\tilde{\boldsymbol{\eta}}$ are to be obtained from Eq. (17). The kth eigenvector $\tilde{\boldsymbol{\eta}}_k$ can be shown to be

$$\tilde{\boldsymbol{\eta}}_k = \mathbf{A}_k^{-1} \mathbf{d}_m^*, \tag{22}$$

where $\mathbf{A}_k$ denotes the right side of expression (18) for $\lambda = \lambda_k$.

Proof. The correctness of this statement can be shown by substituting expression (22) into Eq. (17):

$$(\mathbf{A}_k + \lambda_k \mathbf{d}_m^* \mathbf{d}_m^{*\mathrm{T}})\mathbf{A}_k^{-1} \mathbf{d}_m^* = \mathbf{d}_m^* + \lambda_k \mathbf{d}_m^* \mathbf{d}_m^{*\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{d}_m^* = \mathbf{d}_m^*(1 + \lambda_k \mathbf{d}_m^{*\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{d}_m^*) = 0.$$

The right side is equal to zero, because the kth eigenvalue λ_k satisfies the characteristic equation given by Eq. (21).

Hence, by Eq. (22), one has an explicit analytical expression for the eigenvectors of the modified system, in the modal space. The kth eigenvector in the physical space can immediately be given via

Eq. (3) as

$$\bar{\mathbf{q}}_k = \mathbf{\Phi}\tilde{\boldsymbol{\eta}}_k = \mathbf{\Phi}\mathbf{A}_k^{-1}\mathbf{d}_m^*. \tag{23}$$

The complex frequency response matrix, also referred to as receptance matrix of the mechanical system in Eq. (1) is defined as

$$\mathbf{H}(\omega) = (-\omega^2\mathbf{M} + \mathrm{i}\omega\tilde{\mathbf{D}} + \mathbf{K})^{-1}, \tag{24}$$

where ω denotes the forcing frequency. An expansion of the damping matrix $\tilde{\mathbf{D}}$ in the form

$$\tilde{\mathbf{D}} = \sum_{k=1}^{n} \lambda_k \bar{\bar{\mathbf{d}}}_k \bar{\bar{\mathbf{d}}}_k^{\mathrm{T}} \tag{25}$$

is possible, where λ_k and $\bar{\bar{\mathbf{d}}}_k$ denote the k-th eigenvalue and eigenvector respectively [7]. This in turn, can be rewritten as

$$\tilde{\mathbf{D}} = \sum_{k=1}^{n} \bar{\mathbf{d}}_k \bar{\mathbf{d}}_k^{\mathrm{T}}, \tag{26}$$

with $\bar{\mathbf{d}}_k = \sqrt{\lambda_k}\,\bar{\bar{\mathbf{d}}}_k$. This leads to the following expression for the damping matrix of the modified system given in Eq. (2)

$$\mathbf{D} = \sum_{k=1}^{n} \bar{\mathbf{d}}_k \bar{\mathbf{d}}_k^{\mathrm{T}} + \sum_{i=1}^{m} \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}} = \sum_{i=1}^{m+n} \bar{\mathbf{d}}_i \bar{\mathbf{d}}_i^{\mathrm{T}}, \tag{27}$$

where $\bar{\mathbf{d}}_{n+1} = \mathbf{d}_1, \ldots, \bar{\mathbf{d}}_{n+m} = \mathbf{d}_m$ are introduced.

Introduce further

$$\bar{\mathbf{D}} = [\bar{\mathbf{d}}_1, \ldots, \bar{\mathbf{d}}_{n+m}] \tag{28}$$

such that the matrix $\mathbf{D}$ in Eq. (27) can be written as

$$\mathbf{D} = \bar{\mathbf{D}}\,\bar{\mathbf{D}}^{\mathrm{T}}. \tag{29}$$

Hence, the receptance matrix of modified system can be represented in the form

$$\bar{\mathbf{H}}(\omega) = (-\omega^2\mathbf{M} + \mathrm{i}\omega\bar{\mathbf{D}}\,\bar{\mathbf{D}}^{\mathrm{T}} + \mathbf{K})^{-1}. \tag{30}$$

If at this point use is made of the so-called Sherman–Morrison–Woodbury formula [8], repeated for a special case in Ref. [9],

$$\bar{\mathbf{H}}(\omega) = \mathbf{H}_\circ(\omega)[\mathbf{I} - (\mathrm{i}\omega\bar{\mathbf{D}})\,(\mathbf{I} + \bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega)(\mathrm{i}\omega\bar{\mathbf{D}}))^{-1}\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega)] \tag{31}$$

is obtained, where $\mathbf{H}_\circ(\omega)$ denotes

$$\mathbf{H}_\circ(\omega) = (-\omega^2\mathbf{M} + \mathbf{K})^{-1}, \tag{32}$$

i.e., the receptance matrix of the undamped system. Hence, formula (31) gives the receptance matrix of the modified system, in terms of the receptance matrix of the undamped system and the modified damping matrix.

Sometimes, it is desirable to express the receptance matrix $\bar{\mathbf{H}}(\omega)$ in terms of the receptance matrix of the original damped system. With the definition

$$\bar{\bar{\mathbf{D}}} = [\mathbf{d}_1, \ldots, \mathbf{d}_m] \tag{33}$$

and via Eq. (2), the receptance matrix $\bar{\mathbf{H}}(\omega)$ can be formulated as

$$\bar{\mathbf{H}}(\omega) = [(-\omega^2\mathbf{M} + \mathrm{i}\omega\tilde{\mathbf{D}} + \mathbf{K}) + \mathrm{i}\omega\bar{\bar{\mathbf{D}}}\,\bar{\bar{\mathbf{D}}}^{\mathrm{T}}]^{-1}. \tag{34}$$

Now, Sherman–Morrison–Woodbury formula yields

$$\bar{\mathbf{H}}(\omega) = \mathbf{H}(\omega)[\mathbf{I} - (\mathrm{i}\omega\bar{\bar{\mathbf{D}}})\,(\mathbf{I} + \bar{\bar{\mathbf{D}}}^{\mathrm{T}}\mathbf{H}(\omega)(\mathrm{i}\omega\bar{\bar{\mathbf{D}}}))^{-1}\bar{\bar{\mathbf{D}}}^{\mathrm{T}}\mathbf{H}(\omega)] \tag{35}$$

with

$$\mathbf{H}(\omega) = [(-\omega^2\mathbf{M} + \mathrm{i}\omega\tilde{\mathbf{D}} + \mathbf{K})^{-1}, \tag{24}$$

representing the receptance matrix of the original damped system. Having obtained the eigencharacteristics and the receptance matrix of the modified mechanical system, in the next section, the corresponding sensitivities will be determined.

2.3. Calculation of the sensitivities of the eigencharacteristics and the receptance matrix

Let it be assumed that α denotes some damping-related parameter upon which the additional part of the damping action in Eq. (16) depends, such that

$$\mathbf{D}_1^* = \mathbf{D}_1^*(\alpha), \quad \mathbf{d}_m^* = \mathbf{d}_m^*(\alpha). \tag{36}$$

If the characteristic equation (21) is differentiated partially with respect to α,

$$\lambda_k' := \frac{\partial\lambda_k}{\partial\alpha} = \frac{-\lambda_k[\mathbf{d}_m^{*\mathrm{T}'}\mathbf{A}_k^{-1}\mathbf{d}_m^* - \lambda_k\mathbf{d}_m^{*\mathrm{T}}\mathbf{A}_k^{-1}\mathbf{D}_1^{*'}\mathbf{d}_m^* + \mathbf{d}_m^{*\mathrm{T}}\mathbf{A}_k^{-1}\mathbf{d}_m^{*'}]}{\mathbf{d}_m^{*\mathrm{T}}\mathbf{A}_k^{-1}\mathbf{d}_m^* - \lambda_k\mathbf{d}_m^{*\mathrm{T}}\mathbf{A}_k^{-1}[2\lambda_k\mathbf{I} + \mathbf{D}^* + \mathbf{D}_1^*]\mathbf{d}_m^*} \tag{37}$$

is obtained where a prime denotes partial derivative with respect to the parameter α. Hence, the above formula gives the sensitivity of the eigenvalue λ_k with respect to α.

The sensitivity of the eigenvector $\bar{\mathbf{q}}_k$ with respect to the parameter α can be obtained in a straightforward manner, by differentiating expression (23) with respect to α partially to get

$$\bar{\mathbf{q}}_k' := \frac{\partial\bar{\mathbf{q}}_k}{\partial\alpha} = -\mathbf{\Phi}\mathbf{A}_k^{-1}[\lambda_k'(2\lambda_k\mathbf{I} - \mathbf{D}^* - \mathbf{D}_1^*)\mathbf{d}_m^* + \lambda_k\mathbf{D}_1^{*'}\mathbf{d}_m^* - \mathbf{d}_m^{*'}]. \tag{38}$$

The sensitivity of the receptance matrix with respect to the parameter α can be obtained simply by differentiating equation (31) with respect to α to obtain

$$\begin{aligned}\bar{\mathbf{H}}'(\omega) := \frac{\partial\bar{\mathbf{H}}(\omega)}{\partial\alpha} = \mathbf{H}_\circ(\omega)[&-\mathrm{i}\omega\bar{\mathbf{D}}'\mathbf{B}\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega) + \mathrm{i}\omega\bar{\mathbf{D}}\mathbf{B}(\bar{\mathbf{D}}^{\mathrm{T}'}\mathbf{H}_\circ(\omega)(\mathrm{i}\omega\bar{\mathbf{D}}) \\ &+ (\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega)\,(\mathrm{i}\omega\bar{\mathbf{D}}'))\mathbf{B}\bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega) - \mathrm{i}\omega\bar{\mathbf{D}}\mathbf{B}\bar{\mathbf{D}}^{\mathrm{T}'}\mathbf{H}_\circ(\omega)],\end{aligned} \tag{39}$$

where

$$\mathbf{B} = (\mathbf{I} + \bar{\mathbf{D}}^{\mathrm{T}}\mathbf{H}_\circ(\omega)\,(\mathrm{i}\omega\bar{\mathbf{D}}))^{-1} \tag{40}$$

is introduced.

Since the sensitivities of the eigencharacteristics and of the receptance matrix have been obtained, now the approximate expressions for the modified values of the eigenvalues, eigenvectors and receptance matrix can be given, if the damping-related parameter α is changed by an amount $\Delta\alpha$ around its nominal value α:

$$\lambda_{k_{mod}} \approx \lambda_k(\alpha) + \lambda_k'(\alpha)\Delta\alpha, \tag{41}$$

$$\bar{\mathbf{q}}_{k_{mod}} \approx \bar{\mathbf{q}}_k(\alpha) + \bar{\mathbf{q}}_k'(\alpha)\Delta\alpha, \tag{42}$$

$$\bar{\mathbf{H}}(\omega)_{mod} \approx \bar{\mathbf{H}}(\omega) + \bar{\mathbf{H}}'(\omega)\Delta\alpha. \tag{43}$$

3. Numerical evaluations

This section is devoted to the testing of the reliability of the expressions obtained. The simple system in Fig. 1 is taken as an illustrative example. It consists of a vibrational system with four d.o.f. in which every mass is acted upon by an inertial viscous damper and an additional relative viscous damper is attached between the masses m_2 and m_3. It is essentially the same mechanical system as in Ref. [1] except that the damping matrix here is non-diagonal and the present system is non-proportionally damped. The physical parameters are as follows: $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$, $m_4 = m$ with $m = 3$ kg; $k_1 = k$, $k_2 = 2k$, $k_3 = 4k$, $k_4 = k$ and $k_5 = k$ with $k = 2$ N/m. Further: $\bar{c}_1 = \bar{c}$, $\bar{c}_2 = 2\bar{c}$, $\bar{c}_3 = \bar{c}$, $\bar{c}_4 = 2\bar{c}$ and $\bar{c}_5 = 0.1\bar{c}$, with $\bar{c} = 1$ N/(m/s). It is further assumed that two relative viscous dampers of constants $c_1 = 2\bar{c}$ and $c_2 = 4\bar{c}$ are to be added between the masses m_1, m_2 and m_3, m_4, respectively, as depicted in dashed lines in Fig. 1.

The mass, stiffness and damping matrices of the original system are

$$\mathbf{M} = \mathbf{diag}(3,6,9,3), \quad \mathbf{K} = \begin{bmatrix} 6 & -4 & 0 & 0 \\ -4 & 12 & -8 & 0 \\ 0 & -8 & 10 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \quad \tilde{\mathbf{D}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2.1 & -0.1 & 0 \\ 0 & -0.1 & 1.1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

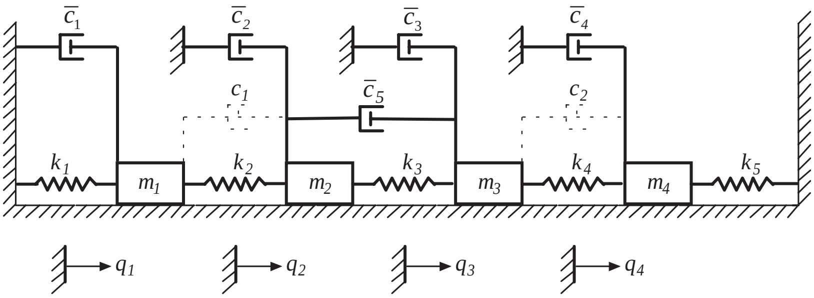

Fig. 1. Sample system with four degrees of freedom.

The damping matrix of the modified system given in Eq. (2) reads as

$$\mathbf{D} = \tilde{\mathbf{D}} + \mathbf{d}_1\mathbf{d}_1^{\mathrm{T}} + \mathbf{d}_2\mathbf{d}_2^{\mathrm{T}}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2.1 & -0.1 & 0 \\ 0 & -0.1 & 1.1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + \sqrt{c_1}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\sqrt{c_1}[1\ \ -1\ \ 0\ \ 0] + \sqrt{c_2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\sqrt{c_2}[0\ \ 0\ \ 1\ \ -1]$$
$$= \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 4.1 & -0.1 & 0 \\ 0 & -0.1 & 5.1 & -4 \\ 0 & 0 & -4 & 6 \end{bmatrix}.$$

The solution of the eigenvalue problem of the undamped system yields

$$\mathbf{\Phi} = \begin{bmatrix} 0.16702739 & 0.23699340 & -0.38616296 & 0.31646087 \\ 0.23370715 & 0.12999193 & -0.06605818 & -0.30130723 \\ 0.24349274 & -0.04719554 & 0.18178137 & 0.12864878 \\ 0.13539191 & -0.48650735 & -0.27631637 & -0.04429703 \end{bmatrix},$$
$$\mathbf{\Omega}^2 = \mathbf{diag}(0.13438064, 1.26866073, 1.77191603, 3.26948704)$$

for the modal matrix and the matrix of the squares of the eigenfrequencies, respectively.

The eigenvalues λ of the modified system in Fig. 1 are given in Table 1. The complex numbers in the first column are the eigenvalues obtained directly by solving the eigenvalue problem as indicated in Ref. [1]. The complex numbers in the second and third columns are obtained by solving Eqs. (12) and (21) respectively. All numerical operations are carried out with MATLAB. The agreement of the numbers in the three columns is excellent, justifying clearly the validity of the alternative forms of the characteristic equations (12) and (21).

In order to gain insight into how accurately the eigenvectors can be obtained by the present method, the eigenvectors of the system in Fig. 1 are given in Table 2 according to the state-space representation $\bar{\mathbf{q}}_k^{\mathrm{T}} = [\tilde{\mathbf{y}}_k^{\mathrm{T}} \lambda_k \tilde{\mathbf{y}}_k^{\mathrm{T}}]$.

The eigenvectors in the first column are obtained directly by solving the eigenvalue problem of the system in Fig. 1. The eigenvectors in the second column are determined by using Eq. (23). The agreement here is excellent.

Table 1
Eigenvalues λ of the modified system in Fig. 1

	Direct solution of the eigenvalue problem	From Eq. (12)	From Eq. (21)
$\lambda_{1,2}$	$-0.145960 \pm 0.342156i$	$-0.145960 \pm 0.342156i$	$-0.145960 \pm 0.342156i$
$\lambda_{3,4}$	$-1.140530 \pm 0.381823i$	$-1.140530 \pm 0.381823i$	$-1.140530 \pm 0.381823i$
$\lambda_{5,6}$	$-0.239151 \pm 1.268421i$	$-0.239151 \pm 1.268421i$	$-0.239151 \pm 1.268421i$
$\lambda_{7,8}$	$-0.599359 \pm 1.613134i$	$-0.599359 \pm 1.613134i$	$-0.599359 \pm 1.613134i$

Table 2
Eigenvectors of the modified system in Fig. 1

	Direct solution of the eigenvalue problem	From Eq. (23)
$\tilde{\mathbf{y}}_{1,2}$	1.000000	1.000000
	$1.410651 \mp 0.064320i$	$1.410651 \mp 0.064320i$
	$1.454578 \mp 0.037187i$	$1.454578 \mp 0.037187i$
	$0.843922 \pm 0.161608i$	$0.843922 \pm 0.161608i$
$\tilde{\mathbf{y}}_{3,4}$	1.000000	1.000000
	$2.619564 \mp 2.017422i$	$2.619564 \mp 2.017422i$
	$3.556430 \mp 4.976448i$	$3.556430 \mp 4.976448i$
	$-13.853560 \pm 22.065139i$	$-13.853560 \pm 22.065139i$
$\tilde{\mathbf{y}}_{5,6}$	1.000000	1.000000
	$0.384640 \pm 0.286630i$	$0.384640 \mp 0.286630i$
	$-0.417956 \mp 0.174735i$	$-0.417956 \mp 0.174735i$
	$-0.376708 \pm 0.058175i$	$-0.376708 \pm 0.058175i$
$\tilde{\mathbf{y}}_{7,8}$	1.000000	1.000000
	$-0.557704 \pm 0.299015i$	$-0.557704 \pm 0.299015i$
	$0.105876 \mp 0.205547i$	$0.105876 \mp 0.205547i$
	$-0.093687 \mp 0.178362i$	$-0.093687 \mp 0.178362i$

Table 3
Eigenvalues of the modified system in Fig. 1, if the damping coefficient is changed slightly by an amount Δc_2 around its nominal value $c_2 = 4$

Δc_2	From Eq. (21)	From Eq. (41)
0	$-0.145960 \pm 0.342156i$	$-0.145960 \pm 0.342156i$
	$-1.140530 \pm 0.381823i$	$-1.140530 \pm 0.381823i$
	$-0.239151 \pm 1.268421i$	$-0.239151 \pm 1.268421i$
	$-0.599359 \pm 1.613134i$	$-0.599359 \pm 1.613134i$
0.001	$-0.145967 \pm 0.342158i$	$-0.145953 \pm 0.342067i$
	$-1.140746 \pm 0.381153i$	$-1.140651 \pm 0.381689i$
	$-0.239147 \pm 1.268418i$	$-0.239154 \pm 1.268361i$
	$-0.599362 \pm 1.613132i$	$-0.599368 \pm 1.613077i$
0.003	$-0.145979 \pm 0.342160i$	$-0.145939 \pm 0.341890i$
	$-1.141179 \pm 0.379809i$	$-1.140895 \pm 0.381421i$
	$-0.239139 \pm 1.268411i$	$-0.239160 \pm 1.268241i$
	$-0.599369 \pm 1.613126i$	$-0.599385 \pm 1.612963i$
0.005	$-0.145991 \pm 0.342163i$	$-0.145925 \pm 0.341712i$
	$-1.141613 \pm 0.378460i$	$-1.141138 \pm 0.381154i$
	$-0.239131 \pm 1.268404i$	$-0.239167 \pm 1.268121i$
	$-0.599376 \pm 1.613120i$	$-0.599403 \pm 1.612848i$

Table 4
Eigenvectors of the modified system in Fig. 1, if the damping coefficient is changed slightly by an amount Δc_2 around its nominal value $c_2 = 4$

Δc_2	$\tilde{\mathbf{y}}_{1,2}$ from Eq. (23)	$\tilde{\mathbf{y}}_{1,2}$ from Eq. (42)
0	1.000000 1.410651 ∓ 0.064320i 1.454578 ∓ 0.037187i 0.843922 ± 0.161608i	1.000000 1.410651 ∓ 0.064320i 1.454578 ∓ 0.037187i 0.843922 ± 0.161608i
0.001	1.000000 1.410650 ∓ 0.06432i 1.454576 ∓ 0.037196i 0.843955 ± 0.161672i	1.000000 1.410651 ∓ 0.064320i 1.454579 ∓ 0.037187i 0.843749 ± 0.161597i
0.003	1.000000 1.410650 ∓ 0.064332i 1.454572 ∓ 0.037215i 0.844021 ± 0.161801i	1.000000 1.410651 ∓ 0.064320i 1.454580 ∓ 0.037186i 0.843403 ± 0.161576i
0.005	1.000000 1.410649 ∓ 0.064340i 1.454568 ∓ 0.037234i 0.844087 ± 0.161930i	1.000000 1.410651 ∓ 0.064320i 1.454581 ∓ 0.037186i 0.843056 ± 0.161555i
Δc_2	$\tilde{\mathbf{y}}_{3,4}$ from Eq. (23)	$\tilde{\mathbf{y}}_{3,4}$ from Eq. (42)
0	1.000000 2.619564 ∓ 2.017422i 3.556430 ∓ 4.976448i −13.853560 ± 22.065139i	1.000000 2.619564 ∓ 2.017422i 3.556430 ∓ 4.976448i −13.853560 ± 22.065139i
0.001	1.000000 2.623597 ∓ 2.016467i 3.567372 ∓ 4.975970i −13.897999 ± 22.059577i	1.000000 2.619564 ∓ 2.017422i 3.556419 ∓ 4.976430i −13.851506 ± 22.063981i
0.003	1.000000 2.631689 ∓ 2.014526i 3.589345 ∓ 4.974931i −13.987238 ± 22.048092i	1.000000 2.619564 ∓ 2.017422i 3.556397 ∓ 4.976393i −13.847395 ± 22.061664i
0.005	1.000000 2.639818 ∓ 2.012541i 3.611438 ∓ 4.973778i −14.076958 ± 22.036119i	1.000000 2.619564 ∓ 2.017422i 3.556375 ∓ 4.976356i −13.843281 ± 22.059346i

Table 4 (*continued*)

Δc_2	$\tilde{\mathbf{y}}_{5,6}$ from Eq. (23)	$\tilde{\mathbf{y}}_{5,6}$ from Eq. (42)
0	1.000000 0.384640 ± 0.286630i −0.417956 ∓ 0.174735i −0.376708 ± 0.058175i	1.000000 0.384640 ± 0.286630i −0.417956 ∓ 0.174735i −0.376708 ± 0.058175i
0.001	1.000000 0.384649 ± 0.286631i −0.417955 ∓ 0.174728i −0.376719 ± 0.058132i	1.000000 0.384640 ± 0.286630i −0.417957 ∓ 0.174735i −0.376746 ± 0.058162i
0.003	1.000000 0.384666 ± 0.286634i −0.417952 ∓ 0.174712i −0.376742 ± 0.058045i	1.000000 0.384640 ± 0.286630i −0.417957 ∓ 0.174735i −0.376824 ± 0.058136i
0.005	1.000000 0.384683 ± 0.286636i −0.417950 ∓ 0.174697i −0.376765 ± 0.057958i	1.000000 0.384640 ± 0.286630i −0.417958 ∓ 0.174735i −0.376902 ± 0.058109i
Δc_2	$\tilde{\mathbf{y}}_{7,8}$ from Eq. (23)	$\tilde{\mathbf{y}}_{7,8}$ from Eq. (42)
0	1.000000 −0.557704 ± 0.299015i 0.105876 ∓ 0.205547i −0.093687 ∓ 0.178362i	1.000000 −0.557704 ± 0.299015i 0.105876 ∓ 0.205547i −0.093687 ∓ 0.178362i
0.001	1.000000 −0.557706 ± 0.299006i 0.105870 ∓ 0.205533i −0.093649 ∓ 0.178386i	1.000000 −0.557704 ± 0.299015i 0.105876 ∓ 0.205547i −0.093709 ∓ 0.178369i
0.003	1.000000 −0.557710 ± 0.298988i 0.105858 ∓ 0.205507i −0.093572 ∓ 0.178432i	1.000000 −0.557704 ± 0.299015i 0.105875 ∓ 0.205547i −0.093753 ∓ 0.178382i
0.005	1.000000 −0.557715 ± 0.298969i 0.105846 ∓ 0.205480i −0.093495 ∓ 0.178478i	1.000000 −0.557704 ± 0.299015i 0.105875 ∓ 0.205548i −0.093797 ∓ 0.178395i

Now, one can test the reliability of the sensitivity-based formulae (41)–(43). To this end, the system in Fig. 1 is denoted as "nominal" system and it is assumed that the damping constant c_2 changes by an amount Δc_2 around its nominal value $c_2 = 4$, due to some reason, which in turn causes a modification of the system. Hence the damping related parameter α is chosen as c_2.

The eigenvalues of the system, thus modified are calculated by solving the characteristic equation (21) numerically and then using the sensitivity-based formula (41), considering Eq. (37). The results are given in Table 3. The complex numbers in the first column are "exact" values obtained from Eq. (21), whereas those of the second column come from the approximate formula (41). As can be seen from the table, the agreement of the numbers in both columns is very good, especially for small values of Δc_2. This in turn means that formula (41) gives accurate approximations for the eigenvalues of the modified system, without having to resolve the characteristic equation (21) with the parameters of the modified system. In a similar manner, the eigenvectors of the modified system are calculated first by Eq. (23) and then by the sensitivity-based formula (42), considering Eq. (38). The results are collected in Table 4. The vectors in the first columns are "exact" eigenvectors obtained from Eq. (23) and those in the second columns come from Eq. (42), considering Eq. (38). One sees clearly, that the agreement of the vectors in both columns is very good. This means that formula (42) gives accurate approximations for the eigenvectors of the modified system. It is observed that also formula (43) gives very good approximate results for the receptance matrix of the modified system, but the numerical results are not given here due to space limitations.

4. Conclusions

This study is concerned with a non-proportionally damped linear discrete mechanical system with n d.o.f. to which m additional viscous dampers are attached, for some reason. Making use of a recently developed formula, the $n \times n$ characteristic determinant of the above-described system is reduced to a much smaller determinant of order $m \times m$, where $m \ll n$ is a frequently encountered case in practice. Besides, an alternative form of the characteristic equation and an explicit analytical expression for the eigenvectors of the modified system are given. Further, the eigenvalue, eigenvector and receptance matrix sensitivities with respect to a damping-related parameter are derived.

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