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Letter to the Editor

A field method in the study of weakly non-linear two-degree-of-freedom oscillatory systems

Ivana Kovacic

Faculty of Technical Sciences, Department of Mechanics, Trg Dositeja Obradovica 6, 21121 Novi Sad, Yugoslavia

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1. Introduction

The problem of finding the motion of a weakly non-linear oscillator with two degrees of freedom has been widely considered and solved by using various approximative methods [1–3]. The aim of this note is to show that a field method can also serve this purpose. The following extension of the field method to such problems contributes to the fact about its generality and applicability to a wide variety of systems: non-conservative dynamical systems [4,5] (for whose study it was primarily developed), weakly non-linear systems with one degree of freedom [6–8] and non-holonomic systems [9].

The field method theory is based on the assumption that one of the state variables of the system (generalized co-ordinate or momentum) can be expressed as a function of the other state variables and time. As a consequence, the problem of the direct integration of the equations of motion is replaced by finding the complete solution of a partial differential equation. The sought solution for the motion is available from this complete solution by applying simple algebraic operations.

2. Extension of the method

Consider a special type of two-degree-of-freedom system:

$$\begin{aligned} \dot{x} &= p, \\ \dot{p} &= -\omega_1^2 x + \varepsilon F_1(x, y, p, z, t), \\ \dot{y} &= z, \\ \dot{z} &= -\omega_2^2 y + \varepsilon F_2(x, y, p, z, t), \end{aligned} \tag{1}$$

E-mail address: ivanakov@uns.ns.ac.yu (I. Kovacic).

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where x, y, p, z are state variables, ω_1 and ω_2 are known constant frequencies of the system, F_1 and F_2 are non-linear functions of the state variables and time t, and ε is a small parameter ($0<\varepsilon\ll 1$), while an overdot denotes the differentiation with respect to time.

In accordance with the field method theory and its prior applications to linear systems with two degrees of freedom [4,5], the next step to be performed in order to find the motion of Eq. (1), is to define one of the state variables as only field dependent on the other state variables and time. However, non-linearity of system (1) seems to require the completely different concept of introducing two fields. The reason for exerting this essential modification is the conspicuous transformation of system (1) into two uncoupled subsystems of differential equations for the case when $\varepsilon = 0$. Then, these two subsystems can be observed as two mutually independent ones with one degree of freedom and for each of them one field can be specified.

So, for system (1) each co-ordinate is defined as a field of time t and the corresponding momentum:

$$x = U_1(t,p), \tag{2}$$

$$y = U_2(t,z). \tag{3}$$

Partial differentials of these two expressions in combination with Eq. (1) yield two basic equations:

$$\frac{\partial U_1}{\partial t} + \frac{\partial U_1}{\partial p}[-\omega_1^2 U_1 + \varepsilon F_1(U_1, U_2, p, z, t)] - p = 0, \tag{4}$$

$$\frac{\partial U_2}{\partial t} + \frac{\partial U_2}{\partial z}[-\omega_2^2 U_2 + \varepsilon F_2(U_1, U_2, p, z, t)] - z = 0. \tag{5}$$

Closed forms of conditioned form solutions [5] of the weakly non-linear equations (4) and (5) are seldom obtainable. In order to achieve approximate solutions, the technique of multiple scales in the first approximation is used and two independent variables, 'fast' and 'slow' times [2] are introduced:

$$T = t, \quad \tau = \varepsilon t. \tag{6}$$

Further, both fields U_1 and U_2 and both state variables p and z can be expanded in powers of the small parameter ε:

$$\begin{aligned} U_1(T,p,\varepsilon) &= U_{10}(T,\tau,p_0) + \varepsilon U_{11}(T,\tau,p_1) + \cdots, \\ p(T,\varepsilon) &= p_0(T,\tau) + \varepsilon p_1(T,\tau) + \cdots, \\ U_2(T,z,\varepsilon) &= U_{20}(T,\tau,z_0) + \varepsilon U_{21}(T,\tau,z_1) + \cdots, \\ z(T,\varepsilon) &= z_0(T,\tau) + \varepsilon z_1(T,\tau) + \cdots. \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eqs. (4), (5) and equating coefficients of the same power of ε, the following system of coupled partial differential equations is obtained:

$$\frac{\partial U_{10}}{\partial T} - \omega_1^2 U_{10}\frac{\partial U_{10}}{\partial p_0} - p_0 = 0, \tag{8}$$

$$\frac{\partial U_{11}}{\partial T} - \omega_1^2 U_{11}\frac{\partial U_{11}}{\partial p_1} - p_1 = -\frac{\partial \bar{U}_{10}}{\partial \tau} - \frac{\partial \bar{U}_0}{\partial p_0} F_1(\bar{U}_{10}^*, \bar{U}_{20}^*, p_0, z_0, T, \tau), \tag{9}$$

$$\frac{\partial U_{20}}{\partial T} - \omega_2^2 U_{20} \frac{\partial U_{20}}{\partial z_0} - z_0 = 0, \tag{10}$$

$$\frac{\partial U_{21}}{\partial T} - \omega_2^2 U_{21} \frac{\partial U_{21}}{\partial z_1} - z_1 = -\frac{\partial \bar{U}_{20}}{\partial \tau} - \frac{\partial \bar{U}_{20}}{\partial z_0} F_2(\bar{U}_{10}^*, \bar{U}_{20}^*, p_0, z_0, T, \tau), \tag{11}$$

where $\bar{U}_{10}^*$ and $\bar{U}_{20}^*$ are the so-called solutions along trajectories, which stand for the conditioned form solutions calculated for the value of the first component of the corresponding momentum.

The conditioned form solution of quasi-linear partial differential equation (8) can be assumed in the form [8]:

$$\bar{U}_{10} = \frac{p_0}{\omega_1} \tan(\omega_1 T + C_1) + \frac{a_1(\tau)\cos(\beta_1(\tau) - C_1)}{\cos(\omega_1 T + C_1)}, \tag{12}$$

where the functions $a_1(\tau)$ and $\beta_1(\tau)$ are to be calculated and C_1 is an arbitrary constant.

Applying Vujanovic's theorem [4,5], i.e. its condition about independency of the conditioned form solution on the arbitrary constant $\partial \bar{U}_{10}/\partial C_1 = 0$, the first component of the momentum p_0 in the first approximation is obtained:

$$p_0 = -\omega_1 a_1(\tau) \sin(\omega_1 T + \beta_1(\tau)). \tag{13}$$

Using Eq. (13), the conditioned form solution (12) transforms into the solution along trajectory:

$$\bar{U}_{10}^* = a_1(\tau) \cos(\omega_1 T + \beta_1(\tau)). \tag{14}$$

It has the well-known form for the solution in the first approximation for the motion of a vibrational system, where functions $a_1(\tau)$ and $\beta_1(\tau)$ imply an amplitude and phase of vibrations.

Analogously, the corresponding component of the second field and its momentum are

$$\bar{U}_{20} = \frac{z_0}{\omega_2} \tan(\omega_2 T + C_2) + \frac{a_2(\tau)\cos(\beta_2(\tau) - C_2)}{\cos(\omega_2 T + C_2)}, \tag{15}$$

$$z_0 = -\omega_2 a_2(\tau) \sin(\omega_2 T + \beta_2(\tau)), \tag{16}$$

$$\bar{U}_{20}^* = a_2(\tau) \cos(\omega_2 T + \beta_2(\tau)), \tag{17}$$

while a_2 and β_2 are functions to be determined and C_2 is an arbitrary constant.

On the basis of Eqs. (12) and (15), the complete solutions for the second components of the fields can be taken as

$$U_{11} = \frac{p_1}{\omega_1} \tan(\omega_1 T + C_1) + \frac{D_1(T, \tau)}{\cos(\omega_1 T + C_1)}, \tag{18}$$

$$U_{21} = \frac{z_1}{\omega_2} \tan(\omega_2 T + C_2) + \frac{D_2(T, \tau)}{\cos(\omega_2 T + C_2)}, \tag{19}$$

where D_1 and D_2 are unknown functions.

So, after substituting Eqs. (12)–(19) into Eqs. (9) and (11), the following system is obtained:

$$\begin{aligned}\frac{\mathrm{d}D_i}{\mathrm{d}T} = & -\frac{\mathrm{d}a_i}{\mathrm{d}\tau}\cos(\beta_i - C_i) + a_i\frac{\mathrm{d}\beta_i}{\mathrm{d}\tau}\sin(\beta_i - C_i) \\ & + \frac{\sin(\omega_i T + C_i)}{\omega_i}F_i(\bar{U}_{10}(p_0), \bar{U}_{20}(z_0), p_0, z_0, T, \tau),\end{aligned} \tag{20}$$

where $i = 1, 2$.

The further consideration assumes the elimination of secular terms. However, this procedure depends on the exact forms of the non-linear functions F_1 and F_2. Therefore, their forms will be specified and the solution for the motion will be found completely.

2.1. Example

Consider the weakly non-linear vibrational system (1), when [3]

$$\omega_1 = 1 \approx 2\omega_2, \quad F_1 = y^2, \quad F_2 = 2xy - \sigma y, \tag{21}$$

where σ is a constant expressing nearness of the frequencies ω_1 and ω_2 in a resonant mode.

For this problem, Eqs. (20) become

$$\begin{aligned}\frac{\mathrm{d}D_1}{\mathrm{d}T} = & -\frac{\mathrm{d}a_1}{\mathrm{d}\tau}\cos(\beta_1 - C_1) + a_1\sin(\beta_1 - C_1)\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} \\ & - \sin(T + C_1)a_2^2\cos^2\left(\tfrac{1}{2}T + \beta_2\right),\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\mathrm{d}D_2}{\mathrm{d}T} = & -\frac{\mathrm{d}a_2}{\mathrm{d}\tau}\cos(\beta_2 - C_2) + a_2\sin(\beta_2 - C_2)\frac{\mathrm{d}\beta_2}{\mathrm{d}\tau} \\ & - 4\sin\left(\tfrac{1}{2}T + C_2\right)a_1a_2\cos(T + \beta_1)\cos\left(\tfrac{1}{2}T + \beta_2\right) \\ & + 2\sigma a_2\sin\left(\tfrac{1}{2}T + C_2\right)\cos\left(\tfrac{1}{2}T + \beta_2\right).\end{aligned} \tag{23}$$

Eliminating secular terms from the previous system, gives

$$\begin{aligned}& -\frac{\mathrm{d}a_1}{\mathrm{d}\tau}\cos\beta_1 + a_1\sin\beta_1\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} + \tfrac{1}{4}a_2^2\sin 2\beta_2 = 0, \\ & -\frac{\mathrm{d}a_1}{\mathrm{d}\tau}\sin\beta_1 - a_1\cos\beta_1\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} - \tfrac{1}{4}a_2^2\cos 2\beta_2 = 0, \\ & -\frac{\mathrm{d}a_2}{\mathrm{d}\tau}\cos\beta_2 + a_2\sin\beta_2\frac{\mathrm{d}\beta_2}{\mathrm{d}\tau} - a_1a_2\sin(\beta_2 - \beta_1) - \sigma a_2\sin\beta_2 = 0, \\ & -\frac{\mathrm{d}a_2}{\mathrm{d}\tau}\sin\beta_2 - a_2\cos\beta_2\frac{\mathrm{d}\beta_2}{\mathrm{d}\tau} - a_1a_2\cos(\beta_2 - \beta_1) + \sigma a_2\cos\beta_2 = 0.\end{aligned} \tag{24}$$

After some transformations, system (24) gives the first order differential equations for the amplitude and phase of vibrations:

$$\begin{aligned}& -\frac{\mathrm{d}a_1}{\mathrm{d}\tau} + \tfrac{1}{4}a_2^2\sin(2\beta_2 - \beta_1) = 0, \\ & a_1\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} + \tfrac{1}{4}a_2^2\cos(2\beta_2 - \beta_1) = 0,\end{aligned}$$

$$-\frac{da_2}{d\tau} - a_1 a_2 \sin(2\beta_2 - \beta_1) = 0,$$

$$a_2 \frac{d\beta_2}{d\tau} + a_1 a_2 \cos(2\beta_2 - \beta_1) - \sigma a_2 = 0. \tag{25}$$

These equations are in complete agreement with those given in Ref. [3].

3. Conclusion

In this note, the analytic procedure for obtaining the solution in the first approximation of a weakly non-linear oscillatory system with two degrees of freedom is developed. It is based on the field method, which is combined with the technique of multiple scales. It is shown that for this kind of a system two fields of the state variables have to be defined and, consequently, two basic equations have to be solved. The complete algorithm for finding the first order differential equations for the amplitudes and phases of vibrations are obtained. Thus, being narrowed down to a procedure, the technique of the field method is amenable to being applied to weakly non-linear systems with many degrees of freedom.

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