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Analysis of wave propagation in sandwich plates with and without heavy fluid loading

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Abstract

The paper addresses wave motions in an unbounded sandwich plate with and without heavy fluid loading in a plane problem formulation. A sandwich plate is composed of two identical isotropic skin plies and an isotropic core ply. Several alternative theories for stationary dynamics of such a plate or a beam are derived, including a formulation in the framework of a theory of elasticity applied for a core ply. 'In-phase' and 'anti-phase' wave motions (with respect to transverse deflections of skins) of a sandwich beam are analyzed independently of each other. Dispersion curves obtained by the use of 'elementary' theories are compared with those obtained by the use of an 'exact' theory (which involves the theory of elasticity in a description of wave motion in a core ply) for a plate without fluid loading. It is shown that these simplified models are capable of giving a complete and accurate description of all propagating waves in not too high-frequency range, which is sufficient in practical naval and aerospace engineering. In the case of heavy fluid loading, similar analysis is performed for 'anti-phase' wave motions of a beam. Two simplified theories as well as an 'exact' one are extended to capture fluid loading effects. A good agreement between results obtained in 'elementary' and 'exact' problem formulations is demonstrated. The role of fluid's compressibility in the generation of propagating waves in a sandwich plate is explored. It is shown that, whereas analysis of wave motions in the case of an incompressible fluid predicts an existence of two propagating waves, only one such wave exists when a fluid is sufficiently compressible. The threshold magnitude of the ratio of a sound speed in an acoustic medium to a sound speed in a skin's material is found, which separates these two regimes of wave motions for a given set of parameters of sandwich plate composition.

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1. Introduction

Sandwich plates and shells are widely used in many technical applications, e.g., naval architecture, aerospace or chemical industries, etc. because this composition of a thin-walled structure conveniently combines the properties of a high strength and a low weight. The issues of strength and reliability of such structures have been thoroughly studied in many publications, see, for example Refs. [1,2]. The stability and dynamics of sandwich beams and plates have also been explored in recent publications [3–10]. Since naval sandwich structures are typically exposed to heavy fluid loading, wave propagation and vibrations of sandwich plates in contact with fluid (either at rest or with a mean flow) have been analyzed in the papers [7,9,10].

As discussed in Ref. [10], there are two possibilities to describe dynamics of sandwich plates. Equations of motions of each ply may be formulated in the framework of a theory of elasticity, which should be solved with continuity conditions at the interfaces between plies, see, for example Refs. [3,4]. Alternatively, some hypotheses may be adopted concerning the deformation of an arbitrary cross-section of the whole package of plies and reduced equations are then derived [5–7]. If fluid loading is included, then additional continuity conditions are formulated at fluid–structure interfaces and a wave equation is introduced for an acoustic medium in both the 'exact' and the 'simplified' theories. Naturally, the reduced formulation does not permit the capture of short-wave, high-frequency motions in sandwich plates, but the essential (from the practical viewpoint) features of wave propagation in these structures are related to the audio frequency range. In this range, the length of a propagating wave exceeds the thickness of the whole package of plies by at least 3–4 times. Thus, for low and intermediate frequencies, it is expedient to estimate the actual validity of the simplified theories in describing wave motions in a sandwich plate. In the present paper, the dispersion equations are derived for wave motions in a plane problem formulation, when the dynamics of a plate are reduced to the dynamics of a beam. However, since the analysis involves solutions of homogeneous equations, the results obtained here are entirely applicable for other types of wave motions in a sandwich plate considered in the framework of a spatial problem formulation (e.g., cylindrical waves in polar co-ordinates).

The paper is structured as follows. In Section 2, propagation of waves in sandwich beams without fluid loading is considered in the framework of a theory of elasticity. Due to the natural symmetry of a sandwich plate composition, two classes of wave motions ('in-phase' and 'anti-phase' ones) are analyzed separately and two dispersion equations are derived. Elementary modelling of wave propagation for the same two classes of motions is presented in Section 3. The dispersion curves, which are obtained for a sandwich beam without fluid loading in the framework of these theories, are compared in Section 4. Section 5 addresses an 'exact' theory of 'anti-phase' wave motions of a sandwich beam with heavy fluid loading, whereas in Section 6 two 'elementary' theories for this type of motions are suggested. These 'exact' and 'elementary' theories are formulated for a compressible and for an incompressible fluid. Results of analysis of dispersion equations, derived in Sections 5 and 6 are discussed in Section 7 with special attention paid to the role of the compressibility of a fluid. Finally, in Section 8 conclusions are presented.

2. Formulation of the problem within the framework of a theory of elasticity

Consider an infinitely long sandwich plate consisting of two thin and relatively stiff plies (skins) and a soft core ply between them as is shown in Fig. 1. To investigate wave propagation in such a structure, it is sufficient to use the plane problem formulation (i.e., to consider its cylindrical bending).

As is well known [1,2], mechanical properties of skin and core plies of sandwich plates used, for example, in naval or aerospace structures are very different. Specifically, elastic and geometry parameters of skin plies considered individually are normally those of conventional thin plates, so that their dynamics are adequately described by a standard Kirchhoff theory. However, due to the interaction between skin and core plies, it is not sufficient to take into account only their flexural wave motions. The longitudinal components of displacements should also be included in the analysis of the wave propagation

$$D_1 w_1^{(4)} + \rho_1 h_1 \ddot{w}_1 = q_{w1} + m_1', \tag{1a}$$

$$E_1 h_1 u_1'' - \rho_1 h_1 \ddot{u}_1 = -q_{u1}, \tag{1b}$$

$$D_3 w_3^{(4)} + \rho_3 h_3 \ddot{w}_3 = q_{w3} + m_3', \tag{1c}$$

$$E_3 h_3 u_3'' - \rho_3 h_3 \ddot{u}_3 = -q_{u3}. \tag{1d}$$

Here, $u_k(x,t)$ and $w_k(x,t)$, $k = 1,3$, are the longitudinal and the flexural displacements of the mid-surfaces of skin plies, positive if codirected with the co-ordinate axes in Fig. 1. $q_{wk}(x,t)$ and $q_{uk}(x,t)$, $k = 1,3$ are the distributed longitudinal and transverse forces acting at skin plies, respectively. The distributed moments, $m_k(x,t)$, $k = 1,3$, are also taken into account. $D_k = E_k h_k^3/12(1-v^2)$, $k = 1,\ 3$ is the conventional formulation of cylindrical stiffness, primes and dots denote derivatives on spatial and temporal co-ordinates x and t, respectively. Elastic properties of material of each ply are specified by densities ρ_k, $k = 1,\ 3$, Young's moduli E_k, $k = 1,\ 3$, the Poisson coefficients $v_1 = v_3 = v$. Right sides of Eq. (1) are composed of stresses acting at the interfaces between skin and core plies. In general, they may also contain external driving forces and moments, but as far as propagation of free waves is concerned an external loading is omitted.

As has been discussed in the Introduction, the core ply of a sandwich plate is much thicker and it is composed of material, which is much softer than the skin plies. Thus, an elementary theory of plates is not applicable and dynamics of a core ply should be described by the standard dynamic theory of elasticity, see for example Ref. [11]. In the plane problem formulation, Lamé equations

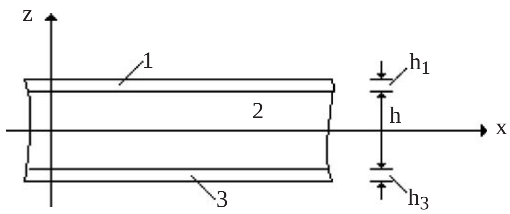


Fig. 1. Sandwich plate composition.

are reduced as follows

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}=0, \tag{2a}$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}=0. \tag{2b}$$

Here $c_1^2=E(1-v)/\rho(1+v)(1-2v)$ and $c_2^2=E/2(1+v)\rho$ are velocities of acoustic and shear waves in the material, respectively. Material density of a core ply, its Young's module and the Poisson coefficient are denoted as ρ, E and v, respectively. Potentials ϕ and ψ are introduced to formulate displacements in the following way:

$$u_2=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z},\quad w_2=\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial z}. \tag{3}$$

Then stresses are defined as [11]

$$\sigma_x=\lambda\Delta\phi+2\mu\left(\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\partial z}\right),$$

$$\sigma_z=\lambda\Delta\phi+2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\psi}{\partial x\partial z}\right),$$

$$\tau_{xz}=\mu\left(2\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right). \tag{4}$$

In these equations, λ and μ are Lamé elastic moduli, defined as

$$\lambda=\frac{vE}{(1+v)(1-2v)},\quad \mu=\frac{E}{2(1+v)}.$$

The system of differential equations (2) should be solved with the following compatibility conditions at the interfaces

$$z=\frac{h}{2}:\quad w_2(x,z,t)=w_1(x,t),\ \ u_2(x,z,t)=u_1(x,t)+\frac{h_1}{2}\frac{\partial w_1(x,t)}{\partial x}, \tag{5a}$$

$$z=-\frac{h}{2}:\quad w_2(x,z,t)=w_3(x,t),\ \ u_2(x,z,t)=u_3(x,t)-\frac{h_1}{2}\frac{\partial w_3(x,t)}{\partial x}. \tag{5b}$$

Since the functions, $u_k(x,t)$, $w_k(x,t)$, $k=1,3$, are defined for the mid-surfaces of skin plies, the continuity conditions at the interfaces for the longitudinal displacements are formulated with the components $\pm(h_k/2)\partial w_k(x,t)/\partial x$, $k=1,3$ (i.e., the angles of rotation due to the flexural motion) taken into account. It is consistent with the governing equations (1) for the skin plies, where distributed moments are also included. This formulation is valid as long as the elementary Kirchhoff theory is applicable to describe wave motion in the skins.

The interfacial distributed forces and moments are formulated as

$$q_{w3}(x,t)=\sigma_z\left(x,-\frac{h}{2},t\right),\quad q_{u3}(x,t)=-\tau_{xz}\left(x,-\frac{h}{2},t\right),\quad m_3(x,t)=\frac{h_3}{2}\tau_{xz}\left(x,-\frac{h}{2},t\right), \tag{6a}$$

$$q_{w1}(x,t) = -\sigma_z\left(x, \frac{h}{2}, t\right), \quad q_{u1}(x,t) = \tau_{xz}\left(x, \frac{h}{2}, t\right), \quad m_1(x,t) = \frac{h_1}{2}\tau_{xz}\left(x, \frac{h}{2}, t\right). \tag{6b}$$

The following scaling is introduced: $x = \bar{x}h,\ z = \bar{z}h,\ u_j = \bar{u}_j h,\ w_j = \bar{w}_j h,\ j = 1,\ 2,\ 3.$

Propagation of a harmonic elastic wave in an infinitely long plate is considered, so that

$$\bar{u}_j = U_j \exp(k\bar{x} - \mathrm{i}\omega t), \quad \bar{w}_j = W_j \exp(k\bar{x} - \mathrm{i}\omega t), \quad j = 1,2,3,$$
$$\phi = \Phi(\bar{z})\exp(k\bar{x} - \mathrm{i}\omega t), \quad \psi = \Psi(\bar{z})\exp(k\bar{x} - \mathrm{i}\omega t). \tag{7}$$

Hereafter bars over non-dimensional variables are omitted, ω is a positive excitation frequency and k is, a priori, a complex wave number. Eqs. (7) are substituted to Eqs. (3), (5) and the problem in elasticity for the core ply is formulated as

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + \left[k^2 + \left(\frac{\omega h}{c_1}\right)^2\right]\Phi = 0, \tag{8a}$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + \left[k^2 + \left(\frac{\omega h}{c_2}\right)^2\right]\Psi = 0, \tag{8b}$$

$$z = \frac{1}{2}: \quad \frac{\mathrm{d}\Phi}{\mathrm{d}z} + k\Psi = h^2 W_1, \quad k\Phi - \frac{\mathrm{d}\Psi}{\mathrm{d}z} = h^2 U_1 + \frac{hh_1}{2}kW_1, \tag{8c}$$

$$z = -\frac{1}{2}: \quad \frac{\mathrm{d}\Phi}{\mathrm{d}z} + k\Psi = h^2 W_3, \quad k\Phi - \frac{\mathrm{d}\Psi}{\mathrm{d}z} = h^2 U_3 - \frac{hh_1}{2}kW_3. \tag{8d}$$

The symmetric composition of a sandwich plate ($h_1 = h_3, E_1 = E_3, \rho_1 = \rho_3$) is considered and therefore it is convenient to identify two uncoupled classes of linear wave motions and analyze them separately. One of them is related to flexural and shear vibrations, which preserve overall thickness of the whole structure, i.e., $W_1 = W_3,\ U_1 = -U_3$. Since lateral displacements of skins are assumed to be the same, this class of waves is referred to as 'in-phase' modes in the present paper. In opposition, by letting $W_1 = -W_3,\ U_1 = U_3$, a kind of 'thickness-changing' motion of a plate is specified. This class of motions may also be called 'anti-phase' modes. It is clear that from the practical viewpoint the 'anti-phase' motions become important only when a core ply is much softer than skin plies.

Consider the case of 'in-phase' wave motions and let $W_1 = W_3, U_1 = -U_3$. A general solution of Eqs. (8a) and (8b) is formulated as

$$\Phi(z) = A\sinh(\gamma_1 z), \quad \gamma_1^2 = -k^2 - \left(\frac{\omega h}{c_1}\right)^2,$$

$$\Psi(z) = B\cosh(\gamma_2 z), \quad \gamma_2^2 = -k^2 - \left(\frac{\omega h}{c_2}\right)^2. \tag{9}$$

The boundary conditions (8c) and (8d) are reduced to

$$A\gamma_1\cosh\left(\frac{\gamma_1}{2}\right) + Bk\cosh\left(\frac{\gamma_2}{2}\right) = W_1 h^2,$$

$$Ak\sinh\left(\frac{\gamma_1}{2}\right)+B\gamma_2\sinh\left(\frac{\gamma_2}{2}\right)=U_1h^2+\frac{hh_1}{2}kW_1,$$

respectively. Solution of this system of algebraic equations is elementary

$$A=\frac{h^2kU_1\cosh\left(\frac{\gamma_2}{2}\right)+\frac{hh_1}{2}k^2W_1\cosh\left(\frac{\gamma_2}{2}\right)+h^2\gamma_2W_1\sinh\left(\frac{\gamma_2}{2}\right)}{k^2\sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)+\gamma_1\gamma_2\cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)},\tag{10a}$$

$$B=\frac{-h^2\gamma_1U_1\cosh\left(\frac{\gamma_1}{2}\right)-\frac{hh_1}{2}\gamma_1kW_1\cosh\left(\frac{\gamma_1}{2}\right)+h^2kW_1\sinh\left(\frac{\gamma_1}{2}\right)}{k^2\sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)+\gamma_1\gamma_2\cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)}.\tag{10b}$$

Two sets of the differential equations of motions of skin plies (1) are reduced to the following two algebraic equations with respect to amplitudes (U_1, W_1):

$$\begin{aligned}&\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4-\rho_1hh_1\omega^2\right]W_1+\left[-\lambda\left(\frac{\omega h}{c_1}\right)^2\sinh\left(\frac{\gamma_1}{2}\right)+2\mu\gamma_1^2\sinh\left(\frac{\gamma_1}{2}\right)+\mu k^2\gamma_1\frac{h_1}{h}\cosh\left(\frac{\gamma_1}{2}\right)\right]A\\&\quad+\left[2\mu k\gamma_2\sinh\left(\frac{\gamma_2}{2}\right)+\frac{1}{2}\frac{h_1}{h}\mu k^3\cosh\left(\frac{\gamma_2}{2}\right)-\frac{1}{2}\frac{h_1}{h}\mu k\gamma_2^2\cosh\left(\frac{\gamma_2}{2}\right)\right]B=0,\\&\left[\frac{E_1h_1}{h}k^2+\rho_1hh_1\omega^2\right]U_1+2\mu\gamma_1\cosh\left(\frac{\gamma_1}{2}\right)A+\left[\mu k^2\cosh\left(\frac{\gamma_2}{2}\right)-\mu\gamma_2^2\cosh\left(\frac{\gamma_2}{2}\right)\right]B=0.\end{aligned}\tag{11}$$

Putting to zero the characteristic determinant of these equations gives a dispersion equation for the 'in-phase' set of modes.

A solution in the case of 'anti-phase' motions ($W_1=-W_3,\ U_1=U_3$) is obtained in a similar way. Elastic potentials are defined as

$$\Phi(z)=\tilde{A}\cosh(\gamma_1z),$$

$$\Psi(z)=\tilde{B}\sinh(\gamma_2z).$$

Their amplitudes are

$$\tilde{A}=\frac{h^2kU_1\sinh\left(\frac{\gamma_2}{2}\right)+\frac{hh_1}{2}k^2W_1\sinh\left(\frac{\gamma_2}{2}\right)+h^2\gamma_2W_1\cosh\left(\frac{\gamma_2}{2}\right)}{k^2\cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)+\gamma_1\gamma_2\sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)},\tag{12a}$$

$$\tilde{B}=\frac{-h^2\gamma_1U_1\sinh\left(\frac{\gamma_1}{2}\right)-\frac{hh_1}{2}\gamma_1kW_1\sinh\left(\frac{\gamma_1}{2}\right)+h^2kW_1\cosh\left(\frac{\gamma_1}{2}\right)}{k^2\cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)+\gamma_1\gamma_2\sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)}.\tag{12b}$$

Then differential equations of motions of skin plies are reduced to the following algebraic equations with respect to amplitudes (U_1, W_1)

$$\left[\frac{E_1 h_1^3}{12(1-\nu^2)h^3}k^4 - \rho_1 h h_1 \omega^2\right] W_1 + \left[\lambda\left(\frac{\omega h}{c_1}\right)^2 \cosh\left(\frac{\gamma_1}{2}\right) + 2\mu\gamma_1^2\cosh\left(\frac{\gamma_1}{2}\right) - \mu k^2\gamma_1\frac{h_1}{h}\sinh\left(\frac{\gamma_1}{2}\right)\right]\tilde{A}$$
$$+\left[2\mu k\gamma_2\cosh\left(\frac{\gamma_2}{2}\right) - \frac{1}{2}\frac{h_1}{h}\mu k^3\sinh\left(\frac{\gamma_2}{2}\right) + \frac{1}{2}\frac{h_1}{h}\mu k\gamma_2^2\sinh\left(\frac{\gamma_2}{2}\right)\right]\tilde{B} = 0,$$
$$\left[\frac{E_1 h_1}{h}k^2 + \rho_1 h h_1\omega^2\right] U_1 - 2\mu\gamma_1\sinh\left(\frac{\gamma_1}{2}\right)\tilde{A} - \left[\mu k^2\sinh\left(\frac{\gamma_2}{2}\right) - \mu\gamma_2^2\sinh\left(\frac{\gamma_2}{2}\right)\right]\tilde{B} = 0. \tag{13}$$

Putting to zero its determinant, a dispersion equation is obtained for the 'anti-phase' set of modes.

The characteristic equations obtained from Eqs. (11) and (13) in both these cases have an infinitely large number of roots, which are either purely real, purely imaginary or complex. In principle, various methods may be used to find these roots, depending on available software and computing facilities. In this paper, the propagation of waves in the sandwich structures used in shipbuilding or aerospace industries is addressed. From the practical viewpoint, it is most important to find the roots, which define propagating wave in the frequency range up to about 10–15 kHz. Then the first roots are not too large and the transcendent dispersion equations may conveniently be transformed to a simple polynomial in k^2 form by expanding hyper-trigonometric functions into power series and elementary algebraic manipulations. The order of this dispersion equation is controlled by a number of terms retained in power series. For each particular value of a frequency parameter, all roots of this approximate polynomial equation are readily found numerically by the use of, for example, the symbolic manipulator *Mathematica* [12]. They are then used one by one as an 'initial guess' to search numerically for the roots of the original dispersion equations. This procedure gives 'refined' values of wave numbers and the accuracy of a polynomial approximation of dispersion equations is readily accessed.

This 'exact' or 'refined' solution of the problem of wave propagation in a sandwich plate is used hereafter to check the validity of simplified models. A set of simplified theories, which may be used to describe several first branches of dispersion curves, is formulated in the following section.

3. Elementary modelling of propagation of waves in sandwich plates

In a number of papers [7–10], an elementary theory of flexural and shear vibrations of sandwich plates has been used. In a certain way, this theory is a generalization of classic Timoshenko theory developed for homogeneous beams in Ref. [13] with a shear angle θ between skin plies introduced as an independent variable in addition to a lateral deflection of the whole package of three plies w (see Ref. [8] for details).

$$\left[2\frac{E_1 h_1^3}{12(1-\nu^2)} + \frac{Eh^3}{12(-1-\nu^2)}\right] w^{(4)} - \frac{Eh}{2(1+\nu)}\left(1+\frac{h}{h_1}\right)^2(\theta' + w'') + (2\rho_1 h_1 + \rho h)\ddot{w}$$
$$-\left(2\frac{\rho_1 h_1^3}{12} + \frac{\rho h^3}{12}\right)\ddot{w}'' = 0,$$

$$\frac{E_1 h_1^3}{2(1-\nu^2)}\theta'' - \frac{Eh}{2(1+\nu)}(\theta + w') - \frac{\rho_1 h_1^3}{2}\ddot{\theta} = 0. \tag{14}$$

A solution of system (14) is sought in the form

$$w = A\exp(kx - \mathrm{i}\omega t), \tag{15a}$$

$$\theta = B\exp(kx - \mathrm{i}\omega t). \tag{15b}$$

Then dispersion equation becomes

$$\begin{aligned}&\frac{1}{12}\left(2+\frac{\gamma}{\varepsilon^3}\right)k^6 - \left[-\frac{1}{12}\left(2+\frac{\gamma}{\varepsilon^3}\right)\Omega_1^2 + \frac{1}{12}\left(2+\frac{\gamma}{\varepsilon^3}\right)(1-\nu)\varepsilon\gamma - \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\Omega_1^2\right.\\&\quad\left.+\frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma\right]k^4 + \left[-\left(2+\frac{\delta}{\varepsilon}\right)\Omega_1^2 + \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\Omega_1^4 - \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)(1-\nu)\varepsilon\gamma\Omega_1^2\right.\\&\quad\left.-\frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma\Omega_1^2\right]k^2 - \left(2+\frac{\delta}{\varepsilon}\right)\Omega_1^4 + \left(2+\frac{\delta}{\varepsilon^3}\right)(1-\nu)\varepsilon\gamma\Omega_1^2 = 0.\end{aligned} \tag{16}$$

This is a bicubic equation and its roots are available in an explicit analytical form. For convenience, the following non-dimensional parameters are introduced in Eq. (16) to describe the internal structure of a sandwich plate: $\varepsilon = h_1/h$ as a thickness parameter, $\delta = \rho/\rho_1$ as a density parameter, $\gamma = E/E_1$ as a stiffness parameter, and $\Omega_1 = \omega h_1/c_{skin}$ as a frequency parameter. Sound speed in the material of a skin ply is $c_{skin} = \sqrt{E_1/\rho_1(1-\nu^2)}$.

In this theory, only 'in-phase' motions of skin plies in transverse direction are considered and this dispersion equation is expected to have the roots, which are reasonably close to the first two roots of the transcendent 'exact' equation, which follows from Eq. (11) in the practically meaningful range of parameters. This aspect will be considered in Section 4.

If a core ply is sufficiently soft, then another type of motions should also be considered, which involves 'anti-phase' wave motions of skin plies in the transverse direction and their simultaneous 'in-phase' motions in the longitudinal direction. This class of wave motions has already been introduced in the previous section in the framework of a theory of elasticity. Similarly to the elementary theory of a sandwich plate, which models its 'in-phase' motions, two separate simple models may be suggested to describe the low branches of dispersion curves in the case of 'anti-phase' wave motions.

If propagation of a dominantly longitudinal wave is modelled, then the lateral components of displacements in all three plies may be put to zero and the longitudinal components of displacements are assumed to be the same in all plies. Then the equation of motion is

$$D_{eq}u_0'' - m_{eq}\ddot{u}_0 = 0. \tag{17}$$

Here an equivalent axial stiffness and an equivalent axial inertia are defined as

$$D_{eq} = 2E_1h_1 + Eh, \quad m_{eq} = 2\rho_1 h_1 + \rho h.$$

A solution of Eq. (17) is sought as

$$u_0(x,t) = U_0 \exp(kx - \mathrm{i}\omega t).$$

Then the roots of dispersion equation are defined as

$$k^2 = -\frac{\rho_1 \omega^2 h^2}{E_1} \frac{\left(1+\frac{1}{2}\frac{\rho}{\rho_1}\frac{h}{h_1}\right)}{\left(1+\frac{1}{2}\frac{E}{E_1}\frac{h}{h_1}\right)}. \tag{18}$$

In contrast to the previous case, to model 'anti-phase' dominantly lateral wave motions the longitudinal displacements in all plies and the core's inertia are ignored. Then the behaviour of a core ply may be reduced to an equivalent distributed spring, which has stiffness $K_{eq} = E/h$. Motions of skin plies are then governed by elementary equations

$$D_1 w_1^{(4)} + \rho_1 h_1 \ddot{w}_1 + K_{eq}(w_1 - w_3) = 0,$$

$$D_1 w_3^{(4)} + \rho_1 h_1 \ddot{w}_3 + K_{eq}(w_3 - w_1) = 0.$$

In the 'anti-phase' motions, the skin plies have the same displacements in opposite directions at each instant of time, $w_3(x,t) = -w_1(x,t)$ and these two equations are reduced as

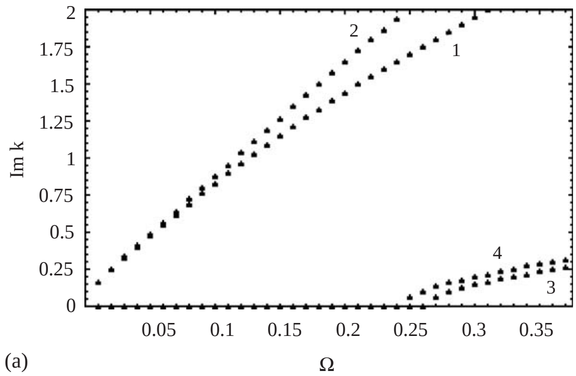
$$D_1 w_1^{(4)} + \rho_1 h_1 \ddot{w}_1 + 2K_{eq} w_1 = 0. \tag{19}$$

The third term in this equation mimics the response of well-known Winckler elastic foundation. As a solution of Eq. (19) is $w_1(x,t) = W_{10} \exp(kx - \mathrm{i}\omega t)$, the following elementary formula is readily obtained for a wave number:

$$k^4 = -24(1-\nu^2)\left(\frac{h}{h_1}\right)^3 \frac{E}{E_1} + 12\left(\frac{h}{h_1}\right)^4 \frac{(1-\nu^2)\rho_1 h_1^2 \omega^2}{E_1}. \tag{20}$$

This equation, as well as Eq. (18), defines branches of dispersion curves, which should qualitatively and quantitatively agree with those obtained from Eqs. (13) formulated by an exact solution of the theory of elasticity.

Summing up these three elementary simplified models, it should be noted that they predict two purely propagating modes, two purely decaying modes and two modes, which cut on at some threshold excitation frequency (wave propagation in one direction, say, from the left to the right is discussed). In particular, a theory for 'in-phase' modes predicts an existence of a dominantly flexural propagating wave, a purely decaying flexural wave and a dominantly shear wave. Their wave numbers are defined by Eq. (16). A theory for 'anti-phase' modes predicts an existence of propagating purely longitudinal wave defined by Eq. (18) and two 'breathing' modes of flexural vibrations with wave numbers given by Eq. (20), respectively. One of them is always evanescent, whereas another one has a cut-on frequency. Now, it should be shown that these theories give a good approximation of all low branches of dispersion curves obtained from a theory of elasticity, i.e., obtained from Eqs. (11) and (13).



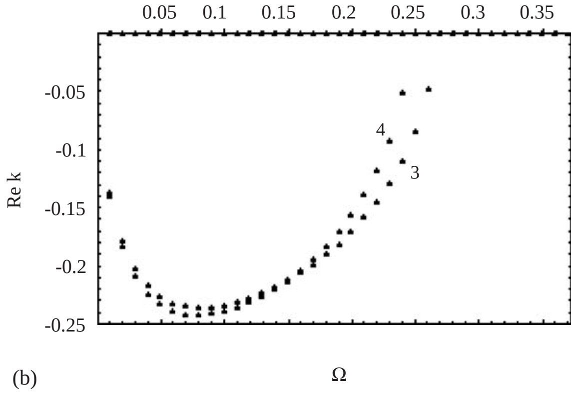


Fig. 2. Dispersion curves (a—imaginary parts; b—real parts) for 'in-phase' wave motions of a plate without fluid loading. Sandwich plate composition: $\varepsilon = 0.1,\ \gamma = 0.01,\ \delta = 0.1,\ v = 0.3$.

4. Dispersion curves for a plate without fluid loading

4.1. 'In-phase' waves

Compare the results obtained for 'in-phase' wave motions of skin plies. In Fig. 2, the roots of dispersion equations (11) and (16) are shown versus frequency parameter $\Omega \equiv \omega h/c_{skin}$ for $\varepsilon = 0.1,\ \gamma = 0.01,\ \delta = 0.1$. Curves 1, 3 and 2, 4 are plotted after 'refined' and 'elementary' theories, respectively. In Fig. 2a, purely imaginary roots are shown. Results attributed here to the 'refined' theory are obtained when transcendent functions are expanded into normal series with 12 terms retained. At low frequencies, there is no mismatch between two curves in describing a purely propagating flexural wave. When $\Omega > 0.15$, curve 1 deviates from curve 2, predicted by an elementary solution. However, it does not mean that an elementary solution loses its accuracy. On the contrary, it means that the polynomial approximation for an exact dispersion equation needs to be refined. Indeed, when the roots of the 'refined' dispersion equation in this polynomial approximation presented by curve 1 are used as an input to find the roots of the original transcendent dispersion equation, which follows from Eq. (11), these exact roots appear to follow curve 2 very closely in the frequency range $0.15 < \Omega < 0.3$. Also, if the computations within the framework of the 'refined' theory are repeated when transcendent functions are expanded into normal series with 20 terms retained, then curve 1 is located much closer to curve 2 than in Fig. 2a. Another two branches (curve 3 is plotted after an 'exact' theory, curve 4 is plotted after an 'elementary' theory) in this figure display a dependence of the purely imaginary wave numbers on frequency parameter for a shear wave. Since this wave is relatively long and has a small wave number, the roots of the dispersion equation obtained in the polynomial approximation do not deviate much from those of the exact dispersion equation, which is derived from Eq. (11). As discussed in Refs. [7,8], this wave has a 'cut-on' frequency and its purely real negative wave number is shown in Fig. 2b with the same notations (e.g., curve 3 is plotted after an 'exact' theory, curve 4 is plotted after an 'elementary' theory). The magnitude of a 'cut-on' frequency predicted by an elementary theory is slightly larger than its magnitude found from a refined theory.

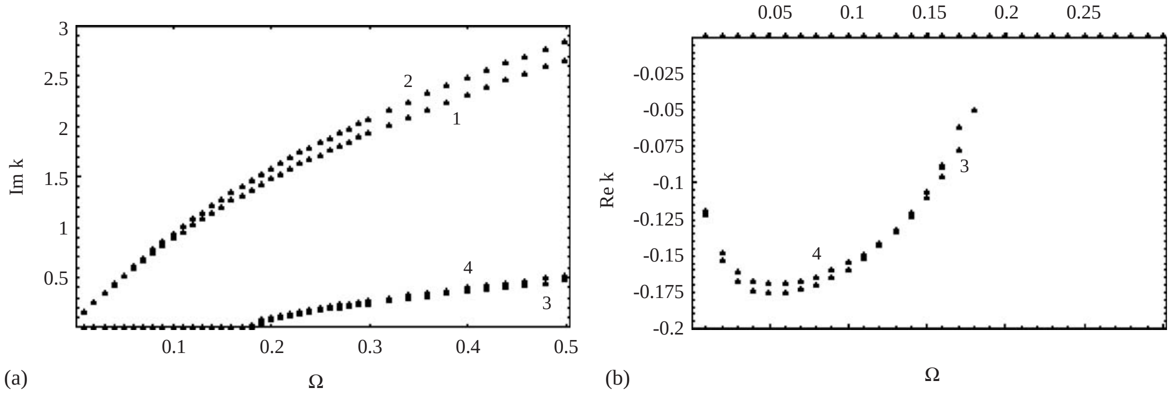


Fig. 3. Dispersion curves (a—imaginary parts; b—real parts) for 'in-phase' wave motions of a plate without fluid loading. Sandwich plate composition: $\varepsilon = 0.2$, $\gamma = 0.01$, $\delta = 0.1$, $\nu = 0.3$.

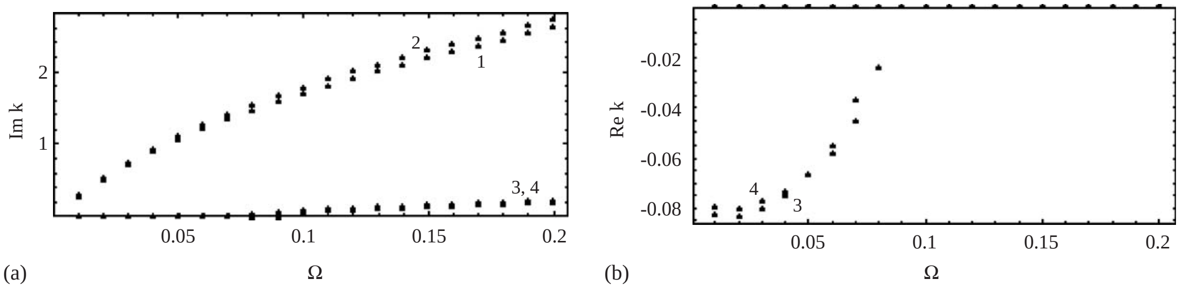


Fig. 4. Dispersion curves (a—imaginary parts; b—real parts) for 'in-phase' wave motions of a plate without fluid loading. Sandwich plate composition: $\varepsilon = 0.1$, $\gamma = 0.001$, $\delta = 0.1$, $\nu = 0.3$.

If a sandwich plate with thinner core ply (e.g., $\varepsilon = 0.2$, $\gamma = 0.01$, $\delta = 0.1$) is considered, then predictions given by these two theories are in better agreement, as is seen in Fig. 3. The same notations are used. In Fig. 3a, two branches of dispersion curves are plotted, which are relevant to flexural (curves 1 and 2) and shear (curves 3 and 4) propagating waves. In Fig. 3b, a dependence of the purely real negative wave number on frequency parameter is shown for a shear wave below 'cut-on' frequency. The agreement between these theories is also very good in the case when a core ply is getting softer. In Fig. 4, dispersion curves are plotted for a plate with $\varepsilon = 0.1$, $\gamma = 0.001$, $\delta = 0.1$ in the same way as before. In the whole range of the magnitudes of the frequency parameter Ω, the pairs of curves 1, 2 and 3, 4 almost merge with each other. The elementary theory has been derived provided that the shear deformation is uniform in any cross-section of the core ply. Apparently, this assumption becomes more realistic as the core ply is getting thinner and softer.

As discussed in Section 3, the refined theory actually describes an infinitely large number of branches of dispersion curves. The curves shown in Figs. 2–4 display a dependence of the first two wave numbers (those with the minimal magnitudes) on a frequency parameter. As the next branch predicted by the refined theory is considered, then it appears that in the low-frequency range this curve describes an attenuated wave with a high decay rate. Besides, it cuts on at a frequency,

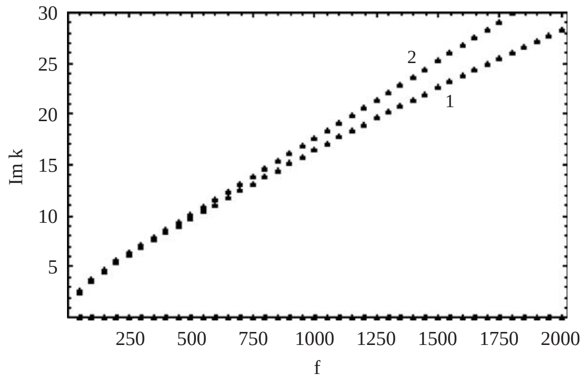


Fig. 5. Dimensional wave number in m^{-1} versus frequency in Hz for a flexural propagating wave in 'in-phase' wave motions of a plate without fluid loading.

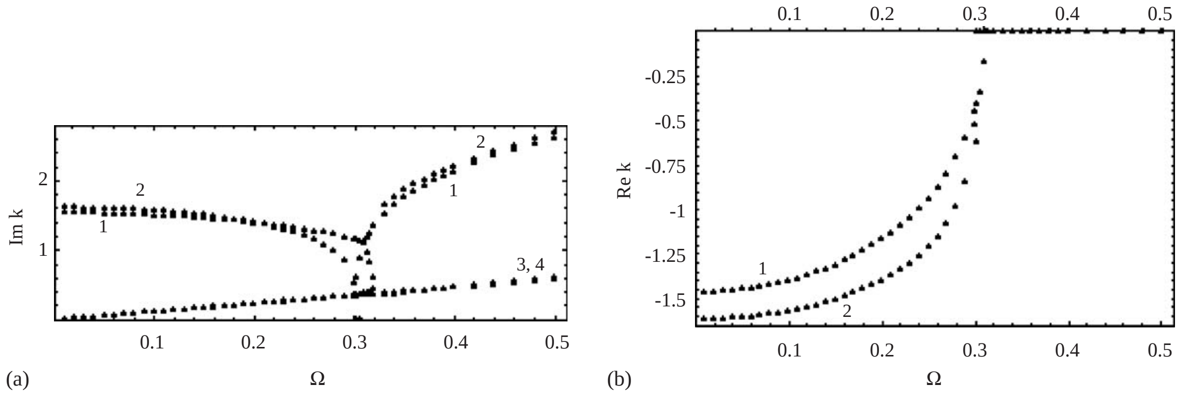


Fig. 6. Dispersion curves (a—imaginary parts; b—real parts) for 'anti-phase' wave motions of a plate without fluid loading. Sandwich plate composition: $\varepsilon = 0.2$, $\gamma = 0.01$, $\delta = 0.1$, $\nu = 0.3$.

which is much higher than the first cut on one. Thus, it should be concluded that the elementary theory suggested in Refs. [7,8] is entirely sufficient to modelling wave propagation in sandwich structures—at least, when a sandwich plate has a set of parameters typical for applications in naval and aerospace industries. To support this statement, consider a sandwich plate with the following dimensional parameters: the skins are of $\rho_1 = 1580\ \mathrm{kg/m^3}$, $E_1 = 9.8 \times 10^9$ Pa, $h_1 = 0.0025$ m, $\nu = 0.3$ and the core is of $\rho = 101\ \mathrm{kg/m^3}$, $E = 94 \times 10^6$ Pa, $h = 0.05$ m, $\nu = 0.3$. The dimensional wave numbers of a flexural propagating wave $\mathrm{K} \equiv kh^{-1}$ in m^{-1} are plotted in Fig. 5 versus excitation frequency in Hz. As before, curves 1 and 2 present results obtained by a use of 'exact' and 'elementary' theories, respectively. This particular set of parameters of a sandwich plate composition is taken from Ref. [14] and the results reported in this reference match these curves very well. The elementary theory of sandwich plates is derived provided that $L_{wave} \gg h$ and it is not aimed at describing short waves. For example (as is seen from Fig. 5), the wave number of $\mathrm{K} = 20\ \mathrm{m}^{-1}$ is relevant to the length of $L_{wave} = \pi/K \approx 0.15\ \mathrm{m} \approx 3(2h_1 + h)$ for this set of parameters. Thus, the elementary theory [7,8] is applicable for analysis of 'in-phase' wave propagation in sandwich plates.

4.2. 'Anti-phase' waves

Now consider propagation of 'anti-phase' waves. In Fig. 6, dispersion curves showing a dependence of the non-dimensional wave number k on the frequency parameter $\Omega \equiv \omega h/c_{skin}$ are plotted for $\varepsilon = 0.2,\ \gamma = 0.01,\ \delta = 0.1$. Imaginary parts of wave numbers are plotted in Fig. 6a. As before, curves 1, 3 and 2, 4 are plotted after the 'exact' theory and the 'elementary' theories, respectively. The lower straight line (designated by indices 3 and 4) is given by both theories almost identically in the whole frequency range. This is a dominantly longitudinal propagating wave described by the elementary equation (18). The 'breathing' dominantly flexural 'anti-phase' wave has the 'cut-on' non-dimensional frequency parameter of about $\Omega \approx 0.3$, see Fig. 6b. The elementary theory suggests that the evanescent wave is transformed into a purely propagating one, when the imaginary and the real parts of the relevant wave number simultaneously cross the frequency axis and the real part vanishes. As is seen from Fig. 6, the 'refined' theory predicts a more complicated behaviour of dispersion curves in the vicinity of the cut-on frequency.

The similar graphs are plotted in Fig. 7 for $\varepsilon = 0.1,\ \gamma = 0.01,\ \delta = 0.1$. The imaginary parts of wave numbers versus the frequency parameter are shown in Fig. 7a. The real negative parts of wave numbers are shown in Fig. 7b. As is seen, the 'elementary' theory gives results, which are in a good agreement with the 'exact' theory in the whole frequency range except for $0.39 < \Omega < 0.46$. This part of the picture is zoomed in Fig. 8. The elementary theory suggests a simple continuous dependence of the both wave numbers on the parameter Ω, see curves 2, 4 in Fig. 8a (the imaginary parts of wave numbers) and curve 2 in Fig. 8b (the real parts of wave numbers). Since these two 'elementary' theories are derived independently, the relevant dispersion curves cross each other. However, such an intersection is impossible when a consistent coupled formulation for both these wave motions is used. Thus, dispersion curves predicted by the 'exact' theory behave in a more complicated way. In particular, two complex-valued wave numbers (their imaginary part is given by curve 11 in Fig. 8a and the real part of one of them is shown by curve 11 in Fig. 8b, another one has the real part with an opposite sign) are split at $\Omega \approx 0.4$. Curve 12, which presents one of two purely imaginary roots matches curve 2 predicted by the elementary theory (see also Fig. 7a). Another branch presented by curve 14 approaches curve 13, which follows very closely

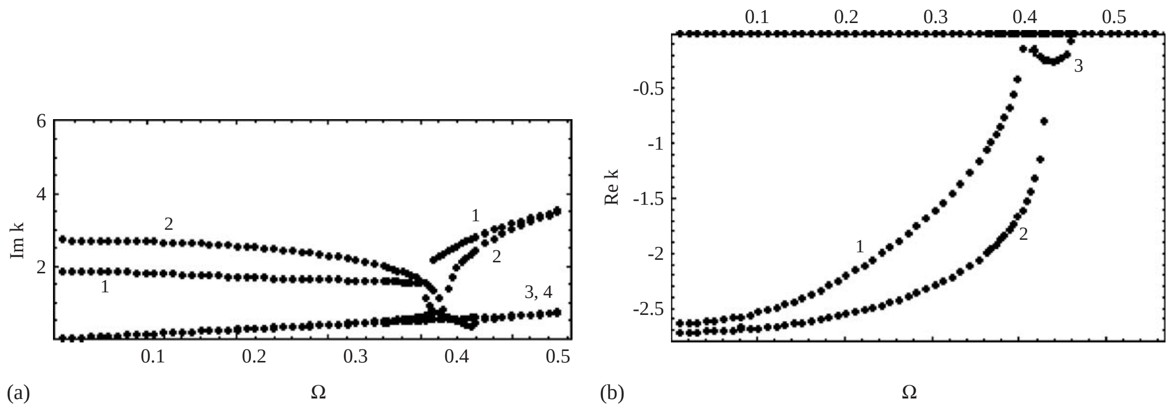


Fig. 7. Dispersion curves (a—imaginary parts; b—real parts) for 'anti-phase' wave motions of a plate without fluid loading. Sandwich plate composition: $\varepsilon = 0.01,\ \gamma = 0.01,\ \delta = 0.1,\ \nu = 0.3$.

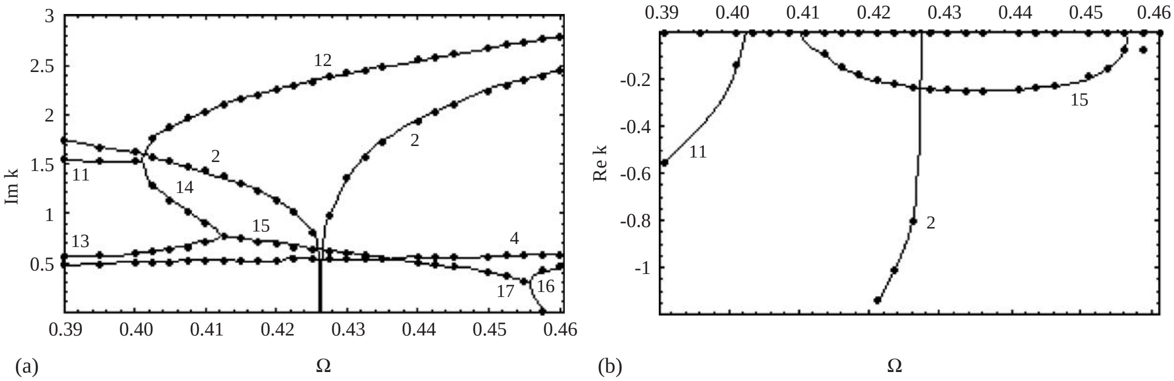


Fig. 8. Zooming of Fig. 7.

the elementary solution (curve 4), but deviates rather sharply at $\Omega \approx 0.4$. As curves 13 and 14 merge at $\Omega \approx 0.41$, they transform into a pair of complex-valued roots, describing non-propagating waves. Their imaginary part is given by curve 15 in Fig. 8a. The real part of one of them is given by curve 15 in Fig. 8b, another one has the positive real part of the same magnitude. These complex-valued roots are in turn split into two purely imaginary roots at $\Omega \approx 0.455$, see curves 16 and 17 in Fig. 8a. Curve 16 matches curve 4 (see also Fig. 7a, where it is designated as curve 3) and therefore recovers the validity of the simplified theory. Curve 17 crosses zero at $\Omega \approx 0.46$ (the cut-off frequency parameter) and this wave transforms to an evanescent type. Thus, the 'refined' theory suggests the existence of a relatively narrow band gap $0.41 < \Omega < 0.455$, where the dominantly longitudinal wave cannot propagate. This 'stop band' is not captured by the 'elementary' theories. On the other hand, there are three propagating waves in frequency bands $0.4 < \Omega < 0.41$ and $0.455 < \Omega < 0.457$. This aspect of wave motions is also not described by the 'elementary' theories. However, 'globally' the elementary dispersion equations (18) and (20) adequately describe propagation of 'anti-phase' waves in a sandwich plate.

Similarly to the case of 'in-phase' wave motions, curves shown in Figs. 6–8 display a dependence of the first two wave numbers (those with the minimal magnitudes) on the frequency parameter Ω. The next branch predicted by the 'exact' theory describes an attenuated wave with a high decay rate and it cuts on at a frequency, which is much higher, that the first cut on one. Thus, 'elementary' theories suggested in this Section adequately model propagation of 'anti-phase' waves in a sandwich plate.

5. Propagation of 'anti-phase' waves in sandwich plates with heavy fluid loading

In papers [7,9,10], propagation of flexural and shear waves in sandwich plates in heavy fluid loading conditions has been thoroughly studied for the 'in-phase' motions of skin plies. As is shown in the previous section, propagation of dominantly longitudinal waves in a sandwich plate with a sufficiently soft core ply may occur at the same frequencies as propagation of flexural waves. Since to the best of the author's knowledge, no work has so far been done on 'anti-phase' motions of a sandwich plate with heavy fluid loading, this case is considered here in detail. To employ the symmetry properties of solution it is assumed that a plate is loaded by an acoustic medium of density ρ_{fl} and sound speed c_{fl} at both sides. Thus, a sandwich plate separates two semi-infinite volumes occupied by the same acoustic medium. Such a problem may be relevant to vibrations of submerged elements of, for example, off-shore structures.

The formulation given in the framework of an elasticity theory is then extended by adding fluid loading terms

$$\begin{aligned}
&D_1 w_1^{(4)} + \rho_1 h_1 \ddot{w}_1 = q_{w1} + m_1' - p_+\left(x, t, \frac{h}{2} + h_1\right),\\
&E_1 h_1 u_1'' - \rho_1 h_1 \ddot{u}_1 = -q_{u1},\\
&D_3 w_3^{(4)} + \rho_3 h_3 \ddot{w}_3 = q_{w3} + m_3' + p_-\left(x, t, -\frac{h}{2} - h_3\right),\\
&E_3 h_3 u_3'' - \rho_3 h_3 \ddot{u}_3 = -q_{u3}.
\end{aligned} \tag{21}$$

An acoustic pressure is defined as

$$p_\pm(x, t, z) = -\rho_{fl}\dot{\varphi}_\pm(x, t, z). \tag{22}$$

Velocity potentials in an acoustic medium are governed by wave equation

$$\Delta\varphi_\pm - \frac{1}{c_{fl}^2}\ddot{\varphi}_\pm = 0. \tag{23}$$

The continuity conditions and the fluid–structure interfaces are formulated as

$$\begin{aligned}
&z = -\frac{h}{2} - h_1: \quad \dot{w}_3(x, t) = \frac{\partial\varphi_-(x, t, z)}{\partial n_-} = \frac{\partial\varphi_-(x, t, z)}{\partial z},\\
&z = \frac{h}{2} + h_1: \quad \dot{w}_1(x, t) = -\frac{\partial\varphi_+(x, t, z)}{\partial n_+} = \frac{\partial\varphi_+(x, t, z)}{\partial z}.
\end{aligned} \tag{24}$$

The displacements and elastic potentials are presented in form (7). Velocity potentials in acoustic medium are sought in the form

$$\varphi_\pm(x, t, z) = \varphi_\pm^{(0)}(z)\exp(kx - \mathrm{i}\omega t).$$

Then the wave equation (23) is reduced to a one-dimensional Helmholtz equation

$$\frac{\mathrm{d}^2\varphi_\pm^{(0)}}{\mathrm{d}z^2} + \left[k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2\right]\varphi_\pm^{(0)} = 0. \tag{25}$$

Solution of this equation is sought as

$$\varphi_\pm^{(0)}(z) = A_+\exp(\mathrm{i}\gamma_{fl}z) + A_-\exp(-\mathrm{i}\gamma_{fl}z), \quad \gamma_{fl} \equiv \sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}.$$

The parameters $A_\pm$ should be selected to satisfy compatibility condition (24) at fluid–structure interface and the radiation condition, which is formulated at infinity for the upper and the lower half-spaces occupied by an acoustic medium. Elementary algebra gives the following expressions for velocity potentials

$$\varphi_+^{(0)}(z) = \frac{\omega h^2}{\gamma_{fl}} W_1 \exp(\mathrm{i}\gamma_{fl} z), \quad z > 0, \tag{26a}$$

$$\varphi_-^{(0)}(z) = \frac{\omega h^2}{\gamma_{fl}} W_3 \exp(-\mathrm{i}\gamma_{fl} z), \quad z < 0. \tag{26b}$$

Then the amplitudes of a contact acoustic pressure at the surfaces of skin plies are formulated via the amplitudes of displacements as

$$p_+ = -\frac{\mathrm{i}\rho_{fl}\omega^2 h^2}{\sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}} W_1, \;\; p_- = \frac{\mathrm{i}\rho_{fl}\omega^2 h^2}{\sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}} W_3, \quad z = 0. \tag{27}$$

The anti-phase motions are considered, $W_1 = -W_3$, $U_1 = U_3$. Then the characteristic determinant becomes

$$\begin{aligned}
&\left[\frac{E_1 h_1^3}{12(1-\nu^2)h^3} k^4 - \rho_1 h h_1 \omega^2 - \mathrm{i}\rho_{fl}\omega^2 h^2 \left(k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2\right)^{-1/2}\right] W_1 \\
&\quad + \left[\lambda\left(\frac{\omega h}{c_1}\right)^2 \cosh\left(\frac{\gamma_1}{2}\right) + 2\mu\gamma_1^2 \cosh\left(\frac{\gamma_1}{2}\right) - \mu k^2 \gamma_1 \frac{h_1}{h} \sinh\left(\frac{\gamma_1}{2}\right)\right] \tilde{A} \\
&\quad + \left[2\mu k \gamma_2 \cosh\left(\frac{\gamma_2}{2}\right) - \frac{1}{2}\frac{h_1}{h}\mu k^3 \sinh\left(\frac{\gamma_2}{2}\right) + \frac{1}{2}\frac{h_1}{h}\mu k \gamma_2^2 \sinh\left(\frac{\gamma_2}{2}\right)\right] \tilde{B} = 0, \\
&\left[\frac{E_1 h_1}{h} k^2 + \rho_1 h h_1 \omega^2\right] U_1 - 2\mu\gamma_1 \sinh\left(\frac{\gamma_1}{2}\right)\tilde{A} - \left[\mu k^2 \sinh\left(\frac{\gamma_2}{2}\right) - \mu\gamma_2^2 \sinh\left(\frac{\gamma_2}{2}\right)\right]\tilde{B} = 0.
\end{aligned} \tag{28}$$

The 'elastic' part of the solution is not affected by the presence of fluid loading, so that parameters $\tilde{A}$, $\tilde{B}$ are defined by formulae (12). Similarly to the case of a sandwich plate without fluid loading, the determinant of these algebraic equations should be put to zero, which yields a transcendent dispersion equation. For each particular value of a frequency parameter, wave numbers should be found as roots of this equation. It is also conveniently transformed to a polynomial form by employing expansions in normal series and each root of the resulting approximate equation is used as an 'initial guess' to find a root of the original dispersion equation numerically. Besides, parameters γ_{fl} related to each root should be checked whether they obey the condition

$$\mathrm{Im}\,\gamma_{fl} \equiv \mathrm{Im}\left(\sqrt{k^2 + \left(\frac{\omega h}{c_{fl}}\right)^2}\right) > 0. \tag{29a}$$

This condition implies the exponential decay of an acoustic wave at infinity, i.e., at $z \to \pm\infty$ for the upper and the lower half-spaces, respectively. If γ_{fl} is purely real (e.g., $\mathrm{Im}\,\gamma_{fl} = 0$), then

Sommerfeld condition holds

$$\operatorname{Re}\gamma_{fl} \equiv \operatorname{Re}\left(\sqrt{k^2+\left(\frac{\omega h}{c_{fl}}\right)^2}\right) > 0. \tag{29b}$$

It implies propagation of acoustic waves from the surface of a plate.

6. An elementary theory for 'anti-phase' wave propagation in fluid-loaded sandwich beam

An elementary theory for propagation of anti-phase waves obtained in Section 3 is readily generalized for the case of heavy fluid loading. The equation for flexural waves is modified as the fluid loading term is added as

$$D_1 w_1^{(4)} + \rho_1 h_1 \ddot{w}_1 + 2K_{eq} w_1 = p\left(x, t, \frac{h}{2} + h_1\right). \tag{30}$$

Its solution is sought in form (7), and formula (27) for a contact pressure is substituted into this equation. After elementary algebraic manipulations the following simple polynomial dispersion equation is obtained

$$\left[k^2+\left(\frac{\omega h}{c_{skin}}\right)^2\left(\frac{c_{skin}}{c_{fl}}\right)^2\right]\left[\frac{1}{12}\left(\frac{h_1}{h}\right)^3 k^4 + 2(1-\nu^2)\frac{E}{E_1} - \frac{h_1}{h}\left(\frac{\omega h}{c_{skin}}\right)^2\right]^2 + \left(\frac{\rho_{fl}}{\rho_1}\right)^2\left(\frac{\omega h}{c_{skin}}\right)^4 = 0. \tag{31}$$

This transformed equation has 10 roots, which should be sorted out depending whether they satisfy the original equation

$$\frac{1}{12}\left(\frac{h_1}{h}\right)^3 k^4 + 2(1-\nu^2)\frac{E}{E_1} - \frac{h_1}{h}\left(\frac{\omega h}{c_{skin}}\right)^2 = \mathrm{i}\left(\frac{\rho_{fl}}{\rho_1}\right)\left(\frac{\omega h}{c_{skin}}\right)^2\left[k^2+\left(\frac{\omega h}{c_{skin}}\right)^2\left(\frac{c_{skin}}{c_{fl}}\right)^2\right]^{-1/2}, \tag{32}$$

and whether the condition

$$\operatorname{Im}\left[\frac{\mathrm{i}\left(\frac{\rho_{fl}}{\rho_1}\right)\left(\frac{\omega h}{c_{skin}}\right)^2}{\frac{1}{12}\left(\frac{h_1}{h}\right)^3 k^4 + 2(1-\nu^2)\frac{E}{E_1} - \frac{h_1}{h}\left(\frac{\omega h}{c_{skin}}\right)^2}\right] > 0, \tag{33}$$

is held.

In the case of a purely longitudinal propagating wave (which is defined by Eq. (17) for a plate without fluid loading), incorporation of the fluid loading term is not so straightforward. Extension of skin plies in the longitudinal direction is not affected by the presence of an acoustic medium, $\varepsilon_x = u'$. Since the dependence of all functions on axial and temporal co-ordinates is taken as $\exp(kx - \mathrm{i}\omega t)$, see Eq. (7), this formula is reduced to $\varepsilon_x = ku$. Then normal longitudinal stresses acting on skin plies in the longitudinal direction are $\sigma_x = E_1\varepsilon_x = kE_1 u$. In the core ply, it is necessary to take into account transverse normal stresses generated by fluid loading, so that Hooke's law is formulated as

$$\varepsilon_x = \frac{1}{E}\sigma_x^{core} + \frac{\nu}{E}p, \tag{34a}$$

$$\varepsilon_z = -\frac{1}{E}p - \frac{v}{E}\sigma_x^{core}. \tag{34b}$$

In these formulae, it is already taken into account that a positive contact pressure is directly transmitted onto a core ply and it produces transverse compression.

Since 'anti-phase' motions of skin plies are considered, $w_3 = -w_1 = -\tilde{w}$ (here the dimensional amplitudes of displacements are introduced), the deformation of a core ply in transverse direction is easily found as

$$\varepsilon_z = -\frac{2\tilde{w}}{h}. \tag{35}$$

Then the following relation between a contact pressure p and the dimensional amplitudes of displacements $\tilde{u} \equiv hu$, $\tilde{w}$ is readily obtained from Eq. (34b)

$$2\tilde{w} = vk\tilde{u} + \frac{1-v^2}{E}ph. \tag{36}$$

Contact pressure p is defined by the formula, which is similar to Eq. (27)

$$p = \frac{\mathrm{i}\rho_{fl}\omega^2 h}{\sqrt{k^2 + (\omega h/c_{fl})^2}}\tilde{w}. \tag{37}$$

If $\tilde{w}$ is excluded from Eq. (36) by using this formula, then the following expression is obtained for a contact pressure:

$$p = -vku\left[\frac{1-v^2}{E} + 2\frac{\mathrm{i}\sqrt{k^2 + (\omega h/c_{fl})^2}}{\rho_{fl}(\omega h)^2}\right]^{-1}. \tag{38}$$

Then longitudinal stresses in the material of a core ply are found to be

$$\sigma_x^{core} = Eku\left[1 + \frac{v^2}{1-v^2}\frac{\rho_{fl}}{\rho_1}\frac{\frac{E_1}{E}\left(\frac{\omega h}{c_{skin}}\right)^2}{2\mathrm{i}\sqrt{k^2 + \left(\frac{c_{skin}}{c_{fl}}\right)^2\left(\frac{\omega h}{c_{skin}}\right)^2 + \frac{\rho_{fl}}{\rho_1}\frac{E_1}{E}\left(\frac{\omega h}{c_{skin}}\right)^2}}\right]. \tag{39}$$

As fluid's density vanishes ($\rho_{fl} \to 0$), this formula merges formula $\sigma_x^{core} = Eku$ used in Section 3.

Apparently, the axial force resultant is defined as $N_x = N_x^{skin} + N_x^{core} = 2h_1\sigma_x^{skin} + h\sigma_x^{core}$. Then the dispersion equation becomes

$$\left\{E_1 h_1 + \frac{1}{2}Eh\left[1 + \frac{v^2}{1-v^2}\frac{\rho_{fl}}{\rho_1}\frac{\frac{E_1}{E}\left(\frac{\omega h}{c_{skin}}\right)^2}{2\mathrm{i}\sqrt{k^2 + \left(\frac{c_{skin}}{c_{fl}}\right)^2\left(\frac{\omega h}{c_{skin}}\right)^2 + \frac{\rho_{fl}}{\rho_1}\frac{E_1}{E}\left(\frac{\omega h}{c_{skin}}\right)^2}}\right]\right\}k^2$$
$$= -\left(\rho_1 h_1 + \frac{1}{2}\rho h\right)(\omega h)^2. \tag{40}$$

To ensure the absence of a growth of an acoustic wave in the fluid's volume, condition (29) must be held. This equation naturally merges the dispersion equation (18) as soon as the fluid loading term disappears, $\rho_{fl} \to 0$.

7. Dispersion curves for 'anti-phase' wave motions in a sandwich plate with heavy fluid loading

Consider a sandwich plate with the following parameters: $\varepsilon = 0.1$, $\gamma = 0.01$, $\delta = 0.1$. This plate is loaded at both sides by an acoustic medium specified by the density ratio $\rho_r \equiv \rho_{fl}/\rho_1 = 0.128$ and the sound speed ratio $c_r \equiv c_{fl}/c_p = 3.4$. This set of fluid loading parameters is relevant to vibrations of a sandwich plate with steel skin plies in water. In Fig. 9, curve 1 presents a dependence of the magnitude of the purely imaginary wave number of a propagating wave on the frequency parameter Ω given by the 'exact' dispersion equation derived in Section 5. Curves 2 and 3 are plotted after formulae (40) and (32), which are derived from simplified theories. These roots of dispersion equations satisfy conditions (29). Besides, curve 4 is the dispersion curve (18) for a propagating, dominantly longitudinal wave in a plate without fluid loading. This curve lies well

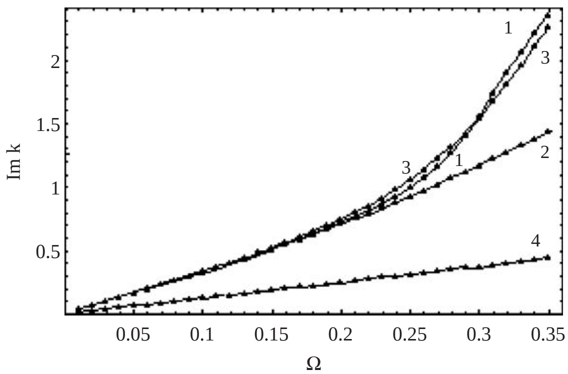


Fig. 9. Purely imaginary wave number versus frequency parameter for 'anti-phase' wave motions of a plate with fluid loading. Sandwich plate composition: $\varepsilon = 0.1$, $\gamma = 0.01$, $\delta = 0.1$, $\nu = 0.3$. Fluid loading $\rho = 0.128$, $c_r = 3.4$.

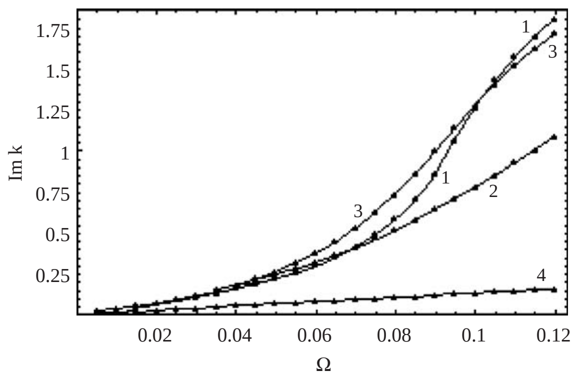


Fig. 10. Purely imaginary wave number versus frequency parameter for 'anti-phase' wave motions of a plate with fluid loading. Sandwich plate composition: $\varepsilon = 0.1$, $\gamma = 0.001$, $\delta = 0.1$, $\nu = 0.3$. Fluid loading $\rho = 0.128$, $c_r = 3.4$.

beyond those plotted for a plate with fluid loading, so that wave motions are strongly affected by the presence of an acoustic medium in this case. It should be noted that there is only one propagating wave in the frequency range $\Omega < 0.35$, see also Fig. 7. At low frequencies, dispersion curves obtained by the use of both the 'elementary' theories merge with the dispersion curve given by an 'exact' theory. When the frequency parameter becomes larger (e.g., $\Omega > 0.2$), the elementary formula (40) derived from the theory of purely longitudinal wave motions of a fluid loaded plate becomes incorrect, whereas formula (32) is still in a perfect agreement with the refined theory.

To present the location of dispersion curves 1, 2, 3 more clearly, it is convenient to consider wave propagation in a plate, which has a softer core ply, e.g., $\varepsilon = 0.1,\ \gamma = 0.001,\ \delta = 0.1$. As it is seen in Fig. 10, curve 2, which displays the solution given by formula (40), follows the exact solution (curve 1) closer than the solution presented by curve 3 for a flexural wave (32) up to $\Omega \approx 0.08$. However, when $\Omega > 0.1$, curve 3 matches the exact solution (curve 1), while curve 2 markedly deviates from them both. It is interesting to note that the longitudinal wave appears to be damped by the presence of an acoustic medium. It has a complex valued wave number $-k_r + \mathrm{i}k_i$. The magnitude of its imaginary part k_i is fairly close to the magnitude of the imaginary part of a wave number, which describes propagation of a wave in a plate without fluid loading (see curve 4). The magnitude of the negative real part $-k_r < 0$ of this wave number is much smaller, $k_r \ll k_i$ and damping is proportional to the magnitude of a sound speed in a fluid.

In the case of an incompressible fluid (i.e., $c_r = 0$), there are two propagating waves in the low-frequency region, rather than only one wave of this type in the case of an acoustic medium of the same density (i.e., $c_r = 3.4$). In Fig. 11a and b, curves 1a, b (sandwich plate parameters are $\varepsilon = 0.1,\ \gamma = 0.01,\ \delta = 0.1$, a fluid is incompressible, $c_r = 0$) present wave numbers predicted by the exact solution (28). Curves 2a, b are plotted after formula (40). Curve 3 in Fig. 11a is plotted after the elementary theory of flexural motions of a fluid-loaded plate, which gives the dispersion equation (32). Curve 4 in Fig. 11b presents the longitudinal wave in a plate without fluid loading, see formula (18). Comparison of curves 1a, b and 2a, b (they are identical in Figs. 11a and b) suggests that the elementary theory resulting in the dispersion equation (40) is capable of accurately predicting the dynamic properties of a sandwich plate in its 'anti-phase' motions for

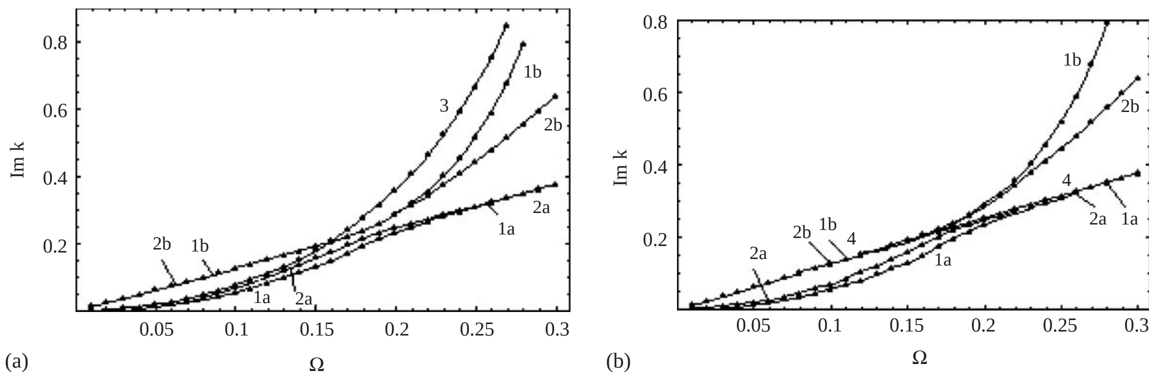


Fig. 11. Purely imaginary wave number versus frequency parameter for 'anti-phase' wave motions of a plate loading by an incompressible fluid (a—comparison with the elementary theory of flexural motions; b—comparison with the theory for longitudinal waves without fluid loading). Sandwich plate composition: $\varepsilon = 0.1,\ \gamma = 0.01,\ \delta = 0.1,\ \nu = 0.3$. Fluid loading $\rho = 0.128,\ c_r = 0$.

not too high frequencies. As is seen, curve 2a matches curve 1a in the whole frequency region and curve 2b matches curve 1b up to $\Omega \approx 0.22$.

Curve 3 in Fig. 11a matches curve 1a (the lower branch of the exact solution) only up to $\Omega \approx 0.08$. This elementary theory becomes fairly inaccurate in the frequency band $0.08 < \Omega < 0.28$. However, for $\Omega > 0.28$, its accuracy is recovered and curve 3 gradually matches another branch of the exact solution, curve 1b. On the other hand, roots of the elementary dispersion equation (18) for a plate without fluid loading fit very well the exact solution for a plate with fluid loading at almost any frequency, see Fig. 11b. Curves 1a and 2a asymptotically tend to the straight line 4 when $\Omega < 0.16$. Respectively, curves 1b and 2b tend to curve 3 when $\Omega > 0.21$. Summing up these results it should be pointed out that elementary approximate solutions given by dispersion curves 3 and 4 are inadequate in the zone (approximately, around $\Omega \approx 0.17$), where they actually cross each other (as it may be seen by combining Fig. 11a and b). As is well known, dispersion curves can never intersect, so it is rather natural that in this 'overlapped' region both simplified theories are incorrect.

To clarify the role of fluid's compressibility, computations have been performed for various values of compressibility parameter c_r. Comparison of graphs plotted in Figs. 9–11 and some additional computations show that curve 3 gradually shifts upwards with the growth in compressibility parameter c_r from $c_r = 0$ to 3.4. At some critical magnitude of this parameter, curve 3 becomes tangent to curve 4. Simultaneously, the relevant roots of the dispersion equation (28) or (40) acquire real parts and the longitudinal wave (which 'originates' from a purely longitudinal propagating wave in a plate without fluid loading) becomes non-propagating. This phenomenon (an existence of the threshold sound speed ratio) has a simple physical explanation. In the case of an incompressible fluid, there is no energy flow from a plate to infinity through an acoustic medium, i.e., in the direction perpendicular to the surface of a vibrating plate. As soon as the fluid becomes sufficiently compressible, the energy transmission ('leakage') from the plate into two semi-infinite volumes of surrounding acoustic medium is developed. Respectively, the structural energy flow disappears and the free dominantly longitudinal wave cannot be supported in a plate with such a fluid loading. Since this energy leakage into an unbounded volume of an acoustic medium increases with the further growth in the magnitude of a compressibility parameter (sound speed ratio) c_r, propagation of a trapped wave is getting suppressed more and more heavily. This explanation is illustrated by the graphs shown in Fig. 12.

In Fig. 12a and b three dispersion curves $k(c_r)$ are plotted presenting wave numbers versus the compressibility parameter for the fixed excitation frequency $\Omega = 0.05$. The parameters of sandwich plate composition are the same as before, i.e., $\varepsilon = 0.1$, $\gamma = 0.01$, $\delta = 0.1$. As is seen from Fig. 12a, there are three purely imaginary roots of the exact dispersion equation (28) for $c_r < 1.25$. Two of them fulfil condition (29a) and are designated here by curves 1 and 2. The third root (curve 3) is located very closely to the first root, so that in Fig. 12a the points of this curve almost merge with the points of curve 1. However, this root does not satisfy the decay condition. In the bifurcation region, curve 1 actually deviates from curve 3, whereas curve 2 tends to curve 3. However, as the imaginary parts of these two wave numbers merge, they acquire the real parts. The second propagating wave is transformed then into the attenuated wave with a complex-valued wave number and its real part is designated by curve 2 in Fig. 12b. The third root, which does not satisfy the decay condition (29a) when $c_r < 1.25$, acquires a positive real part (curve 3) and remains unacceptable, since for $c_r > 1.25$ it presents a wave, which grows exponentially along the plate. Finally, curve 1 in Fig. 12b displays the zero real part of the wave number relevant to a propagating wave.

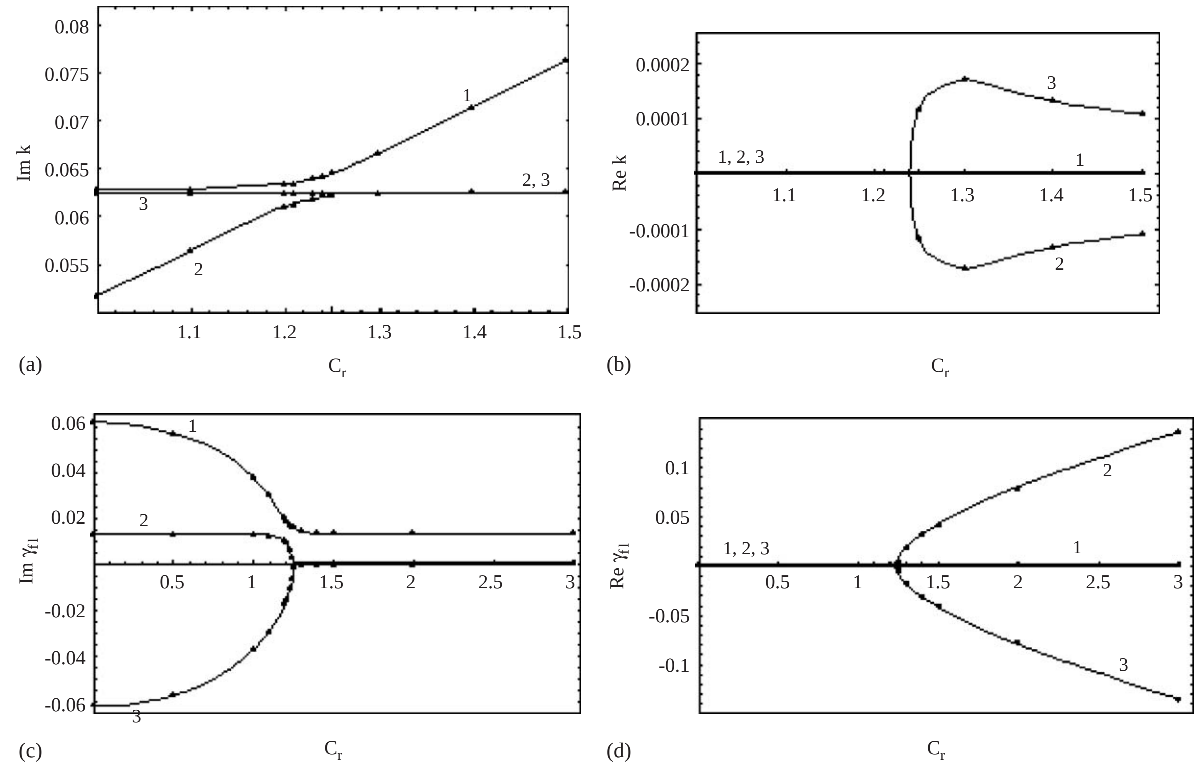


Fig. 12. Wave numbers in a plate and in a fluid versus compressibility parameter for 'anti-phase' wave motions of a plate with fluid loading (a—imaginary parts of k; b—real parts of k; c—imaginary parts of γ_{fl}; d—real parts of γ_{fl}). Sandwich plate composition: $\varepsilon = 0.1$, $\gamma = 0.01$, $\delta = 0.1$, $v = 0.3$. Fluid loading $\rho = 0.128$. Excitation frequency $\Omega = 0.05$.

The dependence of parameter γ_{fl} (which describes propagation of a trapped wave in a fluid) on compressibility parameter c_r is illustrated in Fig. 12c and d for each root of the dispersion equation (28). Curves are labelled in the same order. In Fig. 12c, the imaginary part of the parameter γ_{fl} relevant to the first wave number in Fig. 12a and b is shown by curve 1. As is seen, this is a continuous curve and the decay condition (29a) holds true for any value of c_r. Respectively, as is seen in Fig. 12d, the real part of this parameter is absent at any frequency, so this is always a wave, which decays into an acoustic medium (wave motions of an acoustic medium are trapped in vicinity of a plate). The second propagating wave exists only as long as $c_r < 1.25$. The third root (curve 3) in this region is irrelevant, since it has the negative imaginary part and does not obey the decay condition. Curves 2 and 3 merge at $c_r = 1.25$. For $c_r > 1.25$, parameter γ_{fl} relevant to the second wave number becomes purely real and positive (curve 2). Thus, the acoustic wave propagates into an acoustic medium, whereas propagation of the trapped wave in a plate becomes suppressed, as is shown in Fig. 12b. Respectively, the parameter γ_{fl} calculated for the third root becomes purely negative, see curve 3. In fact, curves 3 in Fig. 12a–d represents the root of the dispersion equation, which is always irrelevant, and they may not be displayed. However, if these curves are omitted, then the transformation of a propagating trapped wave into a decaying one (due to the bifurcation of dispersion curves) becomes less clear.

Summing up the role of the compressibility parameter, it should be noted that the sound speed in the material of skin plies may vary in a rather wide range, depending on the material stiffness. Thus, the inspection into the role of this parameter is relevant, since it controls the energy transportation in a plate with heavy fluid loading. Anyway, at least one propagating trapped wave in a fluid-loaded sandwich plate exists at any excitation frequency independently of the sound speed ratio c_r.

8. Conclusions

Three 'elementary' theories are suggested to describe wave propagation phenomena in sandwich plates without fluid loading. Comparison of dispersion curves obtained by solving characteristic equations derived from these theories with dispersion curves obtained in the 'exact' problem formulation shows that these three theories adequately predict wave motions of a sandwich plate in the whole frequency range of practical interest. In the case of 'anti-phase' motions of a plate with heavy fluid loading, two 'elementary' theories are also suggested and checked against the 'exact' solution. Thus, these simplified theories are proven to be valid for analysis of wave propagation in sandwich plates. In the case of heavy fluid loading, it is shown that a dominantly longitudinal wave may either be propagating or decaying depending on the magnitude of the compressibility parameter (the ratio of a sound speed in a fluid to a sound speed in skin's material). A simple explanation of this transition is given.

Acknowledgements

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