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Journal of Sound and Vibration 271 (2004) 1063–1082

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Analysis of wave propagation in sandwich beams with parametric stiffness modulations

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Received 2 September 2002; accepted 20 March 2003

Abstract

Propagation of flexural and shear waves in an unbounded sandwich beam is considered. This beam is thought of as a 'unit width strip', which is cut from an unbounded sandwich plate in the direction of propagation of a plane flexural or shear wave. The skin plies of such a plate (or, in effect, a beam) are made of a homogeneous material, whereas the core ply is microscopically inhomogeneous. This consists of foam bulk material filled by multiple inclusions whose orientation may be varied in a prescribed manner both in space and in time domains to generate stiffness modulation. At the macro-level, this composition of two 'standard' skin plies with a 'two-phase' core ply manifests itself as a homogenized smart 'dynamic' material having controlled distributed stiffness parameters. Control of the wave motions in a sandwich beam made of this material is then limited to wavelengths, which greatly exceed the characteristic size of cell elements of the micro-structure in the core ply and therefore to comparatively low frequencies. It is shown that asymptotically small stiffness modulations are capable of transforming propagating waves existing in a plate of constant stiffness to non-propagating ones. This mechanism of suppression of wave propagation in sandwich beams always co-exists with suppression of wave propagation due to the presence of internal damping in the material of all plies. The efficiency of the suggested mechanism is therefore estimated by the comparison of decrements of amplitudes of 'almost' propagating waves produced by the stiffness modulation and to the internal damping.

1. Introduction

In many technical applications, it is necessary to suppress the propagation of elastic waves in thin walled structures (e.g., plates and shells) in order to improve their NVH (noise, vibration and

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doi:10.1016/j.jsv.2003.03.005

harshness) characteristics. As is quite typical, e.g., for analysis of dynamics of an aircraft fuselage or a car, a 'loading zone' (where the driving forces are applied) has relatively small dimensions (compared with the dimensions of the whole structure). Since in such a case vibration suppression measures may hardly be performed in the source of excitation, the vibration control and suppression strategy is associated with damping of propagating structural waves. Following Ref. [1], such a situation is specified as the local control of structural waves. Then feedback and feedforward active control strategies (based on measurements of vibration/sound radiation field from primary sources with the use of some 'corrective' secondary ones) are considered as the most efficient tools to suppress the energy transportation.

An alternative way to suppress wave propagation and radiation of sound is associated with the recent advances in material technology, which make it possible to manufacture so-called smart materials. In particular, the concept of 'dynamic materials' as a special type of smart materials designed to suppress vibrations has been suggested and elaborated in recent papers [2–7]. This concept is in effect based on the ideas of 'vibrational rheology' suggested by Blekhman [8]. If a micro-inhomogeneous composite material is considered, then its mechanical 'macro-level' characteristics are usually derived by homogenization (averaging) of the response of its cell elements to 'standard' loading cases at the micro-level. In the literature, for example, in Ref. [9] these elementary cells are considered as immobile so that the 'global' mechanical properties of a smart material are uniquely defined. However, in principle it is possible to introduce some actuators, which provoke the motion (oscillation) of cells in a prescribed manner. Then it appears that the 'global' mechanical properties of a smart 'dynamic' material [10] after homogenization depend on the parameters of these 'hidden motions' (e.g., the frequency and the amplitude of vibrations of cell elements) besides 'static' parameters of cell elements (e.g., dimensions of the cells, their material properties, etc.). This aspect has been thoroughly considered in Refs. [2,6] and is not pursued any further in the present paper.

In Refs. [2,5–7], the active control of resonant standing waves in a plate made of such a smart 'dynamic' material is analyzed. It is shown, that a parametric control introduced as the modulation of stiffness in a sandwich or honeycomb plate of a finite length effectively prevents large amplitude vibrations. The same concept has been used to deal with the active control of wave propagation in Refs. [2–4]. Specifically, a general description of the 'screening phenomenon' of wave propagation in an elastic laminate with modulated stiffness and mass parameters is suggested in Refs. [3,4]. In Ref. [2], the effect of suppression of propagation of flexural waves in an infinitely long beam is analyzed by the method of multiple scales and it is shown that the parametric stiffness modulation may transform propagating flexural waves into evanescent ones.

In the present paper, the concept of parametric stiffness modulation is applied for analysis of suppression of wave propagation in a sandwich beam. This beam is thought of as a 'unit width strip' cut from an unbounded sandwich plate in the direction of propagation of a plane wave. In Section 2, dispersion polynomial is derived for a beam without stiffness modulation and internal damping and two regimes of wave motion are identified depending on the excitation frequency. Then the dynamics of a sandwich beam with stiffness modulation are considered. The method of multiple scales is used to derive amplitude modulation equations in Section 3. In Section 4, suppression of wave propagation in a plate with modulated stiffness is analyzed in three cases—coupled spatial/temporal modulation, purely temporal modulation and purely spatial modulation. The efficiency of the suggested control mechanism in these cases is compared. Finally, Section 5 contains a comparison of the efficiency of suppression of wave propagation due to stiffness modulation and due to internal damping.

2. Wave propagation in a sandwich beam without stiffness modulation

A beam of the sandwich composition, which consists of two symmetrical relatively thin, stiff skin plies and a thick, soft, core ply is considered. All plies are assumed to be isotropic and the following non-dimensional parameters are introduced to describe the internal structure of a sandwich plate: $\varepsilon = h_{skin}/h_{core}$ as a thickness parameter, $\delta = \rho_{core}/\rho_{skin}$ as a density parameter and $\gamma = E_{core}/E_{skin}$ as a stiffness parameter. In a conventional composite plate, Young's module of a core 'static' micro-inhomogeneous material is found by some averaging technique, see Ref. [9] and is independent of time and space. However, if a 'dynamic' material is considered, the Young's module becomes a function of temporal and spatial co-ordinates. In the framework of the theory given in Refs. [11,12], the deformation of a sandwich beam element is governed by two independent variables: a displacement of the mid-surface of the whole element w (which is the same for all plies) and a shear angle θ. The equations of motions of a beam are easily derived from Hamilton's principle (see details in Ref. [12])

$$\frac{1}{12}\left[\left(2+\frac{\gamma}{\varepsilon^3}\right)w''\right]'' - \frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2[\varepsilon\gamma(\theta+w')]' + \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{h}{c_0}\right)^2\ddot{w} - \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{h}{c_0}\right)^2\ddot{w}'' = 0, \tag{1a}$$

$$-\frac{1}{2}\theta'' + \frac{(1-\nu)}{2}\varepsilon\gamma(\theta+w') + \frac{1}{2}\left(\frac{h}{c_0}\right)^2\ddot{\theta} = 0. \tag{1b}$$

This system of linear differential equations is written for non-dimensional variables $w = w_{dim}/h$, $x = x_{dim}/h$. In Eqs. (1), $h \equiv h_{core}$, $c_0 \equiv \sqrt{E_{skin}/\rho_{skin}(1-\nu^2)}$, dots denote derivatives on dimensional time, $\partial/\partial t$, primes denote derivatives on non-dimensional axial co-ordinate, $(\ )' \equiv \partial(\ )/\partial x$.

In this section of the paper a beam with constant stiffness is considered, i.e., $\gamma(x,t) = \gamma_0$ and in Eq. (1a) only differentiation of the functions $w(x,t)$ and $\theta(x,t)$ should be performed. The analysis is restricted by the case of harmonic motions and the solution of these equations, which describes propagation of shear and flexural waves in an infinitely long sandwich beam is formulated as

$$w(x,t) = A_w \exp(\mathrm{i}kx - \mathrm{i}\omega t),$$
$$\theta(x,t) = A_\theta \exp(\mathrm{i}kx - \mathrm{i}\omega t). \tag{2}$$

The non-dimensional frequency parameter $\Omega \equiv (\omega h/c_0)$ and the non-dimensional wave number k are related to each other by the dispersion polynomial

$$\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^6 + \left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)((1-\nu)\varepsilon\gamma_0 - \Omega^2) + 6(1-\nu)\varepsilon\gamma_0\left(1+\frac{1}{\varepsilon}\right)^2 - \Omega^2\left(2+\frac{\delta}{\varepsilon^3}\right)\right]k^4$$
$$+\left[-(1-\nu)\varepsilon\gamma_0\left(\left(2+\frac{\delta}{\varepsilon^3}\right)+6\left(1+\frac{1}{\varepsilon}\right)^2\right)+\Omega^2\left(2+\frac{\delta}{\varepsilon^3}\right)-12\left(2+\frac{\delta}{\varepsilon}\right)\right]\Omega^2k^2$$
$$-\ 12\Omega^2\left(2+\frac{\delta}{\varepsilon}\right)((1-\nu)\varepsilon\gamma_0 - \Omega^2) = 0. \tag{3}$$

For each combination of (k, Ω), which satisfies Eq. (3) the amplitudes of flexural and shear waves are linked by

$$A_w = \mathrm{i} M A_\theta. \tag{4a}$$

Here the modal coefficient is introduced

$$M = \frac{(k^2 + (1-\nu)\varepsilon\gamma_0 - \Omega^2)}{(1-\nu)\varepsilon\gamma_0 k}, \tag{4b}$$

As can be seen from Eq. (4a), the shear component of a displacement vector of a sandwich plate is $\pi/2$ out of phase with the lateral displacement in a propagating wave (i.e., for a real-valued wave number k).

The dispersion polynomial (3) has six roots at each excitation frequency, which are grouped in three pairs. Those having positive or negative imaginary parts specify evanescent waves, which decay in the positive or in the negative direction of an axial co-ordinate, respectively. Purely real positive roots specify waves propagating from the left to the right, whilst those with negative real parts are related to the waves with negative phase velocity. Due to the natural symmetry of the problem, it is sufficient to consider only the waves, propagating or decaying from the left to the right.

In Fig. 1a, the dependence of the non-dimensional purely real wave number k (which defines a propagating wave at any frequency) on the frequency parameter Ω is shown for the following set of parameters of the sandwich beam composition: $\gamma_0 = 0.01$, $\delta = 0.1$, $\varepsilon = 0.05$, $\nu = 0.3$. This is the dominantly flexural wave as can be seen from Fig. 1b, where the modal coefficient M is plotted versus the frequency parameter Ω. In the Kirchhoff plate theory this type of propagating wave always co-exists with an evanescent wave with a purely imaginary wave number of the same magnitude. It is similar in this sandwich plate theory, but, as the evanescent waves do not transport any energy in an infinitely long beam, they are of no specific interest.

However, it is important to note that one more propagating wave may exist in a sandwich beam. The dependence of its non-dimensional purely real wave number k on the frequency parameter Ω is shown in Fig. 2a, while its modal coefficient M is shown in Fig. 2b also as a function of Ω. This is a wave of dominantly shear deformation, generated by the sliding of the skin plies in opposite directions. Its wave number k is smaller than that of the flexural wave. Unlike the

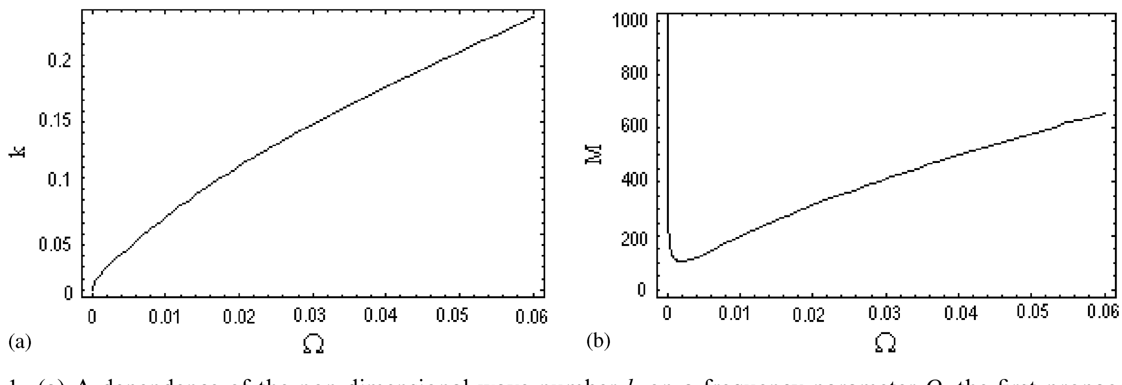

Fig. 1. (a) A dependence of the non-dimensional wave number k on a frequency parameter Ω, the first propagating wave. (b) A dependence of the modal coefficient M on a frequency parameter Ω, the first propagating wave.

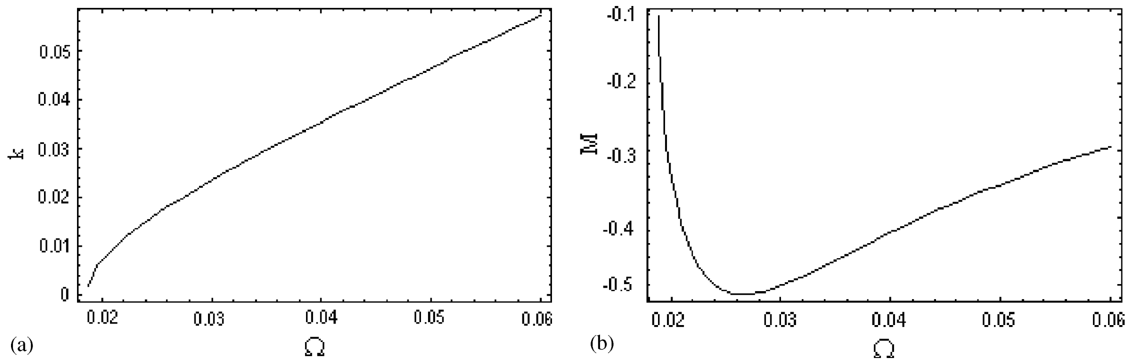

Fig. 2. (a) A dependence of the non-dimensional wave number k on a frequency parameter Ω, the second propagating wave. (b) A dependence of the modal coefficient M on a frequency parameter Ω, the second propagating wave.

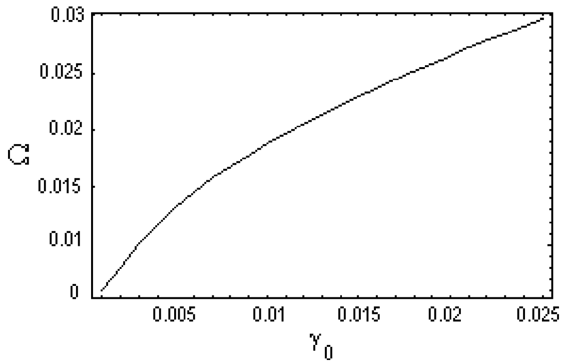

Fig. 3. A dependence of the magnitude of a 'cut-on frequency' on a parameter γ_0.

flexural wave, this wave has a 'cut-on frequency', i.e., a frequency, at which purely imaginary wave number becomes purely real. A dependence of the magnitude of a 'cut-on frequency' on parameter γ_0 of the sandwich beam composition for $\delta = 0.1$, $v = 0.3$ and $\varepsilon = 0.05$ is illustrated in Fig. 3. As it can be seen from this graph, as the parameter γ_0 grows (i.e., as the Young's module of a core ply tends to the Young's module of skin plies), this 'cut-on frequency' grows and the dynamic properties of a sandwich beam become similar to those of a Timoshenko beam. Although it is not a goal of the present paper to analyze in detail validity limits of this sandwich theory, it should be noted that both dispersion curves predicted agree very well with the low branches of dispersion curves obtained by solving the equations of motions derived from the two-dimensional elasticity theory [13,14].

3. Suppression of wave propagation in a sandwich beam by coupled spatial–temporal stiffness modulation

As shown in the previous section, for any excitation frequency there always exist one or (above a 'cut-on frequency') two free waves propagating in a sandwich beam. As it is well known, they transport the energy 'injected' into the structure in a loaded zone, to its remote parts, and therefore generate vibrations and sound radiation there. As discussed in the Introduction, it is necessary to explore the possibilities of suppression of these propagating waves. One of these possibilities is offered by the stiffness modulation performed in both the time and in the space domains. The feasibility of such a modulation in a sandwich or a honeycomb plate has been discussed in Refs. [2,5,6] and this aspect is not pursued any further.

Let the non-dimensional stiffness parameter $\gamma(x,t)$ be decomposed into the sum of 'bulk' constant component γ_0 and some fairly small fluctuating component $\gamma_1(x,t)$, i.e.,

$$\gamma(x,t) = \gamma_0 + \gamma_1(x,t). \tag{5}$$

These fluctuations may be produced by, for example, some small variations in an orientation of the cell elements composing the honeycomb core ply of a three-layered plate, see Ref. [6]. From the practical viewpoint, it is unrealistic to generate large modulations of the stiffness of the core ply, so the following inequality is held

$$|\gamma_1(x,t)| \ll \gamma_0. \tag{6}$$

Thus, asymptotically small perturbations of the stiffness should be considered. Then Eq. (5) is re-written as (μ is a formal small parameter, see Ref. [15])

$$\gamma(x,t) = \gamma_0 + \mu\gamma_1(x,t). \tag{7}$$

An incident free wave specified by the frequency parameter Ω and the wave number $k(\Omega)$ (found from Eq. (3) for a given frequency) is considered. This wave has to be suppressed by the means of stiffness modulation. To solve this problem, the method of multiple scales is used [15] and a lateral displacement and a shear angle are decomposed into regular asymptotic expansions of the same small parameter μ

$$\begin{aligned} w(x,t) &= w_0(x_0,t_0,\xi,T) + \mu w_1(x_0,t_0,\xi,T), \\ \theta(x,t) &= \theta_0(x_0,t_0,\xi,T) + \mu\theta_1(x_0,t_0,\xi,T). \end{aligned} \tag{8}$$

Here t_0, x_0 are the 'fast' spatial and temporal co-ordinates, T, ξ are the 'slow' ones. They are defined as $t_0 = t$, $x_0 = x$, $T = \mu t$, $\xi = \mu x$, respectively.

The harmonic stiffness modulation is introduced as

$$\gamma_1(x,t) = \gamma_1^I \cos \varpi t \cos \eta x. \tag{9}$$

As discussed in the Introduction, a sandwich beam with modulated stiffness may be regarded as a beam made of a 'dynamic material' [10]. Its mechanical properties are then defined not only by the 'static' parameters ε, δ and γ_0, but also by the parameters of 'vibrational rheology', which are the amplitude γ_1^I, the frequency ϖ and the wave number η of stiffness modulation. Such a material has highly adaptive mechanical properties since these parameters may easily be modified in response to any changes in excitation conditions.

The stiffness modulation in the vicinity of the principal parametric resonance is considered

$$\varpi = 2\omega + \mu\sigma_T. \tag{10a}$$

Then the modulation wave number is selected as

$$\eta = 2k(\omega) + \mu\sigma_\xi. \tag{10b}$$

As can be seen from Eq. (10b), ω and k are linked by the dispersion equation (3), whereas the frequency ϖ and the wave number η of stiffness modulation are not. Small deviations from these relations are accounted for by the spatial σ_ξ and the temporal σ_T detuning parameters, see Ref. [15].

The asymptotic expansions (7,8) are substituted into Eqs. (1), which now contain time- and space-dependent stiffness parameter (9) and the terms to the leading order μ^0 are collected. Then the formulation of the problem of propagation of a free wave in a plate without stiffness modulations is obtained, which defines the dependence of the functions $w_0(x_0, t_0, \xi, T)$ and $\theta_0(x_0, t_0, \xi, T)$ on 'fast' co-ordinates (x_0, t_0). To find the dependence of these functions on 'slow' variables, a solution is sought in the form

$$w_0 = A_w(\xi, T)e^{i\varphi} + \bar{A}_w(\xi, T)e^{-i\varphi} + B_w(\xi, T)e^{i\psi} + \bar{B}_w(\xi, T)e^{-i\psi}, \tag{11a}$$

$$\theta_0 = A_\theta(\xi, T)e^{i\varphi} + \bar{A}_\theta(\xi, T)e^{-i\varphi} + B_\theta(\xi, T)e^{i\psi} + \bar{B}_\theta(\xi, T)e^{-i\psi}. \tag{11b}$$

Here $\varphi = kx_0 - \omega t_0$, $\psi = kx_0 + \omega t_0$ and $k = k(\Omega)$ as defined by the dispersion equation (3). The amplitudes of transverse and shear components of the displacement vector are related to each other by the modal coefficient (4), $A_\theta = -(i/M)A_w$. Thus Eqs. (11a,b) actually contain four unknown functions in 'slow' variables.

Following the standard scheme of asymptotic analysis, the problem to the order μ^I is formulated as

$$\begin{aligned}
&\frac{1}{12}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^4 w_1}{\partial x_0^4} - \frac{1}{12}\left(\frac{h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon^3}\right)\frac{\partial^4 w_1}{\partial x_0^2\partial t_0^2} + \left(\frac{h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon}\right)\frac{\partial^2 w_1}{\partial t_0^2} \\
&\quad - \frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2\gamma_0\varepsilon\left(\frac{\partial^2 w_1}{\partial x_0^2} + \frac{\partial \theta_1}{\partial x_0}\right) \\
&\qquad = -\frac{1}{3}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^4 w_0}{\partial x_0^3\partial\xi} + \frac{1}{6}\left(\frac{h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon^3}\right)\frac{\partial^4 w_0}{\partial x_0^2\partial t_0\partial T} + \frac{1}{6}\left(\frac{h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon^3}\right)\frac{\partial^4 w_0}{\partial x_0\partial\xi\partial t_0^2} \\
&\qquad - 2\left(\frac{h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon}\right)\frac{\partial^2 w_0}{\partial t_0\partial T} + (1-\nu)\left(1+\frac{1}{\varepsilon}\right)^2\gamma_0\varepsilon\frac{\partial^2 w_0}{\partial x_0\partial\xi} + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\gamma_0\varepsilon\frac{\partial\theta_0}{\partial\xi} \\
&\qquad - \gamma_1^I\cos\varpi t_0\cos\eta x_0\left[\frac{1}{12\varepsilon^3}\left(\frac{\partial^4 w_0}{\partial x_0^4} - \eta^2\frac{\partial^2 w_0}{\partial x_0^2}\right) - \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\left(\frac{\partial^2 w_0}{\partial x_0^2} + \frac{\partial\theta_0}{\partial x_0}\right)\right] \\
&\qquad - \gamma_1^I\eta\cos\varpi t_0\sin\eta x_0\left[-\frac{1}{6\varepsilon^3}\frac{\partial^3 w_0}{\partial x_0^3} + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\left(\frac{\partial w_0}{\partial x_0} + \theta_0\right)\right], \\
&\frac{1}{2}\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{(1-\nu)}{2}\varepsilon\gamma_0\left(\frac{\partial w_1}{\partial x_0} + \theta_1\right) - \frac{1}{2}\left(\frac{h}{c_0}\right)^2\frac{\partial^2\theta_1}{\partial t_0^2} \\
&\quad = -\frac{\partial^2\theta_0}{\partial x_0\partial\xi} + \frac{(1-\nu)}{2}\varepsilon\gamma_0\frac{\partial w_0}{\partial\xi} + \left(\frac{h}{c_0}\right)^2\frac{\partial^2\theta_0}{\partial t_0\partial T} + \gamma_1^I\cos\varpi t_0\cos\eta x_0\frac{(1-\nu)}{2}\varepsilon\left(\frac{\partial w_0}{\partial x_0} + \theta_0\right).
\end{aligned} \tag{12}$$

A general solution of these equations preserves exactly the same dependence of the functions w_1, θ_1 on 'fast' co-ordinates (x_0, t_0) as

$$\begin{aligned} w_1 &= A_w^1(\xi, T)e^{i\varphi} + \bar{A}_w^1(\xi, T)e^{-i\varphi} + B_w^1(\xi, T)e^{i\psi} + \bar{B}_w^1(\xi, T)e^{-i\psi}, \\ \theta_1 &= A_\theta^1(\xi, T)e^{i\varphi} + \bar{A}_\theta^1(\xi, T)e^{-i\varphi} + B_\theta^1(\xi, T)e^{i\psi} + \bar{B}_\theta^1(\xi, T)e^{-i\psi}. \end{aligned} \tag{13}$$

To ensure a uniform validity of asymptotic expansions (8), see Ref. [15], all secular terms should be removed from the right side of Eqs. (12). The terms containing, for example, $e^{i\varphi}$ are collected in left hand side and right side of Eqs. (12)

$$\begin{aligned} &\left[\frac{1}{12}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^4 - \frac{1}{12}\left(\frac{\omega h}{c_0}\right)^2\left(2+\frac{\delta}{\varepsilon^3}\right)k^2 - \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)^2 + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0 k^2\right]A_w^1 \\ &\quad - i\frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0 k A_\theta^I \\ &\quad = i\left[\frac{1}{3}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^3 - \frac{1}{6}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)^2 k + (1-\nu)\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma k + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0 M\right]\frac{\partial A_w}{\partial \xi} \\ &\qquad + i\left[\frac{1}{6}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right)k^2 + 2\left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right)\right]\frac{\partial A_w}{\partial T} + \frac{\gamma_1^I}{4}e^{i\mu x_0\sigma_\xi - i\mu t_0\sigma_T}\bar{A}_w \\ &\qquad \times\left[-\frac{1}{12\varepsilon^3}(k^4+\eta^2k^2) - \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon(k^2+Mk)\right. \\ &\qquad\qquad \left. + \frac{1}{6\varepsilon^3}\eta k^3 + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon(k+M)\eta\right], \end{aligned} \tag{14a}$$

$$\begin{aligned} &-i\frac{(1-\nu)}{2}\varepsilon\gamma_0 k A_w^1 + \left[-\frac{1}{2}k^2 - \frac{(1-\nu)}{2}\varepsilon\gamma_0 + \frac{1}{2}\left(\frac{\omega h}{c_0}\right)^2\right]A_\theta^1 \\ &\quad = \left[Mk + \frac{(1-\nu)}{2}\varepsilon\gamma_0\right]\frac{\partial A_w}{\partial \xi} + M\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right)\frac{\partial A_w}{\partial T} - i\frac{\gamma_1^I}{4}\exp(i\mu x_0\sigma_\xi - i\mu t_0\sigma_T) \\ &\qquad \times \bar{A}_w\frac{(1-\nu)}{2}\varepsilon(k+M). \end{aligned} \tag{14b}$$

The principal determinant of this algebraic system equals zero, so each of the complementary determinants should also vanish. This gives the amplitude modulation equation, which is cumbersome and not presented here in the interest of brevity.

Similar equations are obtained by collecting the terms containing $e^{-i\varphi}$, $e^{i\psi}$ and $e^{-i\psi}$. The equations formulated with respect to A_w, $\bar{A}_w$ are combined to obtain the following differential equation in slow temporal and spatial co-ordinates

$$a^2\frac{\partial^2 A_w}{\partial \xi^2} + 2ab\frac{\partial^2 A_w}{\partial \xi \partial T} + b^2\frac{\partial^2 A_w}{\partial T^2} + i\left(a\frac{\partial A_w}{\partial \xi} + b\frac{\partial A_w}{\partial T}\right)(\sigma_T b - \sigma_\xi a) - c^2 A_w = 0, \tag{15}$$

$$a = f_1\left[Mk + \frac{(1-\nu)}{2}\varepsilon\gamma_0\right] - \frac{(1-\nu)}{6}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\varepsilon\gamma_0 k^4 + \frac{(1-\nu)}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\varepsilon\gamma_0 k^2\left(\frac{\omega h}{c_0}\right)^2$$

$$-\frac{(1-\nu)^2}{2}\left(1+\frac{1}{\varepsilon}\right)^2(\varepsilon\gamma_0)^2k^2 - \frac{(1-\nu)^2}{4}\left(1+\frac{1}{\varepsilon}\right)^2(\varepsilon\gamma_0)^2Mk,$$

$$b = f_1\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right)M - \frac{(1-\nu)}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\varepsilon\gamma_0 k^3\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right) - (1-\nu)\varepsilon\gamma_0 k\left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)\left(\frac{h}{c_0}\right),$$

$$c = \frac{\gamma_1^I}{8}(1-\nu)\varepsilon[f_1(k+M) - f_2\gamma_0 k],$$

$$f_1 = \frac{1}{12}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^4 - \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)^2k^2 - \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)^2 + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0 k^2,$$

$$f_2 = -\frac{1}{12\varepsilon^3}(k^4+\eta^2k^2) - \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon(k^2+Mk) + \frac{1}{6\varepsilon^3}\eta k^3 + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\eta(k+M).$$

This equation describes modulation of a wave travelling with the positive phase velocity, i.e., from the left to the right. A similar equation is formulated for the wave travelling in opposite direction.

$$a^2\frac{\partial^2 B_w}{\partial\xi^2} - 2ab\frac{\partial^2 B_w}{\partial\xi\partial T} + b^2\frac{\partial^2 B_w}{\partial T^2} - \mathrm{i}\left(a\frac{\partial B_w}{\partial\xi} + b\frac{\partial B_w}{\partial T}\right)(\sigma_T b - \sigma_\xi a) - c^2B_w = 0. \tag{16}$$

Due to the symmetry of the beam, all results of the analysis of Eq. (15) are entirely applicable to the solution of this equation, so only the modulation of waves propagating from the left to the right is considered.

The dimensional detuning parameter σ_T is used in Eqs. (15,16). It is defined as $\sigma_T = \bar{\sigma}_T(c_0/h)$, where $\bar{\sigma}_T$ is the non-dimensional detuning parameter. Hereafter, the new coefficient $\bar{b} = b/(h/c_0)$ is introduced. A wave-type solution of the linear modulation equation (15) is sought as

$$A_w = \hat{A}_w e^{\mathrm{i}K\xi - \mathrm{i}\varpi T}. \tag{17}$$

The dispersion relation between the 'slow' frequency ϖ and the 'slow' wave number K is readily available as Eq. (17) is substituted into Eq. (15)

$$\left(aK - \left(\frac{\varpi h}{c_0}\right)\bar{b}\right)^2 + \left(aK - \left(\frac{\varpi h}{c_0}\right)\bar{b}\right)(\bar{\sigma}_T\bar{b} - \sigma_\xi a) + c^2 = 0. \tag{18}$$

This quadratic in $aK - (\varpi h/c_0)\bar{b}$ equation may have either pure real or complex roots, depending on the sign of the discriminant.

$$D = (\bar{b}\,\bar{\sigma}_T - a\sigma_\xi)^2 - 4c^2. \tag{19}$$

For each of the dispersion curves, this discriminant depends on all the parameters of the sandwich plate composition and on the excitation frequency. If $D > 0$, the roots of Eq. (18) are real and the modulated wave propagates in the beam. On the other hand, if $D < 0$, the roots are complex. Since the frequency of stiffness modulation should be purely real, solution of Eq. (18) in this case gives two complex conjugate wave numbers. However, Eq. (15) is also derived as a combination of two differential equations of the first order in 'slow' variables for two complex conjugate functions A_w, $\bar{A}_w$ and each root should be attributed to one of these functions. Standard technique of tracking the roots (when some damping is introduced in original equations and its magnitude tends to zero) proves that the root with the positive imaginary part is 'generated' by the first term in Eq. (11a), whereas the root with the negative imaginary part is related to the second term. Then the first term in Eq. (11a) has the following form

$$A_w(\xi, T)e^{i\varphi} = \hat{A}_w e^{iK\xi - i\varpi T} e^{ikx_0 - i\omega t_0} = \hat{A}_w \exp[(ik + i\mu K)x_0 - (i\omega + i\mu\varpi)t_0].$$

Thus, for a given excitation frequency ω, the modulation of the stiffness of a core ply performed at frequency $\varpi = 2\omega$ and spatially distributed as $\eta = 2k(\omega)$ transforms a free wave of wave number $k(\omega)$, propagating from the left to the right into a decaying one, which has a complex-valued wave number $\mu\,\mathrm{Im}[K]\mathrm{i} + \mu\,\mathrm{Re}[K] + k$. Exactly the same result is obtained for the second term in Eq. (11a).

Immediately following from Eq. (19), propagation of waves is suppressed at any frequency in the case of a precise tuning ($\sigma_\xi = \bar{\sigma}_T = 0$). The condition $\bar{b}\bar{\sigma}_T - a\sigma_\xi = 0$ holds true not only when $\sigma_\xi = \bar{\sigma}_T = 0$, but also when $\bar{\sigma}_T/\sigma_\xi = a/b$. This observation relaxes the necessity of precise tuning of the stiffness modulation. It is also easily seen that, as c is proportional to γ_1^I, in the absence of stiffness modulation the condition $D > 0$ always holds and no suppression of wave propagation is possible. One more simple observation is related to the symmetry of the function D with respect to the detuning parameters σ_ξ and $\bar{\sigma}_T$: a simultaneous change in signs of these parameters does not affect D. It is also clear that when any of these parameters become large, while another one does not, the determinant becomes positive. Thus, an imperfect tuning of spatial or temporal modulation results in the transformation of the modulated wave (which could be evanescent in the case of a perfectly tuned stiffness modulation) back to a travelling type.

4. Analysis of efficiency of wave propagation control by the coupled spatial and temporal stiffness modulation

To judge whether it is possible to suppress propagation of a given wave at a given frequency, it is sufficient just to determine the sign of the discriminant D, which depends upon (besides the parameters of a sandwich plate composition) the magnitudes of detuning parameters σ_ξ, $\bar{\sigma}_T$, the amplitude of stiffness modulation and the excitation frequency. As discussed, in the case of perfect tuning, this sign is automatically negative and the suppression of wave propagation is always possible. Thus, the role of detuning parameters appears to be crucial for assessment of efficiency of this method of suppression of wave propagation and should be explored in more details. In Fig. 4, a dependence of the discriminant D on a frequency parameter $\Omega \equiv (\omega h/c_0)$ is shown for $\gamma_0 = 0.01$, $\delta = 0.1$, $\varepsilon = 0.05$, $v = 0.3$, $\gamma_1^I = 0.1\gamma_0$ for two combinations of detuning parameters, $\sigma_\xi = \bar{\sigma}_T = 0.0001$ (curve 1) and $\sigma_\xi = \bar{\sigma}_T = 0.001$ (curve 2). Graphs in Fig. 4a,b are plotted for the dominantly flexural wave in the 'low frequency' range (i.e., the frequency range where only one propagating wave exists). Due to the detuning, this wave is suppressed, not in the whole frequency range, but only when $\Omega > 0.00047$ for $\sigma_\xi = \bar{\sigma}_T = 0.0001$ (curve 1, Fig. 4a) and when $\Omega > 0.0013$ for $\sigma_\xi = \bar{\sigma}_T = 0.001$ (curve 2, Fig. 4b). As can be seen from comparison of these graphs, the

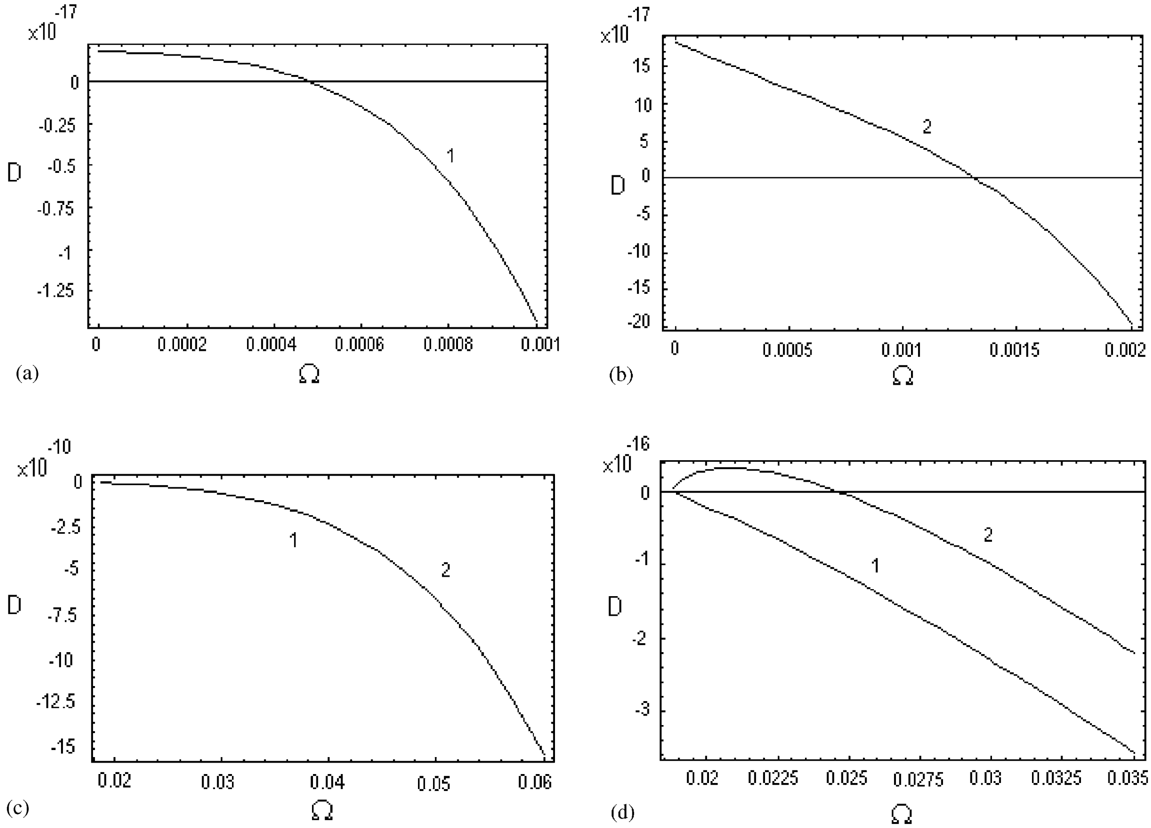

Fig. 4. (a) A discriminant versus a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.0001$, the first propagating wave in the 'low frequency' range. (b) A discriminant versus a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.001$, the first propagating wave in the 'low frequency' range. (c) A discriminant versus a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.0001$—curve 1, $\sigma_\xi = \bar{\sigma}_T = 0.001$—curve 2, the first propagating wave in the 'high frequency' range. (d) A discriminant versus a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.0001$—curve 1, $\sigma_\xi = \bar{\sigma}_T = 0.001$—curve 2, the second propagating wave in the 'high frequency' range.

increase in detuning parameters narrows the frequency range, where suppression of wave propagation is possible.

In the 'high frequency' range, two propagating waves exist. The curve in Fig. 4c is plotted in this range for the dominantly flexural wave. As can be seen, the dominantly flexural wave may be suppressed by the stiffness modulation at any frequency within this range and there is no difference between these two particular cases (curves 1 and 2 merge). The curve in Fig. 4d is plotted for the dominantly shear wave. This wave is suppressed in the whole frequency range for $\sigma_\xi = \bar{\sigma}_T = 0.0001$, whereas for $\sigma_\xi = \bar{\sigma}_T = 0.001$, it is suppressed when $\Omega > 0.0246$.

As can be seen, the suggested mechanism of suppression of wave propagation is rather sensitive to the tuning of stiffness modulation. Another issue, which is very important from the practical viewpoint, is the assessment of the decay rate of a given wave generated by a given stiffness modulation. In fact, some material losses always exist in any real sandwich structure, and the suppression of wave propagation due to the suggested mechanism should exceed the suppression effect due to the presence of material losses in order to be recognised as a tool for vibration control. This assessment is based on calculations of decrements of vibrations and is discussed in Section 6.

5. Purely temporal stiffness modulation and purely spatial stiffness modulation

In previous sections, stiffness modulation has been considered as performed simultaneously in space and time domains. However, it is entirely relevant to assume that the stiffness of a beam varies either only in time or only in space. Following the terminology suggested in Ref. [10], in the former case a 'dynamic material' is still considered, in the latter one a 'static material' is considered. The purely temporal stiffness modulation is defined as

$$\gamma_1(x,t) = \gamma_1^I \cos \varpi t. \tag{20}$$

The same technique as outlined in Section 3 is applied and the following amplitude modulation equation for the wave propagating from the left to the right is obtained

$$(a_T^2 - d_T^2)\frac{\partial^2 A_w}{\partial \xi^2} - \mathrm{i}\sigma_T b_T (a_T + d_T)\frac{\partial A_w}{\partial \xi} - 2b_T d_T \frac{\partial^2 A_w}{\partial \xi \partial T} - \mathrm{i}\sigma_T b_T^2 \frac{\partial A_w}{\partial T} - b_T^2 \frac{\partial^2 A_w}{\partial T^2} + c_T^2 A_w = 0, \tag{21}$$

$$\begin{aligned} a_T =& f_1\left[Mk + \frac{(1-\nu)}{2}\varepsilon\gamma_0\right] + \frac{(1-\nu)}{12}\left(2 + \frac{\delta}{\varepsilon^3}\right)\varepsilon\gamma_0 k^2 \left(\frac{\omega h}{c_0}\right)^2 \\ & - \frac{(1-\nu)^2}{2}\left(1 + \frac{1}{\varepsilon}\right)^2 (\varepsilon\gamma_0)^2 k^2 - \frac{(1-\nu)^2}{4}\left(1+\frac{1}{\varepsilon}\right)^2 (\varepsilon\gamma_0)^2 Mk, \quad b_T = b, \end{aligned}$$

$$c_T = \frac{\gamma_1^I}{4}(1-\nu)\varepsilon\left[f_1(k+M) - \frac{1}{12\varepsilon^3}\gamma_0 k^5\right], \quad d_T = -\frac{(1-\nu)}{6}\varepsilon\gamma_0\left(2 + \frac{\gamma_0}{\varepsilon^3}\right)k^4.$$

As mentioned in Section 3, a similar equation is obtained for the waves propagating in the opposite direction, so that it is sufficient to analyse only this one. A wave-type solution of this linear modulation equation is sought in form (17) and a dispersion relation between the 'slow' frequency ϖ and the 'slow' wave number K is obtained:

$$K^2(a_T^2 - d_T^2) + K\left(2\bar{b}d_T\left(\frac{\varpi h}{c_0}\right) - \bar{\sigma}_T\bar{b}(a_T + d_T)\right) + \left(\frac{\varpi h}{c_0}\right)\bar{b}^2\sigma_T - \left(\frac{\varpi h}{c_0}\right)^2\bar{b}^2 - c_T^2 = 0. \tag{22}$$

Unlike the case of the coupled spatial and temporal stiffness modulation, this equation cannot be reduced to a simple quadratic equation with respect to some linear combination of 'slow' parameters K and $(\varpi h/c_0)$. Indeed, these parameters now are not considered as independent of each other, because the arguments of the exponent in formula (17) have to satisfy the original dispersion equation (3) up to order μ^1 terms. In other words, by performing purely temporal stiffness modulation, the certain point at a given branch of dispersion curve is defined. Then the modulated wave is formulated as a tangent to this branch in the vicinity of this point.

The equations $\tilde{\omega} = \omega + \mu\varpi$ and $\tilde{k} = k + \mu K$ are substituted into the dispersion equation (3) and then the terms to order μ^1 are collected. This gives the following algebraic equation in K and $(\varpi h/c_0)$:

$$K = -\bar{\Lambda}\left(\frac{\varpi h}{c_0}\right). \tag{23}$$

In formula (23), $\bar{\Lambda} = Z_2/Z_1$

$$\begin{aligned}
Z_1 = & -\frac{1}{4}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^5 - \frac{1}{6}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)(1-\nu)\varepsilon\gamma k^3 + \frac{1}{6}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^3\left(\frac{\omega h}{c_0}\right)^2 \\
& + \frac{1}{6}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)^2 k^3 + \frac{1}{12}(1-\nu)\varepsilon\gamma_0\left(2+\frac{\delta}{\varepsilon^3}\right)k\left(\frac{\omega h}{c_0}\right)^2 - \frac{1}{12}k\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)^4 \\
& + \left(2+\frac{\delta}{\varepsilon}\right)k\left(\frac{\omega h}{c_0}\right)^2 - (1-\nu)\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0 k^3 + \frac{1}{2}\left(\frac{\omega h}{c_0}\right)^2(1-\nu)\varepsilon\gamma_0 k\left(1+\frac{1}{\varepsilon}\right)^2, \\
Z_2 = & \frac{1}{12}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)k^4\left(\frac{\omega h}{c_0}\right) - \frac{1}{6}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)^3 k^2 + \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)k^4 \\
& + \frac{1}{12}(1-\nu)\varepsilon\gamma_0\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{\omega h}{c_0}\right)k^2 - \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)^3 + k^2\left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right) \\
& + (1-\nu)\varepsilon\gamma_0\left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right) - \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{\omega h}{c_0}\right)^3 + \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\gamma_0\left(\frac{\omega h}{c_0}\right)k^2.
\end{aligned}$$

Substitution of Eq. (23) into Eq. (22) gives the quadratic in K equation

$$K^2\left(a_T^2 - \left(d_T + \frac{\bar{b}}{\bar{\Lambda}}\right)^2\right) - K\bar{b}\bar{\sigma}_T\left(a_T + d_T + \frac{\bar{b}}{\bar{\Lambda}}\right) - c_T^2 = 0. \tag{24}$$

This may have either pure real or complex roots, depending on the sign of the discriminant

$$D = \bar{\sigma}_T^2\bar{b}^2\left(a_T + d_T + \frac{b}{\bar{\Lambda}}\right)^2 + 4c_T^2\left(a_T^2 - \left(d_T + \frac{\bar{b}}{\bar{\Lambda}}\right)^2\right). \tag{25}$$

The same set of parameters of the sandwich plate composition as in all previous calculations is used. A dependence of the discriminant D on the frequency parameter $\Omega \equiv (\omega h/c_0)$ is shown in the case of perfect tuning ($\bar{\sigma}_T = 0$) for the dominantly flexural wave in Fig. 5a and for the dominantly shear wave in Fig. 5b, respectively. As it can be seen, both these waves are suppressed in the whole frequency range. However, the effect of suppression of wave propagation vanishes at very small magnitudes of detuning parameter (for example, $\bar{\sigma}_T = 0.0001$), i.e., the purely temporal stiffness modulation is less 'robust' in suppression of wave propagation, than the coupled spatial–temporal modulation.

A case of purely spatial stiffness modulation is considered similarly. This is defined as

$$\gamma_1(x,t) = \gamma_1^I \cos\eta x_0. \tag{26}$$

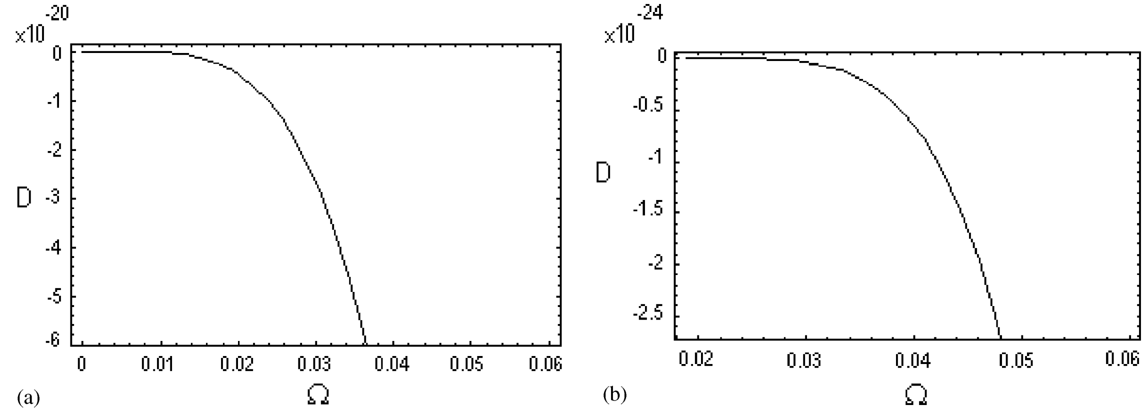

Fig. 5. (a) A discriminant versus a frequency parameter Ω, $\bar{\sigma}_T = 0$, the first propagating wave. (b) A discriminant versus a frequency parameter Ω, $\bar{\sigma}_T = 0$, the second propagating wave.

The following amplitude modulation equation is obtained for the wave propagating from the left to the right

$$(a_S^2 - d_S^2)\frac{\partial^2 A_w}{\partial \xi^2} + \mathrm{i}\sigma_\xi(a_S^2 - d_S^2)\frac{\partial A_w}{\partial \xi} - 2b_S d_S\frac{\partial^2 A_w}{\partial \xi \partial T} + \mathrm{i}\sigma_\xi b_S(a_S - d_S)\frac{\partial A_w}{\partial T} - b_S^2\frac{\partial^2 A_w}{\partial T^2} - c_S^2 A_w = 0, \tag{27}$$

$$a_s = f_1\frac{(1-v)}{2}\varepsilon\gamma - \frac{(1-v)}{6}\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\varepsilon\gamma_0 k^4 + \frac{(1-v)}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\varepsilon\gamma_0 k^2\left(\frac{\omega h}{c_0}\right)^2 - \frac{(1-v)^2}{2}\left(1+\frac{1}{\varepsilon}\right)^2(\varepsilon\gamma_0)^2 k^2 - \frac{(1-v)^2}{4}\left(1+\frac{1}{\varepsilon}\right)^2(\varepsilon\gamma_0)^2 Mk, \quad b_S = b,$$

$$c_s = \frac{\gamma_1^I}{4}(1-v)\varepsilon[f_1(k+M) - f_2\gamma_0 k], \quad d_s = f_1 Mk.$$

The dispersion relation between the 'slow' frequency ϖ and the 'slow' wave number K is formulated as

$$K^2(a_s^2 - d_s^2) + K\left(\sigma_\xi(a_s^2 - d_s^2) + 2\bar{b}d_s\left(\frac{\varpi h}{c_0}\right)\right) - \sigma_\xi\bar{b}\left(\frac{\varpi h}{c_0}\right)(a_s - d_s) - \left(\frac{\varpi h}{c_0}\right)^2\bar{b}^2 + c_s^2 = 0. \tag{28}$$

Substitution of Eq. (23) into Eq. (28) gives the quadratic in K equation:

$$K^2\left(a_s^2 - \left(d_s + \frac{\bar{b}}{\bar{\Lambda}}\right)^2\right) + K\sigma_\xi\left(a_s^2 - d_s^2 + \frac{\bar{b}}{\bar{\Lambda}}(a_s - d_s)\right) + c_s^2 = 0. \tag{29}$$

This equation may have either pure real or complex roots, depending on the sign of the discriminant

$$D = \sigma_\xi^2\left(a_s^2 - d_s^2 + \frac{\bar{b}}{\bar{\Lambda}}(a_s - d_s)\right)^2 - 4c_s^2\left(a_s^2 - \left(d_s + \frac{\bar{b}}{\bar{\Lambda}}\right)^2\right). \tag{30}$$

In Eqs. (29), (30) the parameter $\bar{\Lambda}$ is defined from Eq. (23) and it has exactly the same form as in the case of purely temporal modulation. Unlike both previous cases, this discriminant has appeared to be positive even in the case of perfect tuning (i.e., $\sigma_\xi = 0$) at any frequency for the same set of parameters of the sandwich plate composition as in all previous calculations.

6. Comparison of the efficiency on suppression of wave propagation by stiffness modulation and by an internal damping

As shown in the previous section, the parametric stiffness modulation may suppress propagation of flexural and shear waves in a sandwich beam. However, the energy dissipation due to the material losses plays an important role in analysis of the dynamics of sandwich plates with stiffness modulation [5,7]. Actually, the material losses also transform waves propagating in a beam with no other form of damping into evanescent ones. The rate of decay of these 'almost propagating' evanescent waves is controlled by the magnitude of an internal damping coefficient and it may conveniently be quantified by the logarithmic decrement of their amplitudes. Thus, to estimate the practical relevance of suppression of wave propagation by the parametric stiffness modulation, it is necessary to compare decrements of 'almost propagating' waves (those which would propagate if there were no damping or no stiffness modulation) generated by the material losses and by the stiffness modulation.

The problem in vibrations of a sandwich beam with the internal energy dissipation and without stiffness modulation is formulated as follows:

$$\frac{1}{12}\left(2+\frac{\gamma}{\varepsilon^3}\right)(w'''' + \tilde{\chi}\dot{w}'''') - \frac{(1-v)}{12}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\gamma(\theta' + \tilde{\chi}\dot{\theta}' + w'' + \tilde{\chi}\dot{w}'') + \left(2+\frac{\delta}{\varepsilon}\right)\left(\frac{h}{c_0}\right)^2 \ddot{w}$$
$$- \frac{1}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\left(\frac{h}{c_0}\right)^2 \ddot{w}'' = 0,$$
$$-\frac{1}{2}(\theta'' + \tilde{\chi}\dot{\theta}'') + \frac{(1-v)}{2}\varepsilon\gamma(\theta + \tilde{\chi}\dot{\theta} + w' + \tilde{\chi}\dot{w}') + \frac{1}{2}\left(\frac{h}{c_0}\right)^2 \ddot{\theta} = 0. \quad (31)$$

Here $\tilde{\chi}$ is a frequency-dependent coefficient introduced in a stress–strain relation (ε_x, γ_{xy} are strains, σ_x, τ_{xy} are stresses)

$$\sigma_x = E\left(\varepsilon_x + \tilde{\chi}\frac{\partial \varepsilon_x}{\partial t}\right), \quad \tau_{xz} = G\left(\gamma_{xz} + \tilde{\chi}\frac{\partial \gamma_{xz}}{\partial t}\right).$$

The Young's module is defined as $\tilde{E} = E(1 - i\tilde{\chi}\omega) = E(1 - i\chi)$ for the stationary vibrations. For simplicity, the energy dissipation coefficient χ is taken as frequency-independent.

In this case, the coefficients of the dispersion polynomial (3) of a sandwich beam become complex-valued and all its purely real roots also become complex-valued. As is easy to show, the roots with positive real parts (for a beam with no damping) acquire positive imaginary parts and vice versa. It simply means that all waves decay in direction of their propagation. The rate of decay is defined by a ratio of the magnitude of the imaginary part of a given root to the magnitude of its real part

$$\Delta_\chi = 2\pi \frac{k_i}{k_r}. \tag{32}$$

In Sections 3–5, the aim was just to see whether a given wave at a given frequency remains propagating or becomes evanescent for each of three types of stiffness modulation. Therefore the roots of Eqs. (18), (24) and (29) have not been computed. Now for a given pair (k, Ω) the parameter $\bar{\Lambda}$ should be found and the quadratic in K Eq. (18) is solved for the case of the coupled spatial–temporal stiffness modulation and Eq. (24) for the case of the purely temporal stiffness modulation. If the wave propagation is suppressed, then the wave number becomes complex-valued and the decrement is

$$\Delta_m \approx 2\pi \frac{K_i}{k_r}. \tag{33}$$

This decrement is computed for a beam without damping and is proportional to the amplitude of stiffness modulation. Apparently, the vibration suppression by the stiffness modulation makes sense only if the following inequality is held

$$\Delta_m > \Delta_\chi. \tag{34}$$

Otherwise, the wave propagation is more effectively suppressed by the 'natural' material losses, than by the 'artificial' stiffness modulation.

To estimate the balance between these two mechanisms of suppression of wave propagation, decrements (32,33) are calculated for a sandwich plate with the following set of parameters: $\gamma_0 = 0.05$, $\delta = 0.1$, $\varepsilon = 0.05$, $v = 0.3$. In Fig. 6a, a dependence of the decrements Δ_χ (curves 1–3) and Δ_m (curve 4) on the frequency parameter $\Omega \equiv (\omega h/c_0)$ is shown for the flexural propagating wave. As can be seen from this graph, the suggested mechanism of suppression of wave propagation is competitive with the 'natural' mechanism of decay of these waves due to material losses. It is important to notice that stiffness modulation is particularly efficient at low frequencies, i.e., exactly when material losses become insignificant due to reduction in velocity of deformation. It is well known that suppression of low frequency wave propagation presents

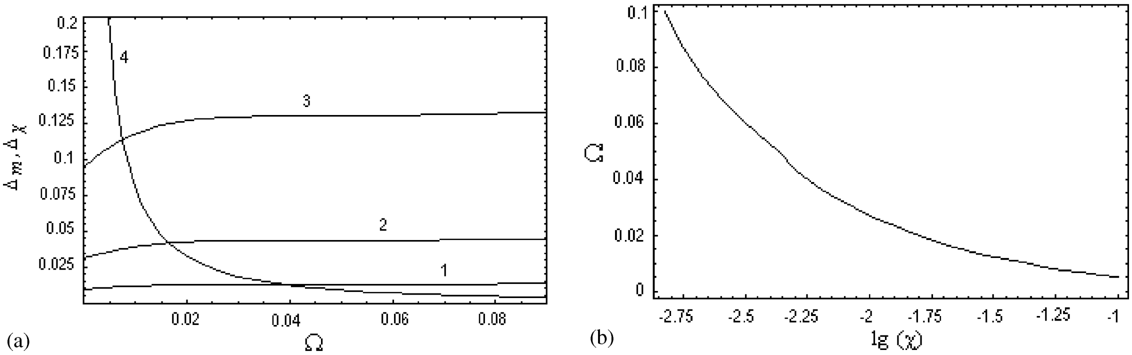

Fig. 6. (a) A dependence of the decrements Δ_χ and Δ_m on a frequency parameter Ω, $\chi = 6 \times 10^{-3}$, $\gamma_1^I = 0$ (curve 1), $\chi = 2 \times 10^{-2}$, $\gamma_1^I = 0$ (curve 2), $\chi = 6 \times 10^{-2}$, $\gamma_1^I = 0$ (curve 3), $\chi = 0$, $\sigma_\xi = \bar{\sigma}_T = 0.0001$, $\gamma_1^I = 0.1\gamma_0$ (curve 4), the first propagating wave. (b) A dependence of the 'equivalent damping coefficient' on a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.0001$, $\gamma_1^I = 0.1\gamma_0$, the first propagating wave.

serious difficulties in various applications. Therefore the stiffness modulation may offer a meaningful alternative to existing methods of low-frequency vibration control. As pointed out in Section 3, perfectly tuned coupled spatial and temporal stiffness modulation can suppress wave propagation at an arbitrary low frequency. However, if modulation is performed with some detuning, then (see Fig. 4a) suppression is possible only starting from a certain frequency. In the case illustrated in Fig. 6a, this frequency is very low indeed ($\Omega \approx 0.0009$) and there is still a broad frequency band, where stiffness modulation heavily suppresses wave propagation.

In Fig. 6b, results presented in Fig. 6a are summarized with results of similar computations for the same sandwich beam at other values of χ. The curve in this figure presents a dependence of the 'threshold' frequency parameter Ω on the internal damping coefficient χ taken in a logarithmic scale. This curve is plotted for the following set of parameters of stiffness modulation $\gamma_1^I = 0.1\gamma_0$, $\sigma_\xi = \bar{\sigma}_T = 0.0001$. This threshold is defined as a frequency parameter, at which the condition $\Delta_m = \Delta_\chi$ holds true. Stiffness modulation gives larger suppression of wave propagation than internal damping when a point $(\lg(\chi), \Omega)$ in this graph lies below this curve. For example, if $\chi = 10^{-2}$ and $\Omega = 0.01$, then it is practical to use stiffness modulation, but if $\chi = 10^{-2}$ and $\Omega = 0.04$, then it is not. The curve shown in Fig. 6b may also be thought of as the curve, which gives the magnitude of an 'equivalent damping coefficient' generated by the given parameters of 'vibrational rheology'.

Similar graphs are presented in Fig. 7a, b for the dominantly shear wave above its 'cut-on frequency'. Although the functions $\Delta_\chi(\Omega)$ (curves 1–3) and $\Delta_m(\Omega)$ (curve 4) have different shapes for a shear wave, than they have for a flexural wave, qualitatively the same conclusion is derived—the suggested mechanism of suppression of wave propagation is efficient at relatively low frequencies. For a shear wave in the case of precise tuning, its applicability range is bounded by the 'cut-on frequency' from below. These results are summarised in Fig. 7b in the same way as in the previous case. This curve is plotted for the following set of parameters of stiffness modulation $\gamma_1^I = 0.1\gamma_0$, $\sigma_\xi = \bar{\sigma}_T = 0.0001$.

In the case of purely temporal stiffness modulation, the efficiency of suppression of wave propagation is much lower. In Fig. 8a, a dependence of the decrement Δ_χ (curves 1–3) and Δ_m

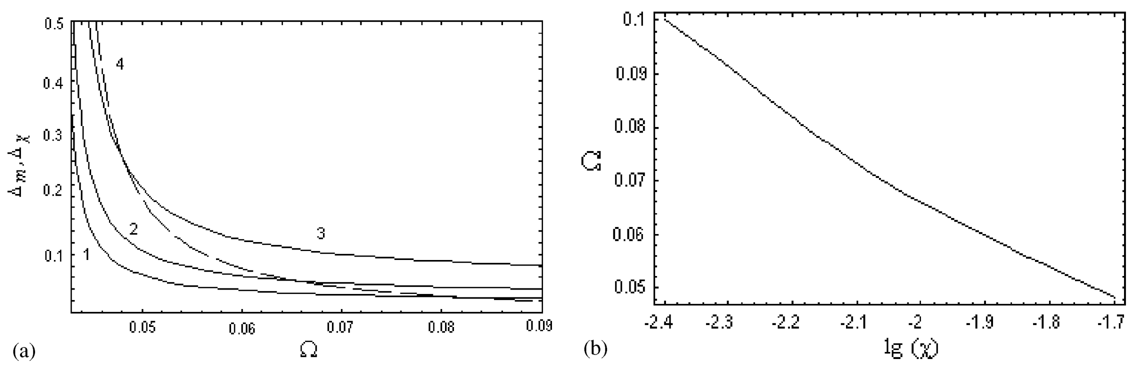

Fig. 7. (a) A dependence of the decrements Δ_χ and Δ_m on a frequency parameter Ω, $\chi = 6 \times 10^{-3}$, $\gamma_1^I = 0$ (curve 1), $\chi = 10^{-2}$, $\gamma_1^I = 0$ (curve 2), $\chi = 2 \times 10^{-2}$, $\gamma_1^I = 0$ (curve 3), $\chi = 0$, $\sigma_\xi = \bar{\sigma}_T = 0.0001$, $\gamma_1^I = 0.1\gamma_0$ (curve 4), the second propagating wave. (b) A dependence of the 'equivalent damping coefficient' on a frequency parameter Ω, $\sigma_\xi = \bar{\sigma}_T = 0.0001$, $\gamma_1^I = 0.1\gamma_0$, the second propagating wave.

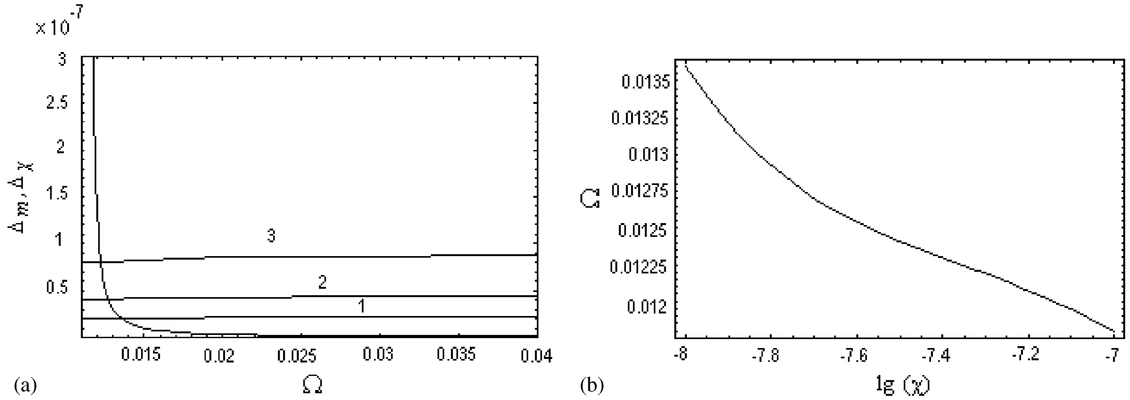

Fig. 8. (a) A dependence of the decrements Δ_χ and Δ_m on a frequency parameter Ω, $\chi = 10^{-8}$, $\gamma_1^I = 0$ (curve 1), $\chi = 2 \times 10^{-8}$, $\gamma_1^I = 0$ (curve 2), $\chi = 4 \times 10^{-8}$, $\gamma_1^I = 0$ (curve 3), $\chi = 0$, $\bar{\sigma}_T = 0$, $\gamma_1^I = 0.1\gamma_0$ (curve 4), the first propagating wave. (b) A dependence of the 'equivalent damping coefficient' on a frequency parameter Ω, $\bar{\sigma}_T = 0$, $\gamma_1^I = 0.1\gamma_0$, the first propagating wave.

(curve 4) on the frequency parameter Ω for the flexural propagating wave is presented for the same sandwich plate. Similarly to Figs. 6b and 7b, the curve in Fig. 8b presents a dependence of the 'threshold' frequency parameter on the internal damping coefficient taken in a logarithmic scale. This curve is plotted for the following set of parameters of stiffness modulation $\gamma_1^I = 0.1\gamma_0$, $\bar{\sigma}_T = 0$. When suppression of shear wave propagation by purely temporal stiffness modulation is considered, it appears that the relevant decrement is extremely small and the effect is very weak. Comparison of scales in Figs. 6b, 7b and 8b suggests that only the coupled spatial–temporal stiffness modulation may be efficient for suppression of wave propagation in sandwich plates. This result supports the discussion given in Refs. [3,4] concerning the necessity to introduce dynamic modulations to suppress wave propagation.

Suppression of wave propagation due to internal damping is associated with irreversible transformation of the wave energy into the heat in the material of a beam. It is well-known that, the heat may be regarded as the energy of hidden very high frequency motions at the micro-level. The mechanism of suppression of wave propagation considered in this paper is qualitatively similar. Parametric stiffness modulation generates the interaction between wave motions at different frequencies and wave numbers, which manifests itself as the energy 'leakage' from an excited mode and therefore its suppression. Although the detailed analysis of the dynamics of a beam at the modulation frequency lies beyond the scope of the present paper, it should be noted that similar mechanism of modal interaction has been introduced in [7] to explain the active control of resonant vibrations of sandwich plates by the stiffness modulation.

7. Conclusions

The stiffness modulation is suggested as a tool to suppress the energy transportation by wave propagation in an unbounded sandwich beam. Theoretical analysis of wave motions in such a beam with asymptotically small modulation of stiffness is performed by the standard technique of the method of multiple scales. Three types of stiffness modulation are considered—coupled spatial and temporal, purely temporal and purely spatial. The following main results are obtained

- Coupled temporal and spatial as well as purely temporal stiffness modulations of fairly small amplitude are capable of transforming propagating waves into non-propagating ones. The effect of suppression exists at any excitation frequency in the case of a perfectly tuned modulation. Coupled spatial and temporal stiffness modulation is more efficient in suppressing wave propagation, than a purely temporal stiffness modulation. Purely spatial stiffness modulation does not suppress wave propagation in all considered examples.
- The efficiency of suppression of wave propagation is shown to be sensitive to the magnitudes of detuning parameters. This mechanism of suppression of wave propagation is most pronounced and 'robust' in the case of coupled spatial and temporal stiffness modulation.
- The suppression of wave propagation due to stiffness modulation is competitive with the suppression of wave propagation due to internal material losses, especially in the low-frequency excitation conditions, when coupled temporal and spatial stiffness modulation produces large 'equivalent damping'.

Acknowledgements

The authors wish to express their sincere gratitude to Professor I.I. Blekhman for the stimulating discussions. The partial financial support from the Russian Academy of Sciences (grant # 01-01-00227) is also gratefully acknowledged.

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