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Letter to the Editor

Vibration of an Euler–Bernoulli stepped beam carrying a non-symmetrical rigid body at the step

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1. Introduction

Publications are available on the transverse vibration of uniform beams carrying rigid bodies of negligible axial dimension. Pan [1] presented a theoretical study on a simply supported beam carrying thin disks *but did not present any results.* Kounadis [2] extended the work in Ref. [1] and tabulated the first three frequencies of a uniform cantilever carrying up to three thin disks in-span. Kim and Dickinson [3] used the Rayleigh–Ritz and the finite element method to study the vibration of a uniform beam carrying thin disks at the ends and two disks in-span. Register [4] considered a resiliently supported uniform beam with thin disks attached at the ends.

Bhat and Wagner [5] studied the transverse vibration of a cantilever carrying a rigid body at the tip taking account of the axial dimension of the body. Liu and Huang [6] and Low [7] tackled systems similar to that in Ref. [5]. Popplewell and Daqing Chang [8] treated the problem in Ref. [5] by the Rayleigh–Ritz method.

Publications on vibration of beams with one-step change in cross-section include Taleb and Suppiger [9] (simply supported), Balasubramanian and Subramanian [10] (cantilever by finite element method), Krishnan et al. [11] (simply supported by finite difference). Jang and Bert [12] considered several combinations of boundary conditions and expressed the frequency equations as fourth order determinant equated to zero. Naguleswaran [13] expressed the frequency equations as second order determinant equated to zero and presented vibratory details like mode shape, position of nodes, etc. Bapat and Bapat [14] used the transfer matrix method to study the transverse vibration of stepped beams carrying particles at the steps but presented results only for uniform beams.

Kopmaz and Telli [15] considered a simply supported two part beam (stepped beam) carrying a symmetrical rigid body, i.e. center of mass at the mid point of the axial width of the body. The frequency equation was expressed as a fourth order determinant equated to zero and the natural

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frequencies were presented in graphical form. The free body diagram of the rigid body in this reference was shown to be incorrect by Naguleswaran [16]. The list of references in [15] was incomplete and in the present paper several relevant references are listed.

Locations of the center of mass of the rigid body within or outside the axial dimension of the body are considered in the present paper. The system parameters are: the step location parameter R_1, the normalized mass per unit length of the two beam portions μ_1 and μ_2, the normalized flexural rigidity ϕ_1 and ϕ_2, the mass parameter δ, moment of inertia parameter Δ and the center of mass offsets ε_1 and ε_2 and combinations of classical clamped (*cl*), pinned (*pn*), sliding (*sl*) and free (*fr*) boundary conditions. Following the method of analysis in Ref. [13], the frequency equations are expressed as second order determinant equated to zero. A scheme to calculate the elements of the determinant and a scheme to evaluate the roots of the frequency equation are presented. Tables of the first three non-zero frequency parameters are presented for selected sets of the system parameters and 16 combinations of classical boundary conditions. The tables demonstrate the trend in the frequency parameter variation as one of the system parameter is varied. The results may be used to judge frequency parameters obtained by numerical methods like Rayleigh–Ritz, finite element method, etc.

2. Theory

Fig. 1a shows the stepped beam carrying a non-symmetrical rigid body at the step and the two co-ordinate systems used in the analysis. The center of mass G of the rigid body is on the neutral axis, its mass is M_B and its moment of inertia is J_B (about axis through G normal to co-ordinate planes). The flexural rigidity, mass per unit length and the length of the portion A_1B_1 are EI_1, m_1

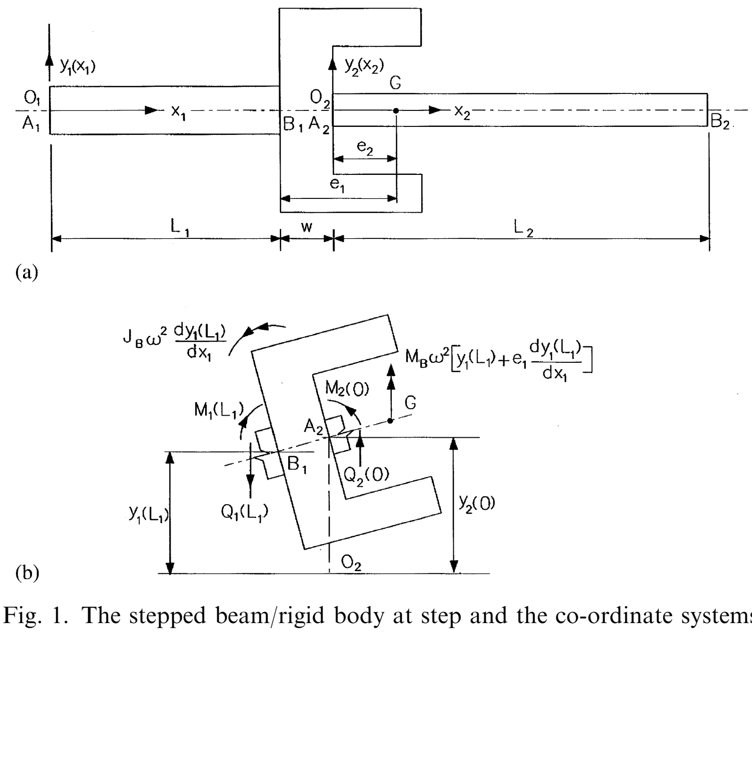


Fig. 1. The stepped beam/rigid body at step and the co-ordinate systems.

and L_1 and of the portion A_2B_2 are EI_2, m_2 and L_2. The center of mass G is offset e_1 from B_1 and e_2 from A_2. Offset e_1 is considered positive if B_1G is in the positive direction of x_1 and e_2 is positive if A_2G is in the positive direction of x_2. The center of mass offsets shown in Fig. 1a are both positive. Combinations of e_1 and e_2 of opposite signs are shown in Fig. 2a. The rigid body of negligible axial width shown in Fig. 2b is the type considered in Refs. [1–4]. The axial width of the rigid body is $w = (e_1 - e_2)$. In practical engineering systems $e_1 > e_2$. Mathematically $e_1 < e_2$ is valid but such cases were not considered. The origins O_1 and O_2 of the two co-ordinate systems used in the analysis coincide with A_1 and A_2 when the beam is at rest.

Consider the free vibration of the system at frequency ω. If the amplitude of vibration at abscissa x_k ($k = 1$ for portion A_1B_1 and $k = 2$ for portion A_2B_2) is $y_k(x_k)$, then based on Euler–Bernoulli theory of bending the bending moment $M_k(x_k)$, shearing force $Q_k(x_k)$ and the mode shape equation are

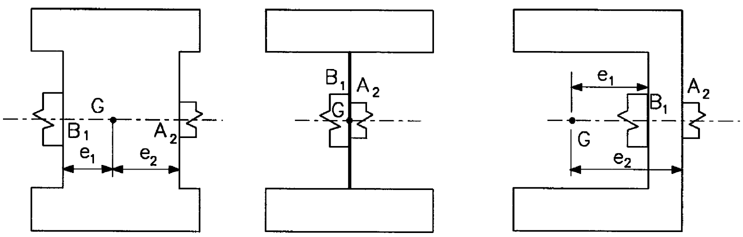
$$M_k(x_k) = EI_k \frac{d^2 y_k(x_k)}{dx_k^2}, \quad Q_k(x_k) = -EI_k \frac{d^3 y_k(x_k)}{dx_k^3},$$
$$EI_k \frac{d^4 y_k(x_k)}{dx_k^4} - m_k \omega^2 y_k(x_k) = 0. \tag{1}$$

Eqs. (1) are normalized relative to a uniform beam of flexural rigidity EI_0, mass per unit length m_0 and length L. Introduce the dimensionless abscissa X_k, co-ordinate $Y_k(X_k)$, operators D_k^n ($n = 1, 2, 3, 4$), mass per unit length ratio μ_k, flexural rigidity ratio ϕ_k, dimensionless bending moment $M_k(X_k)$, shearing force $Q_k(X_k)$ and dimensionless frequency Ω and frequency parameter α_0 defined as follows:

$$X_k = \frac{x_k}{L}, \quad Y_k(X_k) = \frac{y_k(x_k)}{L}, \quad D_k^n = \frac{d^n}{dX_k^n}, \quad \mu_k = \frac{m_k}{m_0}, \quad \phi_k = \frac{EI_k}{EI_0}, \quad M_k(X_k) = \frac{M_k(x_k)L}{EI_0},$$
$$Q_k(X_k) = \frac{Q_k(x_k)L^2}{EI_0}, \quad \Omega_0^2 = \alpha_0^4 = \frac{m_0\omega^2 L^4}{EI_0}, \quad \alpha_k^4 = \frac{m_k \omega^2 L^4}{EI_k} = \frac{\mu_k \alpha^4}{\phi_k}. \tag{2}$$

The nth frequency parameter is denoted $\alpha_{0,n}$. Eqs. (1) in dimensionless form are

$$M_k(X_k) = \phi_k D_k^2[Y_k(X_k)], \quad Q_k(X_k) = -\phi_k D_k^3[Y_k(X_k)],$$
$$\phi_k D_k^4[Y_k(X_k)] - \mu_k \alpha^4 Y_k(X_k) = 0. \tag{3}$$



(a) $e_1 > 0$, $e_2 < 0$ (b) $e_1 = 0$, $e_2 = 0$ (c) $e_1 < 0$, $e_2 < 0$

Fig. 2. Center of mass offset combinations.

The dimensionless mode shape of the portion A_1B_1 is

$$Y_1(X_1) = C_{1,1}\sin\alpha_1X_1 + C_{1,2}\cos\alpha_1X_1 + C_{1,3}\sinh\alpha_1X_1 + C_{1,4}\cosh\alpha_1X_1, \tag{4}$$

where $C_{1,1}$ through to $C_{1,4}$ are constants of integration. Two of the constants may be eliminated when the boundary conditions at A_1 are considered. The mode shape takes the form

$$Y_1(X_1) = AU_1(X_1) + BV_1(X_1), \tag{5}$$

where A and B are constants. In this paper classical clamped (*cl*), pinned (*pn*), sliding (*sl*) or free (*fr*) boundary conditions at A_1 are considered. The functions $U_1(X_1)$ and $V_1(X_1)$ are

$$\begin{aligned}
&\text{if } A_1 \text{ is } cl: && U_1(X_1) = \sin\alpha_1X_1 - \sinh\alpha_1X_1, && V_1(X_1) = \cos\alpha_1X_1 - \cosh\alpha_1X_1,\\
&\text{if } A_1 \text{ is } pn: && U_1(X_1) = \sin\alpha_1X_1, && V_1(X_1) = \sinh\alpha_1X_1,\\
&\text{if } A_1 \text{ is } sl: && U_1(X_1) = \cos\alpha_1X_1, && V_1(X_1) = \cosh\alpha_1X_1,\\
&\text{if } A_1 \text{ is } fr: && U_1(X_1) = \sin\alpha_1X_1 + \sinh\alpha_1X_1, && V_1(X_1) = \cos\alpha_1X_1 + \cosh\alpha_1X_1.
\end{aligned} \tag{6}$$

In the subsequent analysis, the following dimensionless parameters are introduced: center of mass offset parameters ε_1 and ε_2, beam portion length parameters R_1 and R_2, rigid body mass and moment of inertia parameters δ and Δ defined as follows:

$$\varepsilon_1 = \frac{e_1}{L}, \quad \varepsilon_2 = \frac{e_2}{L}, \quad R_1 = \frac{L_1}{L}, \quad R_2 = \frac{L_2}{L}, \quad \delta = \frac{M_B}{m_0L}, \quad \Delta = \frac{J_B}{m_0L^3}. \tag{7}$$

Without loss of generality one may choose

$$R_1 + R_2 = 1. \tag{8}$$

The dimensionless mode shape of the portion A_2B_2 is

$$Y_2(X_2) = C_{2,1}\sin\alpha_2X_2 + C_{2,2}\cos\alpha_2X_2 + C_{2,3}\sinh\alpha_2X_2 + C_{2,4}\cosh\alpha_2X_2. \tag{9}$$

Continuity of deflection and of slope at A_2 and compatibility of forces/moments acting on the rigid body (shown in Fig. 1b) results in the following equations in dimensionless form:

$$\begin{aligned}
&Y_2(0) = Y_1(R_1) + (\varepsilon_1 - \varepsilon_2)D_1[Y_1(R_1)], \quad D_2[Y_2(0)] = D_1[Y_1(R_1)],\\
&\phi_2D_2^2[Y_2(0)] = \phi_1\{D_1^2[Y_1(R_1)] + (\varepsilon_1 - \varepsilon_2)D_1^3[Y_1(R_1)]\} - \Delta\alpha^4D_1[Y_1(R_1)]\\
&\qquad - \delta\varepsilon_2\alpha^4\{Y_1(R_1) + \varepsilon_1D_1[Y_1(R_1)]\},\\
&\phi_2D_2^3[Y_2(0)] = \phi_1D_1^3[Y_1(R_1)] + \delta\alpha^4\{Y_1(R_1) + \varepsilon_1D_1[Y_1(R_1)]\}.
\end{aligned} \tag{10}$$

From the equations which result when Eqs. (5) and (9) are substituted into Eqs. (10), $C_{2,1}$ through to $C_{2,4}$ may be eliminated and the dimensionless mode shape of A_2B_2 expressed as

$$Y_2(X_2) = AU_2(X_2) + BV_2(X_2). \tag{11}$$

The functions $U_2(X_2)$ and $V_2(X_2)$ are

$$\begin{aligned}
&U_2(X_2) = G_1\sin\alpha_2X_2 + G_2\cos\alpha_2X_2 + G_3\sinh\alpha_2X_2 + G_4\cosh\alpha_2X_2,\\
&V_2(X_2) = H_1\sin\alpha_2X_2 + H_2\cos\alpha_2X_2 + H_3\sinh\alpha_2X_2 + H_4\cosh\alpha_2X_2
\end{aligned} \tag{12}$$

in which the coefficients G_1, G_2, G_3 and G_4 are

$$G_1 = \frac{D_1[U_1(R_1)]}{2\alpha_2} - \frac{\phi_1 D_1^3[U_1(R_1)] + \delta\alpha^4\{U_1(R_1) + \varepsilon_1 D_1[U_1(R_1)]\}}{2\phi_2\alpha_2^3},$$

$$G_2 = \frac{U_1(R_1) + (\varepsilon_1 - \varepsilon_2)D_1[U_1(R_1)]}{2} - \frac{\left\{\begin{array}{l}\phi_1\{D_1^2[U_1(R_1)] + (\varepsilon_1 - \varepsilon_2)D_1^3[U_1(R_1)]\}\\ -(\varepsilon_1\varepsilon_2\delta + \Delta)\alpha^4 D_1[U_1(R_1)] - \varepsilon_2\delta\alpha^4 U_1(R_1)\end{array}\right\}}{2\phi_2\alpha_2^2},$$

$$G_3 = \frac{D_1[U_1(R_1)]}{2\alpha_2} + \frac{\phi_1 D_1^3[U_1(R_1)] + \delta_2\alpha^4\{U_1(R_1) + \varepsilon_1 D_1[U_1(R_1)]\}}{2\phi_2\alpha_2^3},$$

$$G_4 = \frac{U_1(R_1) + (\varepsilon_1 - \varepsilon_2)D_1[U_1(R_1)]}{2} + \frac{\left\{\begin{array}{l}\phi_1\{D_1^2[U_1(R_1)] + (\varepsilon_1 - \varepsilon_2)D_1^3[U_1(R_1)]\}\\ -(\varepsilon_1\varepsilon_2\delta + \Delta)\alpha^4 D_1[U_1(R_1)] - \varepsilon_2\delta\alpha^4 U_1(R_1)\end{array}\right\}}{2\phi_2\alpha_2^2}. \quad (13)$$

The coefficients H_1, H_2, H_3 and H_4 are obtained by substituting V for U in above expressions.

3. The frequency equation

Eq. (11) must satisfy the boundary conditions at B_2 and the frequency equation results from this requirement. For classical boundary conditions at B_2, the frequency equations are

$$\begin{array}{ll}\text{if } B_2 \text{ is } cl: & U_2(R_2)D_2[V_2(R_2)] - D_2[U_2(R_2)]V_2(R_2) = 0,\\ \text{if } B_2 \text{ is } pn: & U_2(R_2)D_2^2[V_2(R_2)] - D_2^2[U_2(R_2)]V_2(R_2) = 0,\\ \text{if } B_2 \text{ is } sl: & D_2[U_2(R_2)]D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]D_2[V_2(R_2)] = 0,\\ \text{if } B_2 \text{ is } fr: & D_2^2[U_2(R_2)]D_2^3[V_2(R_2)] - D_2^3[U_2(R_2)]D_2^2[V_2(R_2)] = 0.\end{array} \quad (14)$$

The frequency parameters for the selected set of system parameters $R_1, (R_2 = 1 - R_1)$, ε_1, ε_2, δ, Δ, μ_1, ϕ_1, μ_2, ϕ_2 and the boundary conditions at A_1 and B_2 are the roots of the relevant frequency equation (14).

3.1. The system parameters

The 'reference' beam was chosen with flexural rigidity, mass per unit length and length EI_1, m_1, L and hence $\mu_1 = 1$ and $\phi_1 = 1$. The center of mass offsets were chosen so that $(\varepsilon_1 - \varepsilon_2) \geqslant 0$. For sample calculations the system parameters were chosen from the following list: $\delta = 0.5$, $\Delta = 0.1$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = -0.1$, $R_1 = 0.4$, $R_2 = 1 - R_1$, $d_1 = 1.0$, $d_2 = 0.5$. Three types of step change in cross-section were considered. In Type 1 change in cross-section, both portions are of the same depth but the breadth of A_2B_2 is d_2 while that of A_1B_1 is d_1 and hence $\mu_2 = d_2$ and $\phi_2 = d_2$. In Type 2, both portions are of the same breadth but the depth of A_2B_2 is d_2 and that of A_1B_1 is d_1 and hence $\mu_2 = d_2$ and $\phi_2 = d_2^3$. In Type 3, the breadth and depth of A_2B_2 are d_2 and those of A_1B_1 is d_1 and so $\mu_2 = d_2^2$ and $\phi_2 = d_2^4$.

3.2. Frequency parameter calculations

The functions $U_1(X_1)$ and $V_1(X_1)$ were chosen from the equation set (6) taking account of the boundary conditions at A_1. The derivatives of $U_1(X_1)$ and $V_1(X_1)$ were obtained by straight forward differentiation. For the selected set of system parameters, a trial value of $\alpha_{0,1} = 0.1$ (say) was assumed. The coefficients G_1 through to H_4 were calculated from Eq. (13) to establish $U_2(X_2)$ and $V_2(X_2)$ from Eq. (12). The frequency equation was chosen from equations set (14) taking account of the boundary conditions at B_2 and its right hand side was calculated. The procedure was repeated with increase in the trial $\alpha_{0,1}$ in steps of 0.1, till a sign change in the value of the frequency equation was observed. This indicates a 'range' in which a root lies. The procedure was repeated in this 'range' with change in $\alpha_{0,1}$ of 0.01 to narrow the 'range'. An iterative procedure based on linear interpolation was now invoked to obtain the root to a pre-set accuracy. The search was continued from here for the next root and so on.

The first three frequency parameters $\alpha_{0,1}$, $\alpha_{0,2}$, and $\alpha_{0,3}$ were calculated for 16 combinations of classical boundary conditions and presented in tabular form. The system parameters listed in Section 3.1 were not chosen by design.

Table 1 has the frequency parameters of Type 1, 2 and 3 stepped beams. The system parameters were chosen from the list in Section 3.1. The center of mass of the rigid body was within the axial width of the body, i.e., $\varepsilon_1 > 0$, $\varepsilon_2 < 0$. The frequency parameters of Type 1 beam are greater than those of Type 2 beams. Except for *cl\fr* and *sl\fr* the frequency parameters of Type 2 beams are greater than those of Type 3 beams.

Table 1
The first three non-zero frequency parameters of systems with three types of step change in cross-section

B C	Type 1 step change			Type 2 step change			Type 3 step change		
	$\mu_2 = d_2$, $\phi_2 = d_2$			$\mu_2 = d_2$, $\phi_2 = d_2^3$			$\mu_2 = d_2^2$, $\phi_2 = d_2^4$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	3.1164	4.7158	8.0902	2.3754	4.7048	5.7265	2.2024	4.7146	5.6564
cl\pn	2.5867	4.7145	6.8097	2.0851	4.6752	4.8506	2.0161	4.6647	4.7832
cl\sl	1.7594	4.4180	4.7159	1.6580	3.1516	4.7077	1.7540	3.0076	4.7170
cl\fr	1.5781	3.5935	4.7148	1.5417	2.6047	4.7029	1.6618	2.4762	4.7133
pn\cl	2.9200	3.7492	8.0842	2.0901	3.6919	5.7186	1.8041	3.6858	5.6512
pn\pn	2.2984	3.7304	6.8040	1.6432	3.6893	4.8122	1.4295	3.6848	4.7275
pn\sl	1.1039	3.6705	4.4225	0.7841	3.0892	3.6992	0.7033	2.9653	3.6883
pn\fr	3.3797	3.8451	8.0378	2.4873	3.6945	5.6858	2.3777	3.6869	5.6141
sl\cl	1.6147	3.1429	6.7404	1.3916	2.4481	5.7140	1.2627	2.3023	5.6485
sl\pn	1.2170	2.6968	6.6798	1.0302	2.2517	4.8128	0.9203	2.1893	4.7278
sl\sl	2.0835	4.4187	6.7445	1.9587	3.1556	6.5624	2.0065	3.0102	6.5254
sl\fr	1.8558	3.6194	6.7385	1.7946	2.6464	5.6809	1.8741	2.5137	5.6113
fr\cl	1.1553	3.0175	5.4203	0.8172	2.1344	5.4144	0.6885	1.8366	5.4166
fr\pn	2.4489	5.4203	6.8064	1.7327	4.8122	5.4179	1.4976	4.7275	5.4198
fr\sl	1.4365	4.4027	5.4205	1.0162	3.1140	5.4163	0.8825	2.9768	5.4186
fr\fr	3.5735	5.4205	8.0422	2.5285	5.4134	5.6910	2.4002	5.4155	5.6186

System parameters listed in Section 3.1 ($\mu_1 = 1.0$, $\phi_1 = 1.0$): μ_2 and ϕ_2 as shown in table.

Table 2
As in Table 1, Type 3 beam ($\mu_2 = d_2^2$, $\phi_2 = d_2^4$) but for three different d_2 which are shown in table

B C	$d_2 = 1.0$			$d_2 = 0.8$			$d_2 = 0.25$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
*cl**cl*	3.4961	4.7166	8.2501	2.9472	4.7091	7.2813	1.9271	3.9555	4.7268
*cl**pn*	2.8207	4.7166	7.0013	2.4521	4.7075	6.1464	1.9061	3.2926	4.7258
*cl**sl*	1.6907	4.6500	4.7473	1.6857	4.0372	4.7095	1.7973	2.0875	4.5868
*cl**fr*	1.4221	3.8024	4.7223	1.5194	3.2974	4.7075	1.5313	1.9379	3.9265
*pn**cl*	3.2344	3.8676	8.2415	2.7526	3.7316	7.2754	0.9215	3.6805	3.9560
*pn**pn*	2.5455	3.7892	6.9947	2.1570	3.7187	6.1391	0.7355	3.2909	3.6806
*pn**sl*	1.1895	3.6737	4.6984	1.0206	3.6372	4.0630	0.3742	2.0077	3.6806
*pn**fr*	3.4111	4.0337	8.2038	3.1474	3.7664	7.2364	1.5985	3.6806	3.9271
*sl**cl*	1.7288	3.5345	6.7277	1.5804	2.9815	6.7253	0.6936	2.1172	3.9563
*sl**pn*	1.2486	2.9590	6.7026	1.1912	2.5782	6.1291	0.4911	2.1063	3.2927
*sl**sl*	2.1158	4.6909	6.7315	2.0279	4.0392	6.7377	1.9247	2.1710	4.5935
*sl**fr*	1.7586	3.8629	6.7256	1.8117	3.3286	6.7214	1.5584	2.1208	3.9274
*fr**cl*	1.3682	3.4448	5.4224	1.0980	2.8368	5.4182	0.3447	0.9358	3.9560
*fr**pn*	2.7810	5.4221	6.9974	2.2982	5.4181	6.1413	0.7665	3.2911	5.4207
*fr**sl*	1.6401	4.6659	5.4245	1.3482	4.0167	5.4182	0.4569	2.0091	4.5935
*fr**fr*	3.8068	5.4237	8.2104	3.2679	5.4181	7.2400	1.6024	3.9271	5.4210

Table 3
As in Table 2 but for three different R_1 which are shown in table

B C	$R_1 = 0.2$			$R_1 = 0.5$			$R_1 = 0.8$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
*cl**cl*	2.4483	4.2642	6.9608	2.2593	4.0544	6.7633	3.0017	3.7388	6.3670
*cl**pn*	2.3528	3.5971	6.2845	1.9767	4.0540	5.6495	2.7308	3.0781	6.3612
*cl**sl*	1.8924	2.6135	4.9236	1.6512	3.5307	4.0545	1.4307	3.0133	6.3579
*cl**fr*	1.5754	2.4555	4.2377	1.5944	2.8415	4.0540	1.3578	2.9964	6.3544
*pn**cl*	1.5643	4.2618	5.7352	1.9906	3.2474	6.7606	2.4360	3.7360	5.4630
*pn**pn*	1.2897	3.5798	5.7267	1.5473	3.2430	5.6471	2.2587	2.9344	5.4542
*pn**sl*	0.7117	2.3080	4.9233	0.7044	3.2068	3.5424	0.7717	2.5281	5.4528
*pn**fr*	1.8850	4.2351	5.7353	2.7715	3.2641	6.7149	2.4948	5.4528	6.7437
*sl**cl*	1.1476	2.6229	4.2697	1.3099	2.3205	5.5654	1.3203	3.7360	3.8057
*sl**pn*	0.8104	2.5695	3.5986	0.9905	2.1210	5.5422	1.1605	2.8423	3.7982
*sl**sl*	2.0730	2.7101	4.9322	1.9021	3.5315	5.5668	1.6867	3.7997	7.2636
*sl**fr*	1.6711	2.6288	4.2438	1.8285	2.8548	5.5647	1.5938	3.7950	6.7372
*fr**cl*	0.6696	1.5734	4.2651	0.7043	2.0396	4.5108	0.8219	3.1418	3.7369
*fr**pn*	1.3284	3.5814	6.2998	1.6287	4.5107	5.6480	2.6940	3.2344	6.3406
*fr**sl*	0.8511	2.3087	4.9307	0.9020	3.5193	4.5109	1.0364	3.1807	6.3370
*fr**fr*	1.8954	4.2388	6.9882	2.8179	4.5109	6.7170	3.1689	6.3335	6.7496

The beams in Table 2 are of circular cross-section but with step change in diameter, i.e., Type 3. The diameter of A_2B_2 considered are $d_2 = 1.0$ (uniform beam) or 0.8 or 0.25 and the rest of the system parameters are chosen from the list in Section 3.1. The frequency parameters for $d_2 = 0.5$

is found in Table 1. The frequency parameters decrease with decrease in d_2. This is to be expected because there is an overall decrease in the system stiffness.

The beam in Table 3 is Type 3 but the rigid body location parameters are $R_1 = 0.2$ or 0.5 or 0.8. The frequency parameters for location parameter $R_1 = 0.4$ are tabulated in Table 1.

Table 4 shows the variation in frequency parameters of Type 3 beam systems with $\varepsilon_2 = 0.2$ or 0.0 or -0.2. Note that the axial width of the rigid body with $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.2$ is zero but the center of mass is offset from B_1.

Table 5 illustrates the effect of change in δ on Type 3 beams and Table 6 the effect of change in Δ and shows the expected trend of a decrease in the frequency parameters with an increase in δ or Δ. The system is more sensitive to change in Δ.

In Table 7 rigid bodies of constant width ($\varepsilon_1 - \varepsilon_2 = 0.3$) but with $\varepsilon_1 = 0.5$ or 0.7 or 1.0 are considered. The frequency parameters decrease with increase in ε_1. In Table 8 rigid bodies of width ($\varepsilon_1 - \varepsilon_2 = 0.6$ or 0.9 or 1.1) but with $\varepsilon_2 = -0.1$ are considered.

3.3. 'Conjugate' systems

To reflect the dependence of the frequency parameter on the various system parameters, let it be represented by $\alpha[(i,j), \delta, \Delta, (\varepsilon_1, \varepsilon_2), (R_1, R_2), (d_1, d_2)]$ in which i or $j = 1, 2, 3$ or 4 represent classical *cl*, *pn*, *sl* or *fr* boundary conditions. Clearly

$$\begin{aligned}&\alpha[(i,j), \delta, \Delta, (\varepsilon_1 = a, \varepsilon_2 = b), (R_1 = R, R_2 = 1 - R), (d_1 = 1.0, d_2 = d)]\\&\quad = \alpha[(j,i), \delta, \Delta, (\varepsilon_1 = -b, \varepsilon_2 = -a), (R_1 = 1 - R, R_2 = R), (d_1 = d, d_2 = 1)].\end{aligned} \tag{15}$$

Table 4
As in Table 2 but for three different ε_2 which are shown in table

B C	$\varepsilon_2 = 0.2$			$\varepsilon_2 = 0.0$			$\varepsilon_2 = -0.2$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	2.2467	4.7257	5.6734	2.1571	4.6966	5.6473	2.0677	4.6415	5.6531
cl\pn	2.0210	4.7197	4.7627	2.0098	4.6026	4.8108	1.9922	4.4777	4.8697
cl\sl	1.7173	3.0498	4.7258	1.7899	2.9648	4.7043	1.8560	2.8816	4.6685
cl\fr	1.6273	2.5058	4.7255	1.6967	2.4445	4.6933	1.7660	2.3761	4.6325
pn\cl	1.8746	3.6890	5.6717	1.7265	3.6781	5.6364	1.5477	3.6495	5.6263
pn\pn	1.4696	3.6832	4.7550	1.3852	3.6778	4.7082	1.2827	3.6365	4.6977
pn\sl	0.6920	2.9882	3.6986	0.7143	2.9390	3.6781	0.7344	2.8778	3.6591
pn\fr	2.3976	3.6906	5.6360	2.3571	3.6778	5.5985	2.3146	3.6438	5.5882
sl\cl	1.2297	2.3461	5.6716	1.2945	2.2588	5.6288	1.3489	2.1798	5.6014
sl\pn	0.9098	2.1918	4.7557	0.9301	2.1835	4.7044	0.9474	2.1617	4.6741
sl\sl	1.9628	3.0580	6.5553	2.0466	2.9648	6.4961	2.1072	2.8913	6.4425
sl\fr	1.8327	2.5501	5.6359	1.9154	2.4743	5.5904	1.9952	2.3892	5.5617
fr\cl	0.6515	1.9267	5.4219	0.7302	1.7416	5.4010	0.8261	1.5486	5.3470
fr\pn	1.5511	4.7527	5.4222	1.4383	4.7041	5.4180	1.3043	4.6659	5.4160
fr\sl	0.8658	3.0158	5.4221	0.8974	2.9410	5.4119	0.9187	2.8855	5.3892
fr\fr	2.4315	5.4219	5.6360	2.3697	5.3965	5.6166	2.3154	5.3348	5.6465

Table 5
As in Table 2 but for three different δ which are shown in table

B C	$\delta = 0.1$			$\delta = 1.0$			$\delta = 2.0$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	2.4283	5.1797	5.8491	2.0188	4.4982	5.6270	1.7960	4.3437	5.6127
cl\pn	2.2142	4.6860	5.4150	1.8537	4.4947	4.7010	1.6542	4.3231	4.6955
cl\sl	1.8731	3.1092	5.2745	1.6424	2.9392	4.4990	1.4905	2.8752	4.3503
cl\fr	1.7398	2.6079	5.1613	1.5778	2.3909	4.4985	1.4503	2.3146	4.3441
pn\cl	2.0767	3.8894	5.7292	1.6175	3.6115	5.6270	1.4141	3.5617	5.6117
pn\pn	1.6347	3.8774	4.8280	1.2868	3.6044	4.6995	1.1288	3.5472	4.6826
pn\sl	0.7802	3.0876	3.8983	0.6437	2.8925	3.6237	0.5726	2.8311	3.5849
pn\fr	2.4898	3.8854	5.6977	2.3167	3.6138	5.5882	2.2658	3.5646	5.5718
sl\cl	1.3994	2.4325	5.6871	1.1538	2.2451	5.6270	1.0237	2.2033	5.6090
sl\pn	1.0383	2.2575	4.7921	0.8341	2.1582	4.6984	0.7358	2.1348	4.6761
sl\sl	2.0155	3.1236	6.5264	2.0016	2.9593	6.5221	1.9975	2.9209	6.4773
sl\fr	1.8744	2.6086	5.6532	1.8739	2.4699	5.5882	1.8737	2.4365	5.5690
fr\cl	0.7183	2.0768	5.5327	0.6580	1.7068	5.3346	0.6122	1.5938	5.2741
fr\pn	1.6438	4.7871	5.6955	1.4212	4.6949	5.3373	1.3569	4.6686	5.2876
fr\sl	0.8927	3.0916	5.6482	0.8774	2.9265	5.3363	0.8733	2.8892	5.2816
fr\fr	2.4905	5.5117	5.8535	2.3603	5.3345	5.5891	2.3305	5.2729	5.5786

Table 6
As in Table 2 but for three different Δ which are shown in table

B C	$\Delta = 0.2$			$\Delta = 1.0$			$\Delta = 2.0$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	1.8466	4.4273	5.0396	1.3324	4.1381	5.0396	1.1327	4.0972	5.0396
cl\pn	1.7632	4.1439	4.4830	1.2773	3.9854	4.3395	1.0868	3.9527	4.3291
cl\sl	1.6492	2.5873	4.4411	1.2236	2.5090	4.1636	1.0449	2.4965	4.1249
cl\fr	1.5921	2.1113	4.4233	1.2124	2.0016	4.1317	1.0381	1.9876	4.0905
pn\cl	1.3638	3.3602	5.0238	1.0426	2.9756	5.0188	0.8965	2.9144	5.0182
pn\pn	1.0860	3.3575	4.1890	0.8352	2.9561	4.1847	0.7195	2.8898	4.1842
pn\sl	0.5459	2.5783	3.3618	0.4273	2.4916	3.0080	0.3699	2.4631	2.9604
pn\fr	2.0595	3.3591	4.9888	2.0016	2.9710	4.9836	1.9855	2.9093	4.9830
sl\cl	1.0649	1.9008	5.0187	1.0390	1.3394	5.0065	0.9828	1.1951	5.0047
sl\pn	0.7642	1.8559	4.1837	0.7620	1.2817	4.1685	0.7586	1.0869	4.1664
sl\sl	1.7868	2.5903	5.8190	1.2489	2.5458	5.8042	1.0560	2.5407	5.8019
sl\fr	1.7214	2.1160	4.9832	1.2394	2.0208	4.9704	1.0500	2.0134	4.9686
fr\cl	0.5257	1.3638	5.0102	0.4217	1.1833	4.9585	0.3672	1.1489	4.9470
fr\pn	1.0945	4.1835	5.2602	0.8895	4.1647	5.1124	0.8424	4.1617	5.0926
fr\sl	0.6355	2.5793	5.2531	0.4478	2.5458	5.0961	0.3793	2.5406	5.0747
fr\fr	2.0599	4.9756	5.2702	2.0195	4.9286	5.1551	2.0132	4.9183	5.1431

Eq. (15) and the tables may be used to obtain the frequency parameters of 'conjugate' systems. In the present paper, Eq. (15) was used as a check on the calculations.

Table 7
As in Table 2 but for bodies of constant axial width but different combinations of ε_1 and ε_2 as shown in table

B C	$\varepsilon_1 = 0.5, \varepsilon_2 = 0.2$			$\varepsilon_1 = 0.8, \varepsilon_2 = 0.5$			$\varepsilon_1 = 1.0, \varepsilon_2 = 0.7$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	1.7545	4.9718	5.2510	1.5368	4.9609	5.5463	1.4245	4.9516	5.6671
cl\pn	1.6763	4.1653	5.2157	1.4708	4.1468	5.5099	1.3642	4.1385	5.6243
cl\sl	1.5779	2.5687	5.1944	1.3996	2.5343	5.4345	1.3032	2.5220	5.4978
cl\fr	1.5360	2.0792	4.9374	1.3792	2.0288	4.9237	1.2887	2.0127	4.9137
pn\cl	1.2784	3.9332	5.0176	1.1410	4.0270	5.0247	1.0680	4.0439	5.0339
pn\pn	1.0191	3.9230	4.1880	0.9112	3.9644	4.2412	0.8536	3.9494	4.2786
pn\sl	0.5152	2.5599	3.9351	0.4640	2.5338	4.0376	0.4360	2.5219	4.0616
pn\fr	2.0414	3.9320	4.9823	2.0185	4.0231	4.9907	2.0087	4.0381	5.0010
sl\cl	1.0871	1.9483	5.0114	1.1018	1.7257	5.0047	1.1079	1.6034	5.0033
sl\pn	0.7704	1.9283	4.1735	0.7753	1.7216	4.1654	0.7780	1.6032	4.1643
sl\sl	1.8841	2.5687	5.8145	1.7091	2.5411	5.8070	1.5996	2.5364	5.8044
sl\fr	1.8072	2.1092	4.9754	1.6861	2.0296	4.9684	1.5877	2.0138	4.9670
fr\cl	0.5788	1.2795	5.0085	0.5761	1.1578	4.9960	0.5567	1.1221	4.9905
fr\pn	1.0425	4.1734	5.6608	0.9128	4.1633	5.8036	0.8551	4.1604	5.8558
fr\sl	0.6610	2.5600	5.6350	0.5996	2.5407	5.6918	0.5625	2.5363	5.6909
fr\fr	2.0433	4.9725	5.6663	2.0189	4.9595	5.8172	2.0116	4.9539	5.8753

Table 8
As in Table 2 but for three different axial width ($\varepsilon_1 - \varepsilon_2$) of body

B C	$\varepsilon_1 = 0.5, \varepsilon_2 = -0.1$			$\varepsilon_1 = 0.8, \varepsilon_2 = -0.1$			$\varepsilon_1 = 1.0, \varepsilon_2 = -0.1$		
	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$	$\alpha_{0,1}$	$\alpha_{0,2}$	$\alpha_{0,3}$
cl\cl	3.1164	4.7158	8.0902	1.9243	5.5326	5.6678	1.8725	5.6208	5.7066
cl\pn	2.5867	4.7145	6.8097	1.6261	4.7322	5.5439	1.5506	4.7301	5.6720
cl\sl	1.7594	4.4180	4.7159	1.3130	2.9751	5.5417	1.2188	2.9669	5.6689
cl\fr	1.5781	3.5935	4.7148	1.2780	2.3831	5.5296	1.1903	2.3688	5.5966
pn\cl	2.9200	3.7492	8.0842	1.7523	4.0646	5.6598	1.7368	4.0978	5.6594
pn\pn	2.2984	3.7304	6.8040	1.3337	4.0561	4.7409	1.3113	4.0890	4.7403
pn\sl	1.1039	3.6705	4.4225	0.5449	2.9524	4.0706	0.5102	2.9488	4.1035
pn\fr	3.3797	3.8451	8.0378	2.3513	4.0650	5.6235	2.3452	4.0980	5.6232
sl\cl	1.6147	3.1429	6.7404	1.1559	2.0253	5.6585	1.1183	1.9564	5.6576
sl\pn	1.2170	2.6968	6.6798	0.8886	1.8252	4.7351	0.8766	1.7293	4.7334
sl\sl	2.0835	4.4187	6.7445	1.6302	2.9847	6.5523	1.5281	2.9753	6.5517
sl\fr	1.8558	3.6194	6.7385	1.5742	2.4095	5.6223	1.4835	2.3894	5.6214
fr\cl	1.1553	3.0175	5.4203	0.5694	1.7922	5.6502	0.5394	1.7723	5.6499
fr\pn	2.4489	5.4203	6.8064	1.3974	4.7320	5.8260	1.3671	4.7304	5.8872
fr\sl	1.4365	4.4027	5.4205	0.7142	2.9693	5.8237	0.6703	2.9637	5.8841
fr\fr	3.5735	5.4205	8.0422	2.3723	5.6151	5.8312	2.3632	5.6145	5.8923

The three different offsets ε_1 (with $\varepsilon_2 = -0.1$) are shown in table

3.4. Mode shares

The mode shape corresponding to a natural frequency will consist of three portions (portion B_1A_2 is a straight line). One may choose $A = 1$ and normalize the mode shape with the choice $Y_1(R_1) = 1$ (say). To obtain mode shapes, interactive programs developed by Ilanko [17] are available.

4. Concluding remarks

The transverse vibration of stepped Euler–Bernoulli beam carrying a non-symmetrical rigid body in-span was considered in this paper. It was assumed that the center of mass of the body was on the neutral axis of the beam and within or outside the axial length of the body. The system parameters are: the step location parameter R_1, the normalized mass per unit length of the two beam portions μ_1 and μ_2, the normalized flexural rigidity ϕ_1 and ϕ_2, the mass parameter δ, moment of inertia parameter Δ and the center of mass offsets ε_1 and ε_2 and combinations of classical clamped (*cl*), pinned (*pn*), sliding (*sl*) and free (*fr*) boundary conditions. Mathematically no restriction need be placed on the center of mass offset parameters but in usual engineering applications, $(\varepsilon_1 - \varepsilon_2) \geqslant 0$. The first three frequency parameters are tabulated for 16 combinations of boundary conditions and several sets of system parameters. In each table, one of the system parameter was varied and the trends in frequency parameter changes are commented. The results may be used to judge frequencies of the system obtained by numerical methods.

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