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Letter to the Editor

Large amplitude free vibrations of a uniform spring-hinged beam

Chitaranjan Pany, G. Venkateswara Rao*

Structural Engineering Group, Vikram Sarabhai Space Centre, Trivandrum 695022, India

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1. Introduction

Large amplitude static analysis of cantilever beams with a tip concentrated load, attempted by several authors, are given in Refs. [1–3]. All these studies dealt with assuming infinitely large support rotational stiffness, which in reality is not possible to achieve. For a realistic analysis the rotational stiffness of the support has to be taken care of for both static and dynamic problems.

The linear theory of vibrations predicts the frequencies of natural vibration to be independent of the amplitude of vibration. In many instances, if the amplitude of the vibration is large, then the above statement is not justified due to the non-linear effects involved. Wagner [4] has obtained approximate solutions for the free non-linear oscillations of an initially straight, uniform elastic bar with clamped–free and free–free end conditions. The problem of large amplitude vibrations has been presented for clamped–free and free–free uniform beams in Ref. [5]. Recently, a simple relationship has been presented in Ref. [6] to determine the first mode linear natural frequency of linearly tapered cantilever beam as a function of beam stiffness (small deformation theory), the beam mass, and a mass distribution parameter. A preliminary investigation is successfully presented in Ref. [7], to obtain the large amplitude free vibration characteristics for the first mode of a uniform cantilever beam by replacing the linear stiffness of Ref. [6] by non-linear stiffness.

The purpose of the present paper is to provide the large amplitude fundamental frequency for a spring-hinged uniform beam with a simple, already published [7] modification to the methodology of Ref. [6]. The authors believe that the present study is the first of its kind in literature for obtaining the large amplitude free vibration analyses (first mode) of spring-hinged uniform beams. For this analysis, a polynomial function is derived taking the necessary boundary conditions at the ends of spring-hinged beam. The non-linear static analysis results i.e., the load parameter and the amplitude are compared with those of Ref. [8] for the spring-hinged beam with tip concentrated load, where an elliptical integral approach is used. Based on the results of the load

*Corresponding author. Tel.: +91-471-2564181; fax: +91-471-2704134.
E-mail address: gv_rao@vssc.org (G.V. Rao).

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deflection data, the fundamental frequency is calculated using the simple relation of Ref. [6] modified for large amplitudes, for a range of spring stiffnesses and tip slopes. As a special case, the ratio of the non-linear (large amplitude) period to the linear period (T_{NL}/T_L) versus amplitude (a/L) for a cantilever beam (spring-hinged considering infinite support rotational stiffness) has been found out using the present polynomial function. The present results when compared with those of Wagner [4] and Rao and Rao [5], indicate the efficacy of the proposed simple method.

2. Large deflection analysis of a uniform spring-hinged cantilever beam under tip concentrated load

The governing differential equation of a spring-hinged beam with a tip concentrated load (Fig. 1), in the non-dimensional form is

$$\frac{d^2\theta}{d\xi^2} + \lambda \cos\theta = 0 \tag{1}$$

with the boundary conditions

$$\frac{d\theta}{d\xi} = 0 \quad \text{at } \xi = 0, \tag{2a}$$

$$\theta = \alpha \quad \text{at } \xi = 0, \tag{2b}$$

$$\frac{d\theta}{d\xi} + \gamma\theta = 0 \quad \text{at } \xi = 1, \tag{2c}$$

where the loading parameter $\lambda = PL^2/EI$; P is the vertical tip concentrated load; γ $(= KL/EI)$ is the rotational spring stiffness parameter; ξ $(= s/L)$ is the non-dimensional co-ordinate; α is the tip slope; K is the rotational spring stiffness; θ is the slope of the elastica at any ξ; L is the length of the beam; E is Young's modulus; and I is the area moment of inertia.

It may be noted here that $\gamma = 0$ represents a hinged-free beam and $\gamma \to \infty$ represents a cantilever beam. The present formulation with $\gamma = 0$ is not applicable as this beam contains a rotational

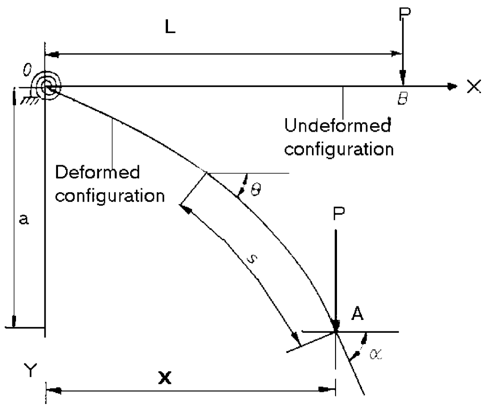

Fig. 1. A uniform spring-hinged cantilever beam undergoing large amplitudes.

rigid-body mode and the first elastic mode is not similar to the first mode obtained for any non-zero value of γ.

Applying the first two boundary conditions represented by Eqs. (2a) and (2b) into Eq. (1), the following expressions θ and $d\theta/d\xi$ can be generated in a three terms polynomial (truncated) form of the variable ξ $(0 \leqslant \xi \geqslant 1)$, using the symbolic algebraic computational software MAPLE [9], as

$$\frac{d\theta}{d\xi} = -\xi\lambda\cos\alpha - \tfrac{1}{6}\xi^3\lambda^2\sin\alpha\cos\alpha + \xi^5\left(\tfrac{-1}{120}\lambda^3\sin^2\alpha\cos\alpha + \tfrac{1}{40}\lambda^3\cos^3\alpha\right), \tag{3}$$

$$\theta = \alpha - \xi^2\frac{\lambda}{2}\cos\alpha - \frac{\lambda^2}{24}\xi^4\sin\alpha\cos\alpha. \tag{4}$$

Even though the three terms presented in the series for θ, $d\theta/d\xi$ are exact, the final expression for the same are approximate to the extent of the neglected high-power terms of ξ, whose coefficients are found to be very small.

Applying the third boundary condition represented by Eq. (2c), on Eqs. (3) and (4), we get

$$\begin{aligned}\phi(\alpha,\lambda,\gamma) = & -\lambda\cos\alpha - \frac{\lambda^2}{6}\sin\alpha\cos\alpha - \frac{\lambda^3}{120}\sin^2\alpha\cos\alpha + \frac{\lambda^3}{40}\cos^3\alpha \\ & + \gamma\left(\alpha - \frac{\lambda\cos\alpha}{2} - \frac{\lambda^2\sin\alpha\cos\alpha}{24}\right) = 0.\end{aligned} \tag{5}$$

When the rotational spring stiffness parameter is very high ($\gamma \to \infty$), Eq. (5) becomes

$$\phi(\alpha,\lambda,\gamma) = 1 - \frac{2\alpha}{\lambda\cos\alpha} + \frac{\lambda\sin\alpha}{12} = 0. \tag{6}$$

Eq. (5) is a cubic equation in λ, and it gives three roots in λ. Similarly Eq. (6) is a quadratic equation and has two roots. The lowest real root of λ is real. The other roots are, in general, complex or not physically feasible. Hence, the lowest root, which is a usable physical solution, is used in the subsequent computations.

The amplitude of the beam can be generated taking real value of λ from Eqs. (5) and (6), as the case may be, from the relation

$$a/L = \int_0^1 \sin(\theta)\, d\xi, \tag{7}$$

where, a is the tip deflection. Here, the first two terms of θ are used in Eq. (7) to find out the amplitude.

3. Fundamental frequency of a spring-hinged uniform cantilever beam

3.1. Linear frequency equation

The fundamental frequency equation of a cantilever beam can be represented as [6]

$$f_L = C\sqrt{\frac{S}{M}}, \tag{8}$$

where f_L is the linear fundamental frequency in Hz, S is the linear stiffness of the beam (P/a) in (N/m), M is the mass of the beam in kg; C is the mass distribution parameter, and for a uniform cantilever beam, $C = 0.323316$, is a standard value found in the text books and also quoted in Ref. [6].

3.2. Large amplitude frequency relation

If the cantilever beam is undergoing large amplitude vibrations, the non-linear fundamental frequency can be approximately calculated by using the non-linear stiffness [7] in the place of the linear stiffness in Eq. (8), as

$$f_{NL} = C\sqrt{\frac{S_{NL}}{M}}, \tag{9}$$

where S_{NL} is the non-linear stiffness, the calculation of which is explained in the next section.

4. Results and discussions

4.1. Determination of load parameter (λ) and amplitude using present polynomial approximation

The approximate non-linear solution for the large deflection of a cantilever beam and spring-hinged beam of length L and with a vertical tip load P are presented explicitly in Eqs. (5)–(7). Eq. (5) is used for the determination of the load parameter (λ) for a spring-hinged cantilever beam for a given value of tip slope (α) and rotational spring stiffness parameter (γ). The lowest real root of λ, is used for the analysis. Eq. (6) has been used to get the load parameter (λ) for a cantilever beam. And Eq. (7) is used for the determination of amplitude (a/L). This equation depends on the tip slope (α) and the load parameter (λ). The load parameter obtained from Eqs. (5) and (6) has been utilized to get corresponding amplitude of a spring-hinged and cantilever beams. Here, the MAPLE [9] software is taken as a computational tool for the determination of the load parameter and the amplitude.

To validate the present polynomial approach, initially Eqs. (6) and (7) are employed to derive the load parameter and the amplitude of a cantilever beam for a range of tip slopes (α). The present results are compared reasonably good accuracy with those of Bisshopp [1] and Rao and Rao [5]. This validates the proposed polynomial approximation. Now, Eqs. (5) and (7) are used to obtain the load parameter and the corresponding amplitude of a spring-hinged cantilever beam with a finite spring stiffness parameter and tip slope values. The results (Table 1) are compared with Rao et al. [8]. The comparison shows that the difference of present results with that of Ref. [8] is very small (within 4%) in the lower range of tip slopes ($\alpha = 10$–60°). And the difference is 5.66% and 7.61% at the tip slopes of 70° and 80°, respectively. The difference of present results with those of Ref. [8] is due to the different methods of analysis of the problem considered.

4.2. Determination of non-linear fundamental frequencies

A uniform spring-hinged cantilever beam (Fig. 1), made of steel having a cross-section of 2.54 cm wide by 0.099 cm deep with a length (L) of 0.693166 m has been considered [10]. The

Table 1
Comparison of load parameter (λ) and amplitude (a/L) for a spring-hinged cantilever beam for a range of spring constant parameter and tip slope

α (deg)	γ	λ			a/L		
		Present	Ref. [8]	% Diff.	Present	Ref. [8]	% Diff.
20	2.0169	0.3677	0.3680	0.0815	0.2869	0.2866	0.1045
30	4.0972	0.7813	0.7834	0.2687	0.3974	0.3952	0.5535
40	6.2982	1.2801	1.2878	0.6015	0.5044	0.4975	1.3679
50	8.7099	1.9325	1.9504	0.9262	0.6075	0.5925	2.4691
60	11.498	2.8800	2.9081	0.9756	0.7065	0.6788	3.9207
70	15.038	4.4924	4.4960	0.0801	0.8017	0.7563	5.6629
80	20.536	8.2336	7.9185	3.827	0.8957	0.8275	7.6141

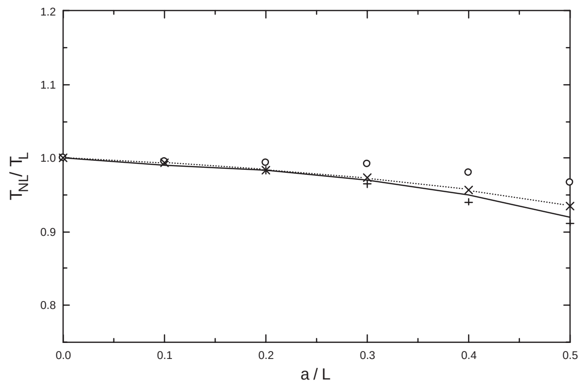

Fig. 2. The ratio of the non-linear period to the linear period (T_{NL}/T_L) versus amplitude (a/L) for a uniform cantilever beam + + +, values are of Ref. [7] using Bisshopp [1] large amplitude data; —, present polynomial function; OOO, Rao and Rao [5]; —×—, Wagner [4].

flexural stiffness (EI) of the beam and mass per unit length ($\bar{m}$) are 0.4107599 N m^2 and 0.196138 kg/m, respectively.

Using the present polynomial approximations the large amplitude fundamental frequency of a cantilever beam has been obtained taking very high spring stiffness parameter ($\gamma \to \infty$), using the mass distribution parameter (C) in the computation. First, using Eqs. (6) and (7), the tip load (P) and the tip amplitude (a/L) are calculated for a given end slope of the cantilever beam. And the corresponding non-linear stiffness (P/a) is determined. Using Eq. (9) the first mode non-linear frequency (f_{NL}) is computed for various end slopes. The fundamental frequencies parameter $\Omega = \sqrt{\bar{m}/(EI)}\,\omega L^2$ (where $\bar{m}$ is the mass per unit length; ω is the radian frequency given by $2\pi f_{NL}$) are compared with those of [5,7]. The ratio of the non-linear (large amplitude) period to the linear period (T_{NL}/T_L) versus amplitude (a/L) for a cantilever beam has been found out using the present polynomial function. The results are shown in Fig. 2. The present results are compared with those of Wagner [4], Rao and Rao [5] and Pany and Rao [7]. The comparison shows that the present (polynomial approach) results are closer to those of Wagner [4] and Pany and Rao [7] than Rao and Rao [5].

Table 2
Non-linear fundamental frequency of a spring-hinged cantilever beam

α (deg)	γ	P	a	S_{NL}	f_{NL}	Ω	% diff. in Ω
20	2.0169	0.3143	0.1988	1.5805	1.1023	2.2996 2.3019[a]	0.097
30	4.0972	0.6679	0.2754	2.4246	1.3653	2.8483 2.8601[a]	0.414
40	6.2982	1.0943	0.3496	2.1298	1.5512	3.2361 3.2680[a]	0.983
50	8.7099	1.6521	0.4210	3.9233	1.7368	3.6232 3.6857[a]	1.724
60	11.498	2.4621	0.4897	5.0275	1.9660	4.1015 4.2047[a]	2.515
70	15.038	3.8406	0.5557	6.9111	2.3051	4.8088 4.9530[a]	2.997
80	20.536	7.0390	0.6208	11.3775	2.9524	6.1592 6.2840[a]	2.025

[a] Lower frequencies values are obtained using load parameter (λ) and amplitude (a/L) of Ref. [8].

Finally, the fundamental frequencies of a spring-hinged cantilever beam for a range of tip slopes and spring stiffness parameters are presented using the same procedure as explained above (i.e., using the mass distribution parameter) in Table 2. Here, the non-linear frequencies have been obtained from the load amplitude data obtained from the present polynomial approach and that of Ref. [8]. The results agree well, within 2% difference, with those of Ref. [8]. It is observed that the non-linear fundamental frequency parameter (Ω) increases with tip slope (α) or amplitude (a/L), which implies that the first mode of vibration of a uniform cantilever beam is having a hardening type of non-linearity.

5. Concluding remarks

In this paper a polynomial function has been derived to get the large deflection data for a uniform cantilever and a spring-hinged cantilever beam with a tip concentrated load. The large amplitude fundamental frequency is obtained for a uniform spring-hinged cantilever beam, based on the above data, using a very simple approach. The present approach gives accurate results, for all practical purposes, with much less computational effort when compared to the other methods [4,5]. The authors believe that the non-linear fundamental frequency of a spring-hinged cantilever beam is presented for the first time in this paper.

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