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Letter to the Editor

A classical perturbation technique that works even when the linear part of restoring force is zero

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1. Introduction

There are various perturbation techniques for constructing analytical approximations to the oscillatory solutions of second order, non-linear differential equations [1–3]. But many of them apply to weekly non-linear cases and require the linear part of restoring force to be non-zero. However, there exist some non-linear oscillation problems in which the linear part of restoring force is zero [1,2]. For this situation, the method of harmonic balance can be used to obtain analytical approximate solutions [1]. But it is difficult to give high order analytical approximate formulae by applying the method. Senator and Bapat [4] presented a method to solve this non-linear oscillation problem. Wu and Li [5,6] also presented an approach which combines the linearization of non-linear oscillation equation with the method of harmonic balance. Recently, Hu [7] pointed out that there exists a classical perturbation technique which is valid for large parameters. The main purpose of this paper is to point out that this classical perturbation technique works when the linear part of restoring force is zero.

2. Comparison of two classical perturbation techniques

Consider the Duffing equation

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denote differentiation with respect to time t and ε is a positive parameter. According to the Lindstedt–Poincaré method [1–3], the solution of Eq. (1) is assumed in the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{2}$$

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and the fundamental frequency ω is given by

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{3}$$

in which the parameters $\omega_i\,(i = 1, 2, \ldots)$ are undetermined. Introducing the substitution $\tau = \omega t$, $\mathrm{d}/\mathrm{d}t = \omega\mathrm{d}/\mathrm{d}\tau$ into Eq. (1), we obtain

$$\omega^2 x'' + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{4}$$

where primes designate differentiation with respect to τ. Substituting Eqs. (2) and (3) into Eq. (4), and setting the coefficients of the resulting expansion in ε to zero, the following relations are obtained:

$$\varepsilon^0: \quad \omega_0^2 x_0'' + \omega_0^2 x_0 = 0, \tag{5}$$

$$\varepsilon: \quad \omega_0^2 x_1'' + \omega_0^2 x_1 = -2\omega_0\omega_1 x_0'' - x_0^3. \tag{6}$$

Taking into account the initial conditions given in Eq. (4), the following formulae can be obtained:

$$x(\tau) = A_0 \cos\tau + \frac{\varepsilon A_0^3}{32\omega_0^2}(\cos 3\tau - \cos\tau) + O(\varepsilon^2), \tag{7}$$

$$\omega = \omega_0 + \frac{3\varepsilon A_0^2}{8\omega_0} + O(\varepsilon^2). \tag{8}$$

If ω_0 is very small, or if we let $\omega_0 \to 0$, then Eqs. (5) and (6) take the form

$$0 = 0, \tag{9}$$

$$0 = -x_0^3. \tag{10}$$

For this case, formulae (7) and (8) are invalid. In order to avoid this defect of the "normal" Lindstedt–Poincaré technique, an "innovative" classical perturbation technique [7] is now resorted to. Instead of expansion (3), we may use the expansion [7–12]

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{11}$$

where, at this point, the ω_i are unknown constants. Substituting Eqs. (2) and (11) into Eq. (1) gives

$$\begin{aligned}(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + (\omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)\\ + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 = 0.\end{aligned} \tag{12}$$

This equation is satisfied by setting the coefficients of the powers of ε equal to zero, resulting in

$$\varepsilon^0: \quad x_0'' + \omega^2 x_0 = 0, \tag{13}$$

$$\varepsilon: \quad x_1'' + \omega^2 x_1 = \omega_1 x_0 - x_0^3, \tag{14}$$

$$\varepsilon^2: \quad x_2'' + \omega^2 x_2 = \omega_2 x_0 + \omega_1 x_1 - 3x_0^2 x_1. \tag{15}$$

Using the normal classical perturbation technique and taking into account the initial conditions given in Eq. (1), we can easily obtain the second approximate solution to Eq. (1) [7]:

$$x = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t), \tag{16}$$

with

$$\omega^2 = \omega_0^2 + \frac{3}{4}\varepsilon A^2 - \frac{3\varepsilon^2 A^4}{128\omega^2}. \tag{17}$$

Solving Eq. (17) for ω gives [7]

$$\omega = \frac{1}{4}\sqrt{8\omega_0^2 + 6\varepsilon A^2 + \sqrt{64\omega_0^2 + 96\omega_0^2\varepsilon A^2 + 30\varepsilon^2 A^4}}. \tag{18}$$

Formulae (16) and (18) are valid even when $\omega_0 = 0$.

3. Two approaches to the problem

In what follows, we will use the ‘‘innovative’’ classical perturbation technique to solve non-linear oscillation problems without the linear part of restoring force. Without the loss of generality, we consider the equation

$$\ddot{x} + x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{19}$$

This equation can be rewritten as

$$\ddot{x} + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{20}$$

which gives Eq. (19) when $\varepsilon = 1$. There are two approaches to solve the problem given by Eq. (20).

Approach 1: First, we solve Eq. (1) instead of Eq. (20). Obviously, Eq. (1) becomes Eq. (20) if $\omega_0^2 = 0$. Therefore, from Eq. (18) we have

$$\omega^2 = \frac{6 + \sqrt{30}}{16}\varepsilon A^2 = 0.71733\varepsilon A^2, \tag{21}$$

which gives

$$\omega = 0.84695A \tag{22}$$

if $\varepsilon = 1$. Eq. (16) still holds and is identical to the second approximate solution obtained by Senator and Bapat [4]. The second approximate period of the oscillation of Eq. (19), as determined by Eq. (22), is

$$T_c = 2\pi/\omega = 7.4186/A. \tag{23}$$

The exact solution to Eq. (19) is [1]

$$x(t) = Acn(At; 1/\sqrt{2}), \tag{24}$$

where *cn* is the Jacobi elliptic function. The exact period of the oscillation is

$$T_e = 7.4163/A \tag{25}$$

and the second approximation obtained by the method of harmonic balance is [1]

$$T_h = 7.3859/A. \tag{26}$$

The corresponding approximate result in Ref. [6] is

$$T_w = 7.4278/A. \tag{27}$$

Obviously, formula (23) is more accurate than formulae (26) and (27). The method in this paper is much simpler than the method of harmonic balance and the Senator–Bapat method.

Approach 2: Eq. (20) can be rewritten in the form

$$\ddot{x} + 0x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{28}$$

Letting $\omega_0^2 = 0$ in Eq. (11) results in

$$0 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots. \tag{29}$$

Substituting Eqs. (2) and (29) into Eq. (28) and setting the coefficients of the powers of ε equal to zero still gives Eqs. (13)–(15). Then, the analysis now proceeds in the usual way. The second approximate solution is still expressed by Eq. (16), with

$$\omega^2 = \frac{3}{4}\varepsilon A^2 - \frac{3\varepsilon^2 A^4}{128\omega^2}. \tag{30}$$

Solving this equation for ω^2 gives Eq. (21).

We can see that approach 2 is simpler than approach 1. Obviously, the two approaches can be applied to non-linear oscillation problems with other kinds of non-linearities. For example, consider a non-linear oscillation equation

$$\ddot{x} + f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{31}$$

where $f(x)$ is an odd non-linear function. Using approach 2, this equation may be written as

$$\ddot{x} + 0x + \varepsilon f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{32}$$

Substituting Eqs. (2) and (29) into this equation and repeating the procedural steps described above, we can easily obtain the approximate solution to Eq. (31).

4. Conclusions

A comparison of two classical perturbation methods has been made. An "innovative" classical method is presented again. It is an effective method for dealing with conservative single-degree-of-freedom systems without linear part of restoring force which cannot be treated by the standard classical perturbation method. It is simple and easy to use and can give excellent approximate results.

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