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Exterior problems of acoustics by fractal finite element mesh

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Abstract

The propagation and attenuation of acoustic waves in an exterior domain is an essential ingredient in the study of acoustic–structure interaction. In this paper the problems of acoustic radiation from an arbitrarily shaped vibrating body in an infinite exterior region are investigated by using a fractal two-level finite element mesh (FEM) with self-similar layers in the media enclosing the conventional FEM for the vibrating body. The fractal two-level FEM has been successfully used in stress intensity factor prediction with self-similar ratio smaller than one so that the mesh converges to the crack tip. In this paper, the similarity ratio is bigger than one so that the mesh extends to infinity. By means of the Hankel functions satisfying automatically Sommerfeld's radiation conditions at infinity, the different unknown nodal pressures in different layers are transformed to some common unknowns of the Hankel coefficients. The final matrix size of the exterior region is equal to the number of terms in the Hankel expansion. The set of infinite number of unknowns of nodal pressure is reduced to a set of small finite number of Hankel's coefficients. All layers have the same unknowns after the transformation. Due to self-similarity, the transformed stiffness matrix of the first layer is proportional to that of the second and so on. Therefore, the stiffness matrices of the infinite layers can be summed by using just one layer. Numerical examples show that this method is efficient and accurate in solving unbounded acoustic problems.

1. Introduction

In modern engineering, the study of acoustic–structure interaction encountered in many kinds of structural systems, especially large systems, such as railway, cloverleaf junction, sports field and concert hall, is of great significance for noise pollution control and reduction, and structure optimization design. An important building block in the study of structural acoustics and fluid–structure interaction is the problems of acoustic radiation and scattering governed by the

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Helmholtz equation or the reduced wave equation in the time-harmonic case. In recent years, a great amount of research work about the problems has been done. Different mathematical modelling tools have been developed to solve the problems, ranging from pure analytical methods, restricted to special geometry cases, to more generally applicable numerical methods such as finite element and boundary element methods [1–18]. Astley [25] reviewed the current formulations for infinite elements. His comment on the domain-based method and the infinite element method is briefly summarized here. The domain-based methods require the truncation of the computational domain at a discrete distance from the radiating or scattering object and the application of an approximate boundary operator to simulate anechoic conditions. The infinite elements do not suffer from the domain truncation problem, but all infinite elements are susceptible to ill-conditioning that places practical restrictions on their effectiveness at high radial orders.

Leung and Su [19–21] and Leung [22] presented a fractal finite element method using self-similar meshes (fractal mesh) that has been very successful to determine the stress intensity factors in elastic fracture mechanics. The method does not create new elements. The key is to make a self-similar finite element mesh (FEM) and use the proportional properties of the entries of the relevant matrices for closed-form summation. It is being implemented within NASTRAN for stress intensity problems [22]. In this paper the method of fractal FEM for the exterior region combined with conventional FEM for the vibrating body is developed to solve the unbounded acoustic problems.

It is appropriate to give a brief description of the fractal mesh used in fracture mechanics here. Consider a body containing a crack. Use a FEM over the body which is separated into two parts, a singular region **S** enclosing the neighbourhood of the crack and a regular region **R** containing the remaining part as shown in Fig 1. The mesh on **R** has no restriction. The mesh on **S** consists of concentric curves with the crack tip as centre. The elements between two concentric curves constitute a layer. The layer stiffness matrices are partially overlapped in the nodal co-ordinates as shown in Fig 2a. After the nodal co-ordinates are transformed to a common set of generalized

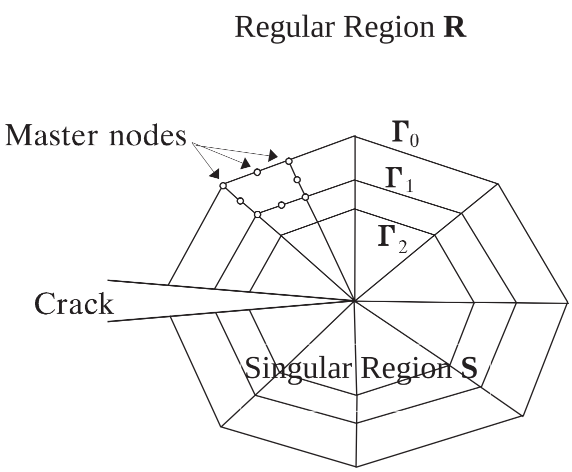

Fig. 1. Construction of fractal mesh.


(a)


(b)

Fig. 2. (a) Stiffness matrix of **R** in nodal co-ordinates plus first five layers of **S** in nodal co-ordinates, matrices in **S** are partially overlapped. (b) Stiffness matrix of **R** in nodal co-ordinates + first five layers of **S** in generalized co-ordinates, matrices in **S** are fully overlapped and can be summed in closed form.

co-ordinates using (e.g., Williams) eigenfunction expansion, the partially overlapped stiffness matrices representing **S** become fully overlapped as shown in Fig 2b. The size of the overlapping matrices is equal to the number of eigenfunctions but independent of the number of layers.

As each entry of the overlapped matrices is proportional to a certain power of the scale factor, the summation over all the layers is similar to the summation over a geometric series. In fact, only one layer is required to be processed. The stiffness matrix of this layer is generated by conventional finite elements. Since the power of the scale factor need not be the same for each entry of the stiffness matrix of a layer, the method provides great flexibility and has been extended to 3-D penny-shaped cracks.

The method has two distinct advantages in formulation. Firstly, no new finite elements are generated so that an engineer can use commercial packages. Secondly, the transformation from nodal displacements to generalized co-ordinates, including the SIF, by means of eigenfunction expansion near the crack tip is new. Other advantages are (i) SIF is a direct output, which does not require post-processing, (ii) great saving in computing time and storage, and (iii) very accurate answer in SIF can be expected.

The order of the global matrix of Fig. 2a is proportional to the number of layers and becomes infinity if an infinite number of layers are used. The order of the transformed global matrix of Fig. 2b is proportional to the number of eigenfunctions taken but independent on the number of layers and remains finite even if an infinite number of layers are used. The method is very different from the singular finite elements which require integration of some singular functions to generate new elements; from the infinite (focus) elements which require the solution of quadratic eigenproblems of the associated matrix difference equation not applicable for 3-D problems; and from the boundary elements which also require fine meshes and clustered unknowns at the singular points.

The paper extends the method to infinite domain for the exterior problems by making the self-similar ratio bigger than one so that the domain covered would extend to infinity. The order of the final matrix is finite and equal to the number of Henkel terms being considered. Sixty terms are taken in the numerical examples. The final matrix is of order 60 and is positive definite and well behaved.

2. Governing equation of acoustic wave and its solution

The unbounded domain is divided by a smooth artificial boundary $\mathbf{S_1}$ into two parts: a finite region $\mathbf{R}_1$ and an infinite region $\mathbf{R}_2$ as shown in Fig. 3. Within $\mathbf{R}_1$ a conventional FEM is employed. In the outer region $\mathbf{R}_2$, the fractal geometrical concept is adopted to achieve the self-similar meshes having similarity ratio bigger than one. The self-similar mesh makes a simple relationship between the dynamic stiffness matrices of two adjacent layers. By means of the Hankel functions satisfying automatically Sommerfeld's radiation conditions at infinity, the unknown nodal pressures on different layers are transformed to some common unknowns of the Hankel coefficients. The size of the final matrix over the exterior region is equal to the number of the Hankel coefficients considered. The set of infinite number of unknowns of nodal pressure is reduced to the set of small finite number of the Hankel coefficients. All layers have the same matrix dimension after the transformation and the respective matrices of each layer are summed. Due to proportionality, the infinite number of layers can be summed in closed form as the entries of each matrix are in geometric series. That is, processing one layer is enough to represent virtually a set of infinite number of layers covering an infinity domain.

Let $\mathbf{R}$ be an unbounded region with an arbitrary vibrating body bounded by $\mathbf{S}$, which is assumed piecewise smooth as shown in Fig. 3. The unbounded region is divided into two parts by

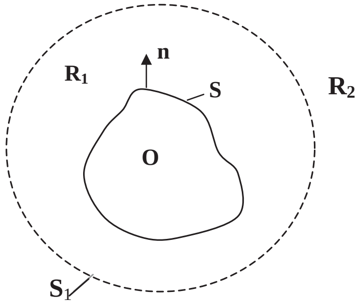

Fig. 3. Geometry of the discussed problem.

a circular artificial boundary $\mathbf{S_1}$, namely $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively. The inner surface **S** experiences a prescribed normal acceleration $a_n(x,t)$ or sound pressure $\bar{p}(x,t)$ assuming no sound sources are present. The acoustical pressure $p(x,t)$ in the exterior region is governed by the linearized wave equation [23]

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial^2 t} \quad \text{in } \mathbf{R}, \tag{1}$$

where ∇^2 is the Laplacian operator and c is the sound speed.

For the time-harmonic case, the solution is of the form

$$p(x,t) = P(x,\omega)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{2}$$

where P is the complex pressure amplitude depending on the radian frequency ω. Substituting Eq. (2) into Eq. (1), one obtains the well-known Helmholtz equation

$$\nabla^2 P + k^2 P = 0 \quad \text{in } \mathbf{R}, \tag{3}$$

where $k = \omega/c$ is the wave number. The appropriate boundary conditions on **S** involve

$$P = \bar{P} \quad \text{or} \quad \nabla P \cdot \mathbf{n} + \rho \bar{a}_n = 0, \tag{4a, b}$$

where $\bar{P}$ is the prescribing acoustic pressure, ρ the mean fluid density, $\bar{a}_n = \bar{a}_n(x,\omega)$ the amplitude of the normal acceleration $a_n(x,t)$. The Sommerfeld radiation condition which ensures that energy propagates in an outward direction is

$$\lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial P}{\partial r} - \mathrm{i}kP\right) = 0, \tag{5}$$

where $d = 2$ or 3 is the number of space dimensions, r is the distance measured from a global origin **O** within the boundary **S**.

The general solution of an exterior 2-D Helmholtz Eq. (3) for outgoing wave, is of the form

$$P(r,\theta) = \sum_m H_m^2(kr)(A_m\cos(m\theta) + B_m\sin(m\theta)), \quad m = 0,1,2,\ldots, \tag{6}$$

whereH_m^2 is m order Hankel function of second kind.

In this paper the fractal two-level FEM are developed to solve the unbounded domain acoustic problems governed by Eqs. (1)–(5).

3. Fractal two-level FEM for exterior acoustic problem

Consider an infinite domain **R** with a piecewise smooth inner boundary **S**. For solving the boundary-value problem (1)–(5), one introduces an artificial boundary $\mathbf{S}_1$ to divide the domain **R** into two parts: $\mathbf{R}_1$ and $\mathbf{R}_2$, as shown in Fig. 3. In the region $\mathbf{R}_1$ the conventional FEM is employed, in the region $\mathbf{R}_2$ the fractal FEM is adopted.

3.1. FE equation in domain $\mathbf{R}_1$

The domain R_1 is discretized according to the conventional finite element method. The interpolations of sound pressure P and the co-ordinates $\mathbf{x}$ for an isoperimetric formulation are expressed by

$$P = [\mathbf{N}(\xi)]\{P\}, \tag{7}$$

$$\mathbf{x} = [\mathbf{N}(\xi)]\{\mathbf{X}\}, \tag{8}$$

where $\mathbf{N}(\xi)$ is the shape function matrix, and $\{P\}$and $\{\mathbf{X}\}$ are the nodal sound pressure and the nodal co-ordinate vectors, respectively. Applying variational method, one transforms problems (3)–(4) to a matrix equation that can be written in the form [24]

$$[\mathbf{K} - k^2\mathbf{M}]\{P\} = \{F\}, \tag{9}$$

with the related coefficients matrices

$$K_{ij} = \sum K_{ij}^{(e)} = \sum \int_{\Omega^e} \nabla N_i \cdot \nabla N_j \, \mathrm{d}\Omega^e, \tag{10}$$

$$M_{ij} = \sum M_{ij}^{(e)} = \sum \int_{\Omega^e} N_i N_j \, \mathrm{d}\Omega^e, \tag{11}$$

$$F_i = \sum F_i^{(e)} = \rho \sum \int_{S^e} N_i \bar{a}_n \, \mathrm{d}S^e, \tag{12}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices of the acoustic medium, respectively, $\{P\}$ is the nodal pressure vector undetermined and $\{F\}$ the acoustic forcing vector on $\mathbf{S}$.

3.2. Fractal FE equation in domain $\mathbf{R_2}$

3.2.1. Transformation of dynamical stiffness matrix and global interpolation function

Using Eq. (6), the nodal pressures $\{P\}$ can be transformed to the generalized co-ordinates $\{a\}$ by,

$$\{P\} = \mathbf{T}\{a\}, \tag{13}$$

where $\mathbf{T}$ is a transformation matrix evaluated from Eq. (6).

Substituting Eq. (13) into variation procedure for deducing FEM formula, one has

$$\mathbf{T}^{\mathrm{T}}[\mathbf{K} - k^2\mathbf{M}]\mathbf{T}\{a\} = \mathbf{T}^{\mathrm{T}}\{F\}, \tag{14}$$

where the superscript 'T' indicates matrix transposition. As the order of $\{a\}$ is far less than that of $\{P\}$, solving Eq. (14) is easier than that of Eq. (8).

3.2.2. Fractal transformation of mass matrix and stiffness matrix

In the region $\mathbf{R_2}$, the fractal geometric concept is adopted for forming self-similar meshes with the ratio α in terms of their length dimensions. Consider two respective elements in the two adjacent fractal layers. Their co-ordinates can be expressed as

$$\begin{aligned} X_i^{(m+1)} &= \alpha X_i^{(m)}, \\ Y_i^{(m+1)} &= \alpha Y_i^{(m)}, \end{aligned} \tag{15}$$

where the superscript denotes the layer number in which the element is located in.

The formulas for calculating the element acoustical stiffness and mass matrices in the local co-ordinate are

$$k_{ij} = \int_{-1}^{1}\int_{-1}^{1}\left[\mathbf{J}^{-1}\cdot\begin{Bmatrix}\partial N_i/\partial\xi\\ \partial N_i/\partial\eta\end{Bmatrix}\right]^{\mathrm{T}}\cdot\left[\mathbf{J}^{-1}\cdot\begin{Bmatrix}\partial N_j/\partial\xi\\ \partial N_j/\partial\eta\end{Bmatrix}\right]\cdot\det(\mathbf{J})\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{16}$$

$$m_{ij} = \int_{-1}^{1}\int_{-1}^{1} N_iN_j\det(\mathbf{J})\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{17}$$

where

$$\mathbf{J} = \begin{bmatrix}\partial x/\partial\xi & \partial y/\partial\xi\\ \partial x/\partial\eta & \partial y/\partial\eta\end{bmatrix}. \tag{18}$$

From Eqs. (7), (15) and (18) one has

$$\mathbf{J}^{(m+1)} = \alpha\mathbf{J}^{(m)}. \tag{19}$$

Substituting Eq. (19) into Eqs. (16) and (17), one obtains

$$[K]^{(m+1)} = [K]^{(m)} \quad \text{and} \quad [M]^{(m+1)} = \alpha^2[M]^{(m)}. \tag{20a, b}$$

Let the nodes on the artificial boundary be the master nodes and those within the region $\mathbf{R_2}$ be the slave nodes with node pressure written as $\{P_m\}$ and $\{P_s\}$, respectively. The transformation is needed only for the slave node pressure. For the first layer, the stiffness matrix $[K]$ and mass matrix $[M]$ are first partitioned with respect to m and s in block form as

$$([K^{\mathrm{f}}]-k^2[M^{\mathrm{f}}])\{P\}^{\mathrm{f}} = \left(\begin{bmatrix}K_{11}^{\mathrm{f}} & K_{12}^{\mathrm{f}}\\ K_{21}^{\mathrm{f}} & K_{22}^{\mathrm{f}}\end{bmatrix} - k^2\begin{bmatrix}M_{11}^{\mathrm{f}} & M_{12}^{\mathrm{f}}\\ M_{21}^{\mathrm{f}} & M_{22}^{\mathrm{f}}\end{bmatrix}\right)\begin{Bmatrix}\{P_m\}^{\mathrm{f}}\\ \{P_s\}^{\mathrm{f}}\end{Bmatrix}, \tag{21}$$

where the superscript 'f' indicates the first layer. The corresponding transformation matrix is

$$\begin{bmatrix}I & 0\\ 0 & T^{\mathrm{f}}\end{bmatrix}.$$

After transformation, one has

$$\left(\begin{bmatrix}K_{11}^{\mathrm{f}} & K_{12}^{\mathrm{f}}T^{\mathrm{f}}\\ T^{\mathrm{f}^{\mathrm{T}}}K_{21}^{\mathrm{f}} & T^{\mathrm{f}^{\mathrm{T}}}K_{22}^{\mathrm{f}}T\end{bmatrix} - k^2\begin{bmatrix}M_{11}^{\mathrm{f}} & M_{12}^{\mathrm{f}}T^{\mathrm{f}}\\ T^{\mathrm{f}^{\mathrm{T}}}M_{21}^{\mathrm{f}} & T^{\mathrm{T}}M_{22}^{\mathrm{f}}T^{\mathrm{f}}\end{bmatrix}\right)\begin{Bmatrix}\{P_m\}^{\mathrm{f}}\\ \{a\}\end{Bmatrix} = \{0\}. \tag{22}$$

For the layer n, the transformed matrix can be written as follows

$$D^i = [T^i]^{\mathrm{T}}[K^{\mathrm{f}} - k^2\cdot\alpha^{2(i-1)}M^{\mathrm{f}}][T^i]. \tag{23}$$

After summing from layer 2 to infinity, one has

$$\bar{D} = \sum_{i=2}^{\infty}[T^i]^{\mathrm{T}}[K^{\mathrm{f}} - k^2\cdot\alpha^{2(i-1)}M^{\mathrm{f}}][T^i], \tag{24}$$

where $[T^i]$ is proportional to the distance r from $\mathbf{O}$ in the form $r^{-1/2}\mathrm{e}^{-r}$ and is related to $\alpha = r_{i+1}/r_i$. Therefore, Eq. (24) represents a convergent series in terms of r and can be summed in closed form.

4. Examples

The performance of the fractal FEM for the sound radiation problems is investigated in this section. Consider the case when the inner boundary is subject to cosine distributed sound pressure $\bar{P}\cos(\theta)$. The fractal ratio α is taken to be 1.15 and Eq. (6) is truncated at the 60th term of the Fourier expansion, $m = 60$. Therefore, the matrix dimension of the exterior region is always equal to 60 independent of the number of layers taken. The frequency is in the range from 35 to 2835 Hz and the sound velocity is 1400 m/s. All the results are plotted as the non-dimension pressure $|P/\bar{P}|$ against the non-dimension wave number kR. In the case of circular vibrating body, the radius of the artificial boundary $\mathbf{S}_1$ is taken to be three times of that of the inner boundary. For the elliptical vibrating body, the radius of the artificial boundary is three times of the length of the half major axis.

Fig. 4 shows the solutions for the circular inner boundary $\mathbf{S}$ with unit radius at points $R = 6$, $\theta = \pi/4$. Fig. 4(a)–(d) are for the fractal layer numbers taken to be 15, 21, 27, and 31, respectively. Correspondingly, the infinity domain $\mathbf{R}_2$ is truncated at the distances $R = 24.5, 56.5, 130.6$ and 228.4, respectively . Figs. 5 and 6 are the results at different points with the fractal layer number taken to be 31. Figs. 7 and 8 are the solutions for the elliptical bodies with the quotients $l_b/l_a = 1.5$and 1.25, respectively. From these figures, one can see that as the unbounded domain is truncated at the distance far enough, the results are almost the same with the value calculated by the wave envelope elements method [5–10].

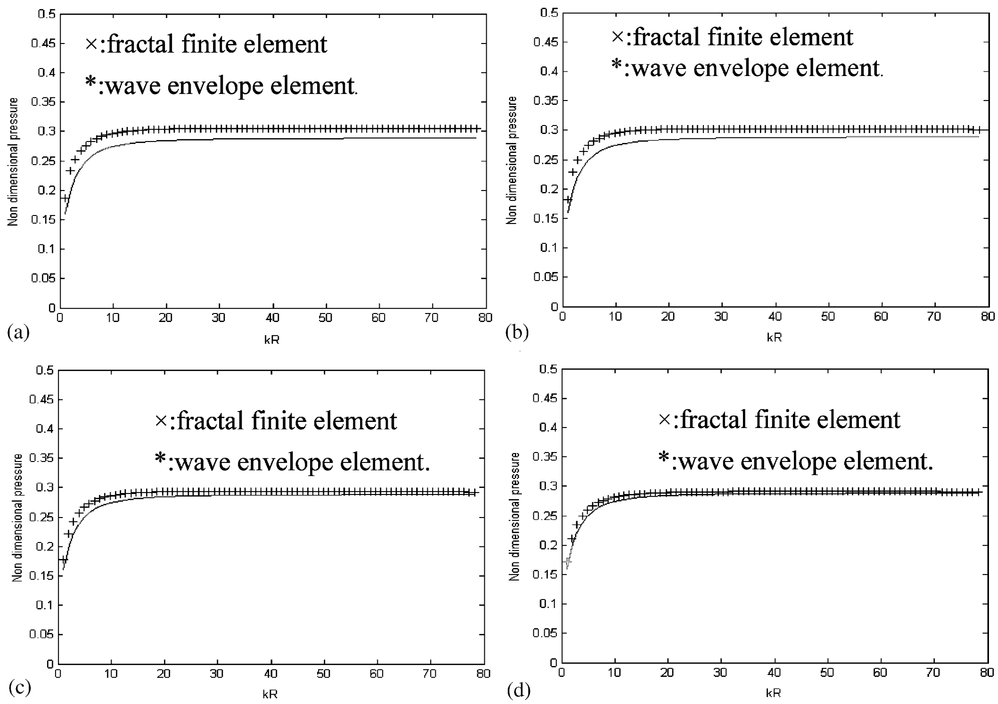

Fig. 4. Radiation pressure amplitude ratio from circular inner boundary at $R = 6$, $\theta = \pi/4$.

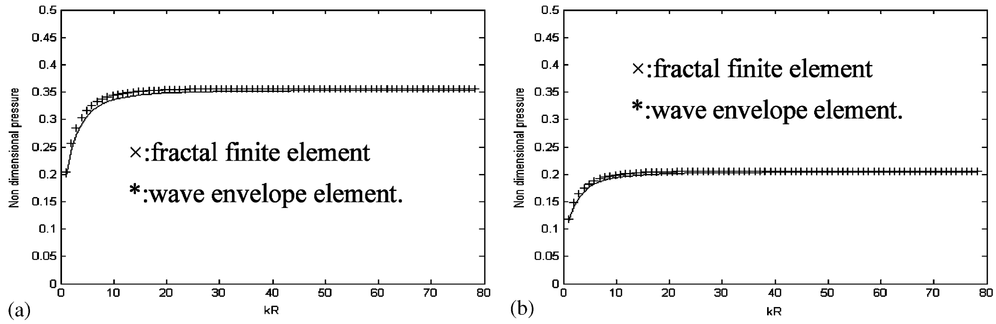

Fig. 5. Radiation pressure amplitude ratio at $R = 6$, (a) $\theta = \pi/6$, (b) $\theta = \pi/3$ from circular inner boundary.

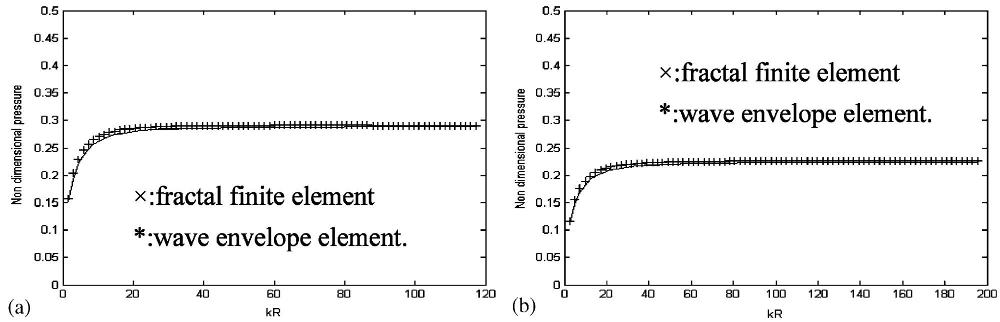

Fig. 6. Radiation pressure amplitude ratio at (a) $R = 9$, (b) $R = 15$, $\theta = \pi/6$ from circular inner boundary.

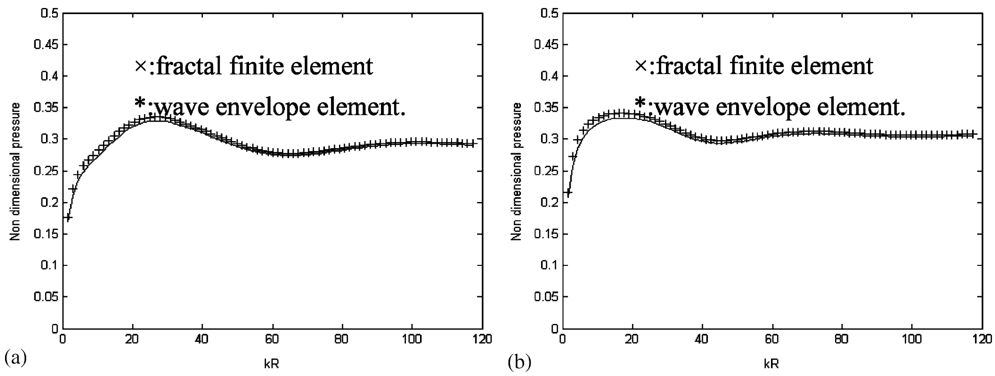

Fig. 7. Non-dimension radiation pressure from elliptical vibrating body with $l_b/l_a = 1.5$ (a) at $R = 9$, $\theta = \pi/4$, (b) at $R = 9$, $\theta = \pi/6$.

5. Conclusion

The fractal two-level finite element method has been shown to be effective in calculating the infinite domain sound problems. This method has two distinct advantages: one is by employing the global interpolation functions satisfying the Sommerfeld radiation condition automatically so that the unknowns can be greatly reduced; and the other is by adopting the fractal geometrical

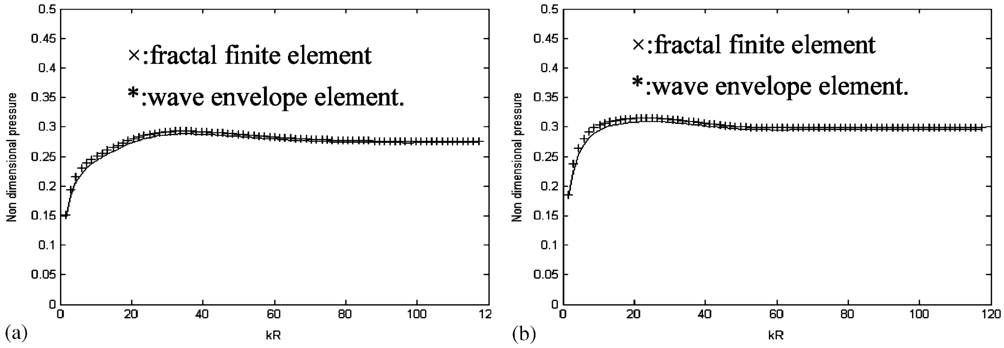

Fig. 8. Non-dimension radiation pressure from elliptical vibrating body with $l_b/l_a = 1.25$ (a) at $R = 9$, $\theta = \pi/4$, (b) at $R = 9$, $\theta = \pi/6$.

concept to form the similar meshes so that the global dynamical matrix can be easily obtained. Several numerical examples are performed with results showing the effectiveness of this method. Further work to extend it to three-dimensional problems is being investigated.

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