Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 272 (2004) 245–265
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness

Roshan Lal*, Shuchita Sharma

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247 667, Uttaranchal, India

Received 28 March 2002; accepted 24 March 2003

Abstract

In this study of the free axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of exponentially varying thickness, on the basis of the classical theory of plates, the Chebyshev collocation technique has been employed to solve the differential equation governing the transverse motion of such plates. The non-homogeneity of the plate material is assumed to arise due to the variation of Young's moduli and density which are assumed to vary exponentially with the radius vector. The effect of various plate parameters such as radii ratio, rigidity ratio, non-homogeneity and thickness variation, on the vibrational characteristics of the plate have been studied for three different boundary conditions. Normalized displacements are presented for a specified plate for the first three modes of vibration. A comparison of results with those available in the literature has been presented.

1. Introduction

Structural components of varying thickness are highly favoured these days due to economy and light-weight considerations. Plates of variable thickness are often encountered in engineering applications and their use in machine design, nuclear reactor technology, naval structures and acoustical components is quite common. In recent years, the development of fibre-reinforced materials and its increasing use, such as in diaphragms used in pressure capsules, circular and annular plates stiffened with radial and circumferential ribs, and plates fabricated out of modern composites (boron-epoxy, glass-epoxy, Kevalar and graphites, etc.) has necessitated the study of the dynamic response of anisotropic plates of variable thickness. The use of such plates as

*Corresponding author. Fax: +91-1332-273560.
E-mail address: rlatmfma@iitr.ernet.in (R. Lal).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00329-8

structural elements in various technological situations demand that the non-homogeneity of the materials should be taken into account for the analysis of plate vibrations. Certain parts in aircraft and rockets have to operate under elevated temperatures which cause non-homogeneity in the material, i.e., the elastic constants of the material become functions of space variables. Further, many structural components possess initial non-homogeneity due to the inclusion of foreign materials or imperfections or being composite material. The material properties in such elastic bodies vary in a continuous manner. Plywood, timber and fibre-reinforced plastic, etc., form an important class of non-homogeneous materials which are used in engineering design.

The work upto 1965, dealing with linear vibrations of non-homogeneous isotropic plates of various geometries has been reported in his monograph by Leissa [1]. Notable contributions made thereafter dealing with various types of non-homogeneity consideration are given in Refs. [2–13]. In Ref. [2], Bose analyzed the vibrations of thin non-homogeneous circular plates with a central hole assuming that the Young's modulus $E = E_0 r$ and density $\rho = \rho_0 r$ of the plate material vary with the radius vector r where E_0, ρ_0 are constants. Biswas [3] has considered a non-homogeneous material for which rigidity$\mu = \mu_0 e^{-\mu_1 z}$and density $\rho = \rho_0 e^{-\mu_1 z}$ both vary exponentially whereμ_0, ρ_0 and μ_1 are constants. In Ref. [4], a fundamental boundary value problem for a plane circular region has been solved with an exponential function in radial direction for non-homogeneity ($\mu = \mu_0 e^{qx}$, where μ_0 is constant, q a small parameter and x the dimensionless radial variable) and the constant Poisson ratio. Recently, Chakraverty and Petyet [5] have studied the vibration of non-homogeneous elliptic plates with radial tapering in Young's modulus and density with the constant Poisson ratio. In a series of papers, Tomar et al. [6–9] have analyzed the dynamic

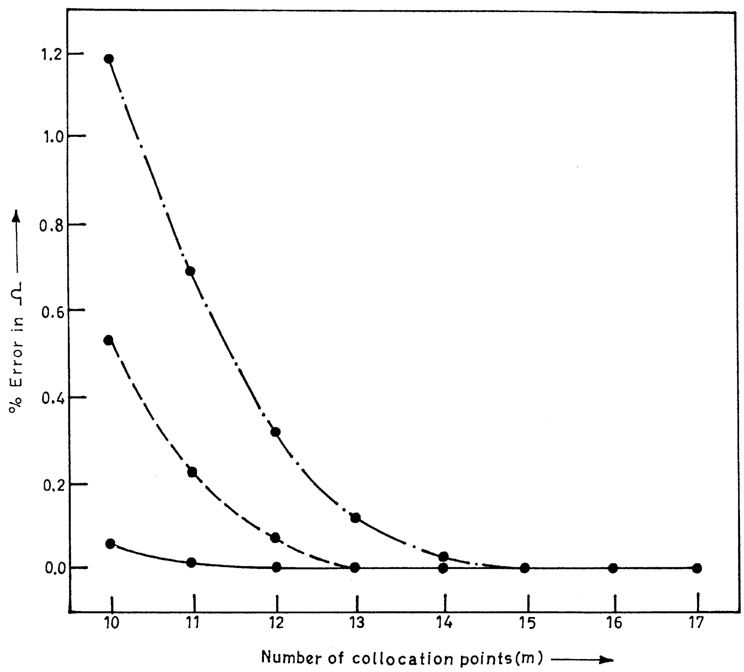

Fig. 1. Percentage error in Ω for C–C plate for $\varepsilon = 0.5$, $p = 5.0$, $\alpha = 0.5$ and $\mu = 0.5$. ——, first mode; ----, second mode; —·—, third mode. Percentage error $= [(\Omega_m - \Omega_{15})/\Omega_{15}] \times 100$, $m = 10(1)17$.

Table 1
Values of frequency parameter for C–C annular plate for $\varepsilon = 0.3$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	32.3461	32.2805	32.2636	32.2954
	II		89.8514	89.7676	89.7509	89.8015
	III		176.8262	176.7379	176.7234	176.7825
	I	1.0	32.6519	32.5781	32.5539	32.5792
	II		90.2536	90.1662	90.1462	90.1936
	III		177.2752	177.1848	177.1683	177.2256
	I	2.0	33.2523	33.1628	33.1244	33.1370
	II		91.0511	90.9565	90.9298	90.9710
	III		178.1686	178.0742	178.0537	178.1073
0.0	I	0.5	44.8383	44.9520	45.1352	45.3895
	II		124.6547	124.8192	125.0773	125.4297
	III		245.3620	245.5462	245.8329	246.2222
	I	1.0	45.2406	45.3462	45.5225	45.7712
	II		125.2016	125.3621	125.6167	125.9659
	III		245.9744	246.1563	246.4408	246.8282
	I	2.0	46.0317	46.1218	46.2848	46.5226
	II		126.2864	126.4393	126.6868	127.0296
	III		247.1933	247.3707	247.6511	248.0345
0.5	I	0.5	62.5870	63.0408	63.6005	64.2711
	II		173.0684	173.6860	174.4354	175.3175
	III		339.8285	340.5077	341.3294	342.2941
	I	1.0	63.1230	63.5706	64.1257	64.7932
	II		173.8117	174.4254	175.1713	176.0504
	III		340.6619	341.3387	342.1582	343.1209
	I	2.0	64.1787	64.6143	65.1605	65.8221
	II		175.2869	175.8930	176.6320	177.5051
	III		342.3214	342.9935	343.8086	344.7673

behaviour of non-homogeneous isotropic plates of variable thickness of different geometries. The non-homogeneity of the material of the plate is assumed to arise due to the variation of Young's modulus and density along one direction as $(E, \rho) = (E_0, \rho_0)\, e^{\beta x}$where E_0, ρ_0 are constants and β is the non-homogeneity parameter. The Poisson ratio is assumed to remain constant. In two significant contributions [10,11], Elishakoff has obtained unusual closed-form solutions for the axisymmetric vibrations of isotropic inhomogeneous clamped [10], free [11] circular plates

Table 2
Values of frequency parameter for C–C annular plate for $\varepsilon = 0.5$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	61.0806	61.0039	60.9742	60.9914
	II		168.8308	168.7282	168.6897	168.7152
	III		331.3912	331.2798	331.2387	331.2676
	I	1.0	61.2668	61.1876	61.1557	61.1709
	II		169.0784	168.9749	168.9355	168.9603
	III		331.6643	331.5524	331.5107	331.5393
	I	2.0	61.6371	61.5529	61.5166	61.5281
	II		169.5724	169.4670	169.4260	169.4491
	III		332.2095	332.0967	332.0541	332.0818
0.0	I	0.5	88.8951	88.9892	89.1522	89.3850
	II		245.8553	245.9860	246.2103	246.5283
	III		482.6829	482.8273	483.0740	483.4231
	I	1.0	89.1591	89.2508	89.4118	89.6431
	II		246.2130	246.3428	246.5661	246.8832
	III		483.0778	483.2216	483.4678	483.8164
	I	2.0	89.6844	89.7713	89.9285	90.1567
	II		246.9268	247.0545	247.2760	247.5915
	III		483.8665	484.0093	484.2544	484.6021
0.5	I	0.5	129.8186	130.2603	130.8072	131.4620
	II		358.1435	358.7421	359.4775	360.3505
	III		702.3736	703.0297	703.8351	704.7898
	I	1.0	130.1962	130.6363	131.1823	131.8366
	II		358.6601	359.2576	359.9922	360.8644
	III		702.9440	703.5995	704.4043	705.3586
	I	2.0	130.9479	131.3848	131.9287	132.5823
	II		359.6908	360.2863	361.0191	361.8898
	III		704.0834	704.7377	705.5413	706.4947

assuming that inertial term/density and stiffness of the plate are the polynomial functions of the radial co-ordinate.

An excellent survey dealing with complicating effects in the vibration of plates by Leissa [12,13] reveals that relatively little work has been done on vibration of plates of variable thickness possessing polar orthotropy. In a recent survey of the literature the authors have found no work dealing with vibration of non-homogeneous polar orthotropic plates of varying thickness.

Table 3
Values of frequency parameter for C–S annular plate for $\varepsilon = 0.3$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	22.8811	22.3279	21.7872	21.2555
	II		73.3806	72.8286	72.3285	71.8800
	III		153.0759	152.5164	152.0209	151.5895
	I	1.0	23.2000	22.6414	22.0966	21.5622
	II		73.7874	73.2325	72.7298	72.2789
	III		153.5300	152.9689	152.4719	152.0392
	I	2.0	23.8232	23.2542	22.7012	22.1614
	II		74.5929	74.0322	73.5242	73.0687
	III		154.4329	153.8686	153.3688	152.9334
0.0	I	0.5	30.2542	29.5386	28.8276	28.1177
	II		100.4407	99.8632	99.3555	98.9181
	III		211.0530	210.5047	210.0441	209.6714
	I	1.0	30.6950	29.9777	29.2675	28.5611
	II		101.0028	100.4228	99.9132	99.4741
	III		211.6786	211.1291	210.6673	210.2936
	I	2.0	31.5566	30.8359	30.1269	29.4269
	II		102.1160	101.5312	101.0176	100.5754
	III		212.9233	212.3710	211.9069	211.5312
0.5	I	0.5	39.8578	38.9002	37.9370	36.9654
	II		137.3503	136.7984	136.3414	135.9808
	III		290.2096	289.7545	289.4190	289.2038
	I	1.0	40.4898	39.5416	38.5924	37.6398
	II		138.1307	137.5778	137.1204	136.7600
	III		291.0723	290.6165	290.2805	290.0650
	I	2.0	41.7241	40.7934	39.8702	38.9531
	II		139.6769	139.1221	138.6639	138.3039
	III		292.7888	292.3316	291.9945	291.7786

However, a number of papers is available in Refs. [14–16] in which the effect of non-homogeneity has been analyzed on the bending behaviour of orthotropic curved plates. Similarly in two recent studies [17,18], the work dealing with free vibrations of inhomogeneous membranes has been reported.

This paper deals with the axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of exponentially varying thickness on the basis of classical plate theory. Based

Table 4
Values of frequency parameter for C–S annular plate for $\varepsilon = 0.5$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	43.0059	42.2627	41.5313	40.8091
	II		137.6838	136.9427	136.2506	135.6074
	III		286.6693	285.9151	285.2216	284.5888
	I	1.0	43.2157	42.4720	41.7406	41.0191
	II		137.9435	137.2020	136.5097	135.8663
	III		286.9522	286.1979	285.5043	284.8714
	I	2.0	43.6321	42.8874	42.1560	41.4357
	II		138.4612	137.7190	137.0262	136.3825
	III		287.5171	286.7626	286.0688	285.4358
0.0	I	0.5	60.5342	59.5060	58.4852	57.4690
	II		198.5647	197.6720	196.8490	196.0955
	III		415.6144	414.7427	413.9582	413.2612
	I	1.0	60.8462	59.8200	58.8021	57.7899
	II		198.9461	198.0535	197.2306	196.4776
	III		416.0279	415.1563	414.3719	413.6751
	I	2.0	61.4654	60.4429	59.4307	58.4263
	II		199.7065	198.8140	197.9916	197.2393
	III		416.8536	415.9821	415.1980	414.5016
0.5	I	0.5	85.0592	83.6097	82.1598	80.7064
	II		286.2149	285.1771	284.2386	283.3997
	III		601.7307	600.7834	599.9616	599.2658
	I	1.0	85.5340	84.0924	82.6521	81.2102
	II		286.7766	285.7399	284.8025	283.9652
	III		602.3353	601.3885	600.5673	599.8721
	I	2.0	86.4757	85.0492	83.6277	82.2084
	II		287.8968	286.8620	285.9271	285.0926
	III		603.5426	602.5967	601.7766	601.0827

upon the above Refs. [3,4,6–9], an exponential type variation in Young's moduli and density has been assumed. This type of orthotropy and non-homogeneity arises during the fibre-reinforced plastic structure which use fibres with different moduli and strength properties. Thus the present study of theoretically investigated vibrational characteristics will be of interest to design engineers. The fourth order linear differential equation with variable coefficients which governs the motion

Table 5
Values of frequency parameter for C–F annular plate for $\varepsilon = 0.3$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	5.9584	5.3872	4.8642	4.3870
	II		33.3063	32.2360	31.1972	30.1872
	III		91.3767	90.3337	89.3522	88.4322
	I	1.0	6.3168	5.7429	5.2175	4.7380
	II		33.7629	32.6957	31.6611	30.6568
	III		91.8560	90.8124	89.8305	88.9103
	I	2.0	6.9779	6.3943	5.8589	5.3692
	II		34.6575	33.5957	32.5688	31.5748
	III		92.8061	91.7613	90.7785	89.8578
0.0	I	0.5	6.7804	6.1218	5.5232	4.9809
	II		43.3180	41.9179	40.5548	39.2255
	III		124.0547	122.7815	121.5943	120.4936
	I	1.0	7.3211	6.6604	6.0600	5.5163
	II		44.0028	42.6141	41.2650	39.9525
	III		124.7385	123.4662	122.2801	121.1809
	I	2.0	8.2926	7.6174	7.0019	6.4426
	II		45.3415	43.9737	42.6502	41.3687
	III		126.0942	124.8235	123.6396	122.5431
0.5	I	0.5	7.7028	6.9607	6.2915	5.6904
	II		56.0175	54.1867	52.3998	50.6530
	III		167.9274	166.4062	165.0057	163.7272
	I	1.0	8.5266	7.7844	7.1152	6.5142
	II		57.0763	55.2758	53.5245	51.8194
	III		168.9116	167.3941	165.9978	164.7238
	I	2.0	9.9497	9.1850	8.4917	7.8646
	II		59.1376	57.3927	55.7066	54.0776
	III		170.8626	169.3525	167.9642	166.6991

of such plates has been solved using Chebyshev collocation method. This method is preferred because Chebyshev polynomials have minimax property, i.e., of all the monic polynomials, the maximum error is minimum (Jain et al. [19, p. 174]; Fox and Parker [20, p.5]). An approximation for deflection function involving fifteen terms gives an accuracy of four decimals for appreciable thickness variation, rigidity parameter, radii ratio and non-homogeneity parameter.

Table 6
Values of frequency parameter for C–F annular plate for $\varepsilon = 0.5$

α	Mode	p	μ			
			−0.5	0.0	0.5	1.0
−0.5	I	0.5	10.7766	10.0085	9.2869	8.6102
	II		62.2031	60.7645	59.3561	57.9760
	III		170.9384	169.5323	168.1848	166.8958
	I	1.0	11.0616	10.2949	9.5751	8.9003
	II		62.5383	61.1041	59.7007	58.3262
	III		171.2639	169.8587	168.5121	167.2242
	I	2.0	11.6091	10.8432	10.1245	9.4512
	II		63.2033	61.7778	60.3840	59.0205
	III		171.9129	170.5096	169.1649	167.8790
0.0	I	0.5	13.5494	12.5653	11.6440	10.7828
	II		86.5034	84.4936	82.5225	80.5877
	III		245.0760	243.2002	241.4098	239.7052
	I	1.0	14.0043	13.0243	12.1076	11.2516
	II		87.0329	85.0328	83.0727	81.1501
	III		245.5679	243.6940	241.9057	240.2033
	I	2.0	14.8672	13.8908	12.9782	12.1267
	II		88.0822	86.1011	84.1622	82.2633
	III		246.5486	244.6787	242.8946	241.1965
0.5	I	0.5	16.9729	15.7253	14.5614	13.4775
	II		119.9339	117.1241	114.3640	111.6502
	III		350.8259	348.3470	345.9931	343.7650
	I	1.0	17.7076	16.4700	15.3170	14.2450
	II		120.7835	117.9945	115.2574	112.5690
	III		351.5728	349.0977	346.7479	344.5240
	I	2.0	19.0781	17.8498	16.7067	15.6451
	II		122.4646	119.7158	117.0232	114.3841
	III		353.0619	350.5946	348.2529	346.0374

2. Mathematical formulation

Consider an annular plate of inner and outer peripheral radii b and a, respectively, referred to a cylindrical polar co-ordinates system (r, θ, z) with its axis as the line $r = 0$ and middle surface as the plane $z = 0$. The small deflection axisymmetric motion of such a homogeneous and polar orthotropic plate of thickness $h(r)$, density $\rho(r)$ is governed by the equation (see Eqs. (11.1) and

Table 7
Values of frequency parameter for C–C, C–S, C–F annular plates for $\mu = 0.5$, $p = 5.0$

	ε	α	Mode		
			I	II	III
C–C plate	0.3	−0.5	34.7619	93.2283	180.6749
		−0.3	39.6731	106.4243	206.1910
		−0.1	45.3324	121.5025	235.2405
		0.0	48.4804	129.8303	251.2375
		0.1	51.8635	138.7328	268.3023
		0.3	59.4120	158.4239	305.9189
		0.5	68.1497	180.9286	348.7035
	0.5	−0.5	62.5849	170.8874	333.6782
		−0.3	72.8088	198.7748	388.0767
		−0.1	84.7514	231.2257	451.2744
		0.0	91.4581	249.3918	486.6058
		0.1	98.7098	268.9887	524.6833
		0.3	115.0348	312.9353	609.9399
		0.5	134.1396	364.0804	708.9407
C–S plate	0.3	−0.5	24.4151	75.8457	156.0191
		−0.3	27.3934	86.1425	177.6146
		−0.1	30.7378	97.8282	202.1165
		0.0	32.5616	104.2486	215.5733
		0.1	34.4950	111.0872	229.9017
		0.3	38.7192	126.1276	261.3937
		0.5	43.4741	143.1836	297.0678
	0.5	−0.5	43.3777	138.5631	287.7549
		−0.3	49.8098	160.5632	334.0532
		−0.1	57.1896	186.0434	387.7171
		0.0	61.2770	200.2563	417.6158
		0.1	65.6542	215.5507	449.9031
		0.3	75.3603	249.7174	521.9467
		0.5	86.4872	289.2737	605.3895
C–F plate	0.3	−0.5	7.4504	35.1489	93.5567
		−0.3	8.1074	39.3299	105.9577
		−0.1	8.8333	44.0108	119.9608
		0.0	9.2253	46.5594	127.6252
		0.1	9.6385	49.2589	135.7677
		0.3	10.5342	55.1534	153.6054
		0.5	11.5329	61.7888	173.7290
	0.5	−0.5	11.6030	62.3890	171.1082
		−0.3	12.9352	71.3893	197.8298
		−0.1	14.4333	81.6710	228.6754
		0.0	15.2524	87.3482	245.8376
		0.1	16.1227	93.4160	264.2737
		0.3	18.0333	106.8333	305.3481
		0.5	20.2003	122.1628	352.7309

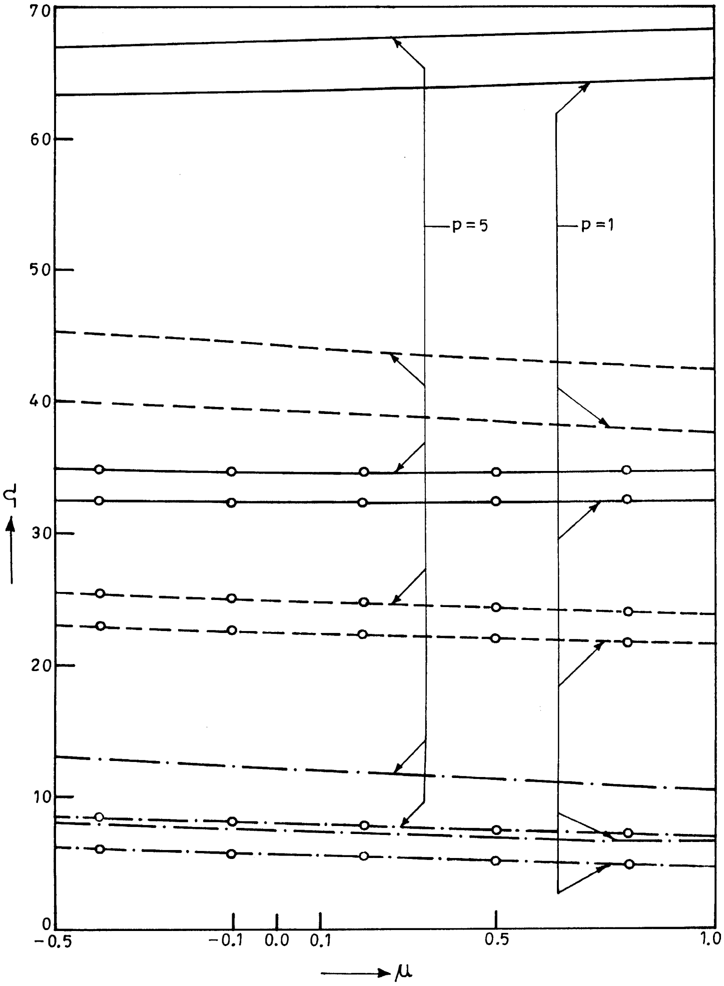

Fig. 2. Natural frequencies for annular plates vibrating in fundamental mode for $\varepsilon = 0.3$ and $\alpha = \pm 0.5$. C-C plate ——; C-S plate - - - - -; C-F plate —·—. $\alpha = 0.5$: ——, - - - - -, —·—; $\alpha = -0.5$: —o—, - -o- -, —·o—·—.

(A. 35) of Ref. [1])

$$D_r w_{,rrrr} + [2(D_r + rD_{r,r})/r]w_{,rrr} + [\{-D_\theta + (2+\nu_\theta) rD_{r,r} + r^2 D_{r,rr}\}/r^2]w_{,rr} + [\{D_\theta - rD_{\theta,r} + r^2 \nu_\theta D_{r,rr}\}/r^3]w_{,r} + \rho h w_{,tt} = 0, \quad (1)$$

where a comma followed by a suffix represents the partial differentiation with respect to that variable and $(D_r, D_\theta) = (E_r, E_\theta)h^3/12(1 - \nu_r\nu_\theta)$ are the flexural rigidities, w the transverse deflection, t the time, $E_r(r)$, $E_\theta(r)$, ν_r, ν_θ the elastic constants in proper directions with $\nu_r E_\theta = E_r \nu_\theta$.

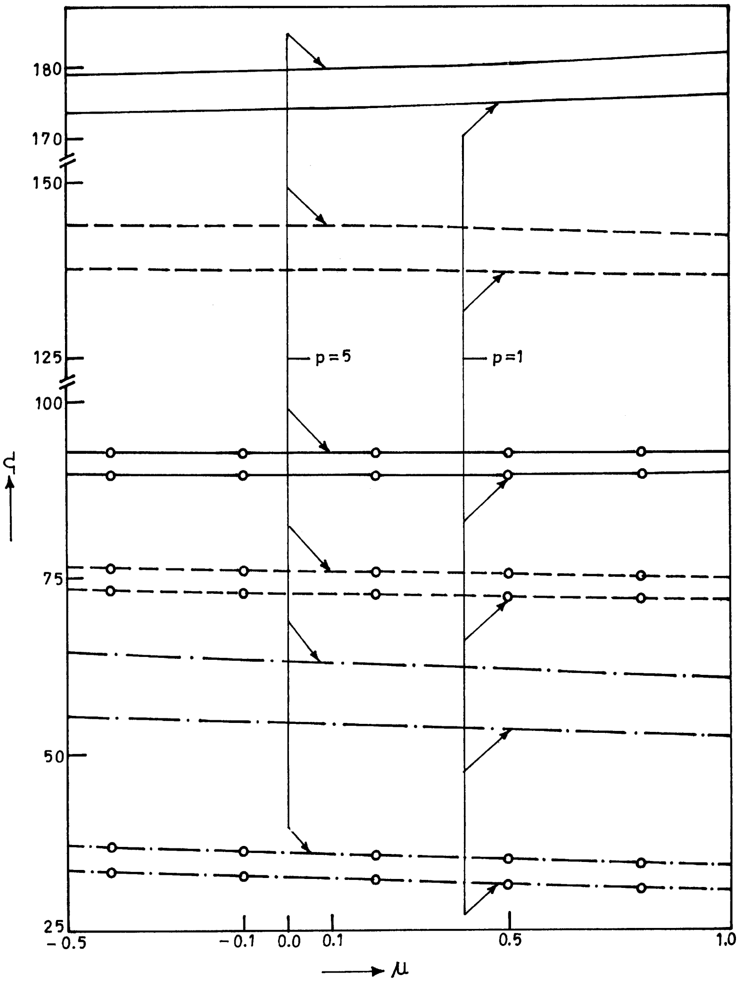

Fig. 3. Natural frequencies for annular plates vibrating in second mode for $\varepsilon = 0.3$ and $\alpha = \pm 0.5$. C-C plate ——; C-S plate -----; C-F plate —·—. $\alpha = 0.5$: ——, -----, —·—; $\alpha = -0.5$: —o—, --o--, —·o—·—.

Introducing the non-dimensional variables $x = r/a$, $\bar{w} = w/a$, $\bar{h} = h/a$ together with exponential variation in thickness along radial direction, i.e., $\bar{h} = h_0 e^{\alpha x}$ and following Refs. [3,4,6–9] for non-homogeneity of the material in radial direction as follows:

$$E_r = E_1 e^{\mu x}, \quad E_\theta = E_2 e^{\mu x}, \quad \rho = \rho_0 e^{\mu x}. \tag{2}$$

Eq. (1) now reduces to

$$P_0 W^{iv} + P_1 W''' + P_2 W'' + P_3 W' + P_4 W = 0, \tag{3}$$

where $\bar{w}(x,t) = W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_0, ρ_0 are the thickness and density of the plate at $x = 0$, μ is the non-homogeneity parameter, α is the taper

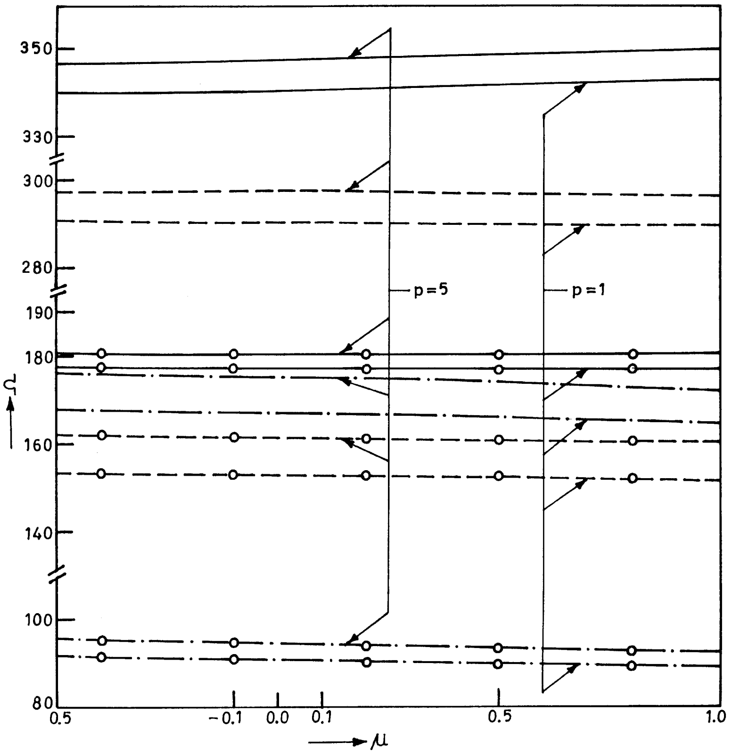

Fig. 4. Natural frequencies for annular plates vibrating in third mode for $\varepsilon = 0.3$ and $\alpha = \pm 0.5$. C-C plate ——; C-S plate -----; C-F plate —·—. $\alpha = 0.5$: ——, -----, —·—; $\alpha = -0.5$: —o—, --o--, —·o—·—.

parameter:

$$P_0 = 1, \;\; P_1 = 2\{1 + (\mu + 3\alpha)x\}/x,$$

$$P_2 = \{-p + (2 + \nu_\theta)(\mu + 3\alpha)x + (\mu + 3\alpha)^2 x^2\}/x^2,$$

$$P_3 = \{p - p(\mu + 3\alpha)x + \nu_\theta(\mu + 3\alpha)^2 x^2\}/x^3,$$

$$P_4 = -\Omega^2/\mathrm{e}^{2\alpha x}, \;\; p = E_2/E_1, \;\; \Omega^2 = 12\rho_0 a^2 \omega^2 (1 - \nu_r \nu_\theta)/(E_1 h_0^2)$$

and E_1, E_2 are moduli in radial and tangential directions at $x = 0$, respectively.

Eq. (3) together with the boundary conditions at the edges $x = \varepsilon$ and $x = 1$, where $\varepsilon = b/a$, constitutes a two-point boundary value problem in the range $(\varepsilon, 1)$ which has been solved by Chebyshev collocation technique.

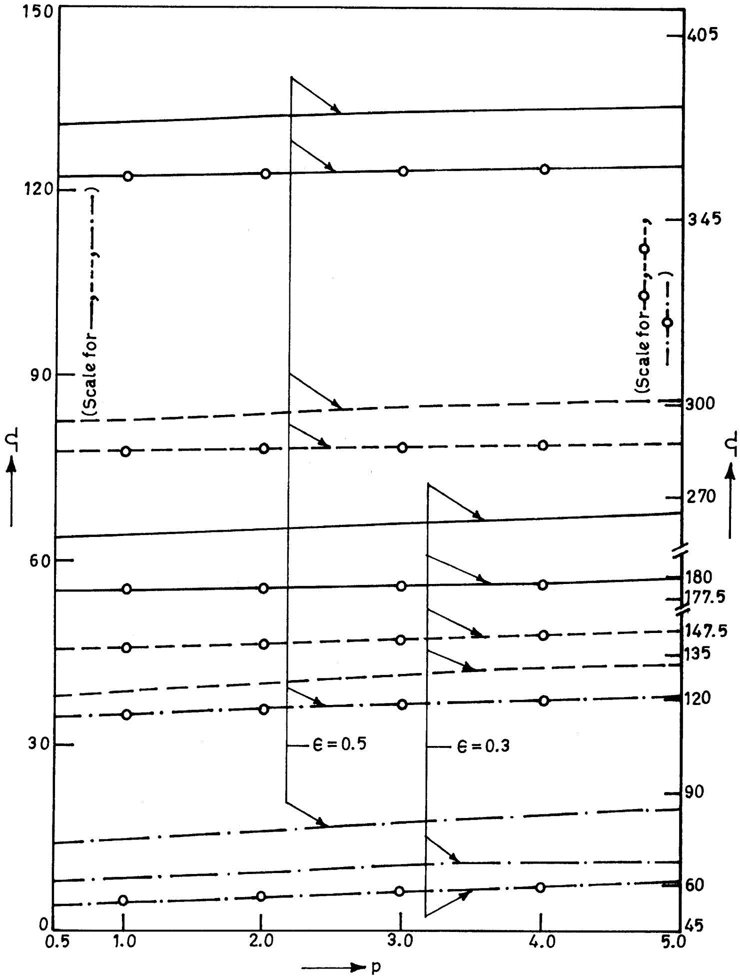

Fig. 5. Natural frequencies for annular plates vibrating in first two modes for $\mu = 0.5$ and $\alpha = 0.5$. C-C plate ——; C-S plate - - - - -; C-F plate —·—. Fundamental mode: ——, - - - - -, —·—; second mode: —o—, - -o- -, —·o—·—.

3. Method of solution

By taking a new independent variable

$$y \equiv \{2x - (1 + \varepsilon)\}/(1 - \varepsilon), \tag{4}$$

the range $\varepsilon \leqslant x \leqslant 1$ is transformed to $-1 \leqslant y \leqslant 1$, the applicability range of the technique. Eq. (3) is now reduced to

$$A_0 W^{iv} + A_1 W'''' + A_2 W'' + A_3 W' + A_4 W = 0, \tag{5}$$

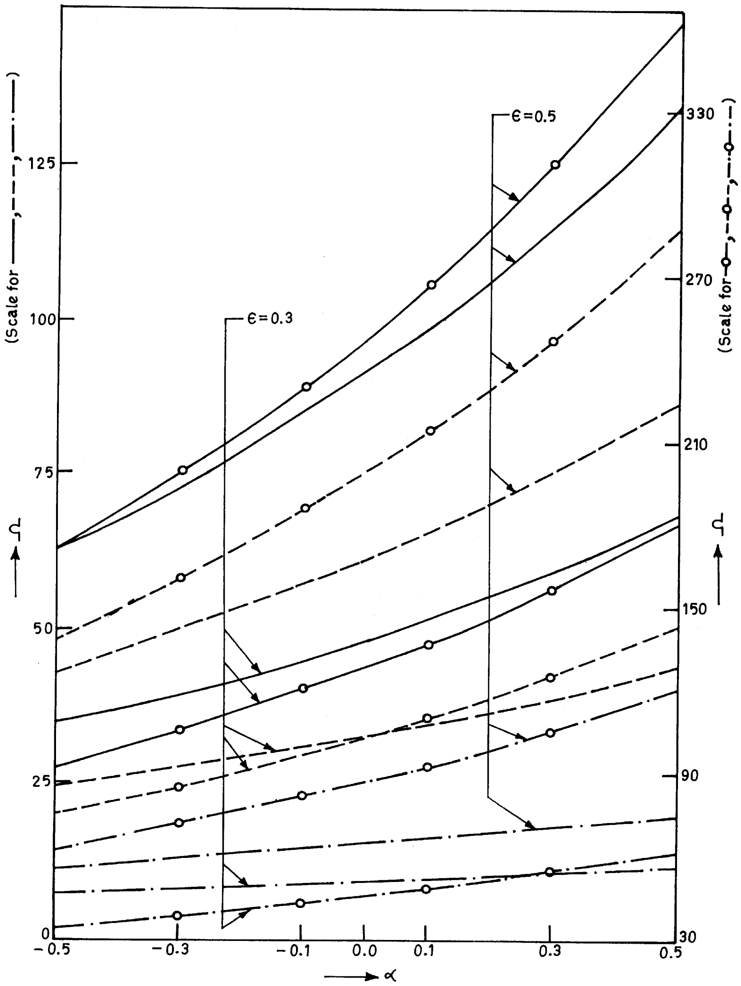

Fig. 6. Natural frequencies for annular plates vibrating in first two modes for $\mu = 0.5$ and $p = 5.0$. C-C plate ——; C-S plate - - - - -; C-F plate —·—. Fundamental mode: ——, - - - - -, —·—; second mode: —o—, - -o- -, —·o—·—.

where $A_i = \eta^{4-i} P_i$, $i = 0, 1, 2, 3, 4$ and $\eta = 2/(1 - \varepsilon)$. According to Chebyshev collocation method (Refs. [20–22]), we assume

$$\frac{d^4 W}{dy^4} = \sum_{k=0}^{m-5} c_{k+5} T_k \tag{6}$$

and its successive integrations lead to

$$W = c_1 + c_2 T_1 + c_3 T_1^1 + c_4 T_1^2 + \sum_{k=0}^{m-5} c_{k+5} \ T_k^4, \tag{7}$$

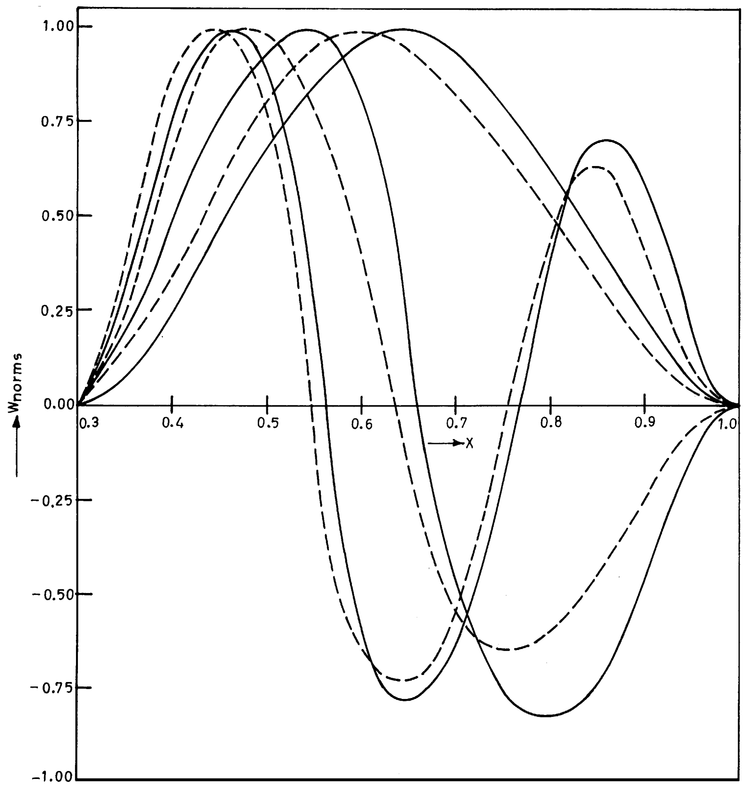

Fig. 7. Normalized displacements of C–C annular plate for the first three modes of vibration for $\varepsilon = 0.3$, $\mu = 0.5$, $p = 5.0$. $\alpha = -0.5$ ——; $\alpha = 0.5$ - - - - -.

where $c_j (j = 1, 2, \ldots, m)$ are unknown constants, $T_k (k = 0, 1, 2, \ldots, m-5)$ are Chebyshev polynomials and T_k^j represents the jth integral of T_k which are defined as

$$T_0^1 = T_1; \; T_1^1 = (T_2 + T_0)/4; \; T_j^1 = \int T_j \, \mathrm{d}y = [T_{J+1}/(j+1) - T_{j-1}/(j-1)]/2, \; j > 1;$$

$$T_j^i = \int T_j^{i-1} \, \mathrm{d}y; \; T_j = 2yT_{j-1} - T_{j-2}, \; T_1 = y, \; T_0 = 1.$$

Substitution of W and its derivatives in Eq. (5) gives an equation in terms of the T's and c's. The satisfaction of this resultant equation at $(m-4)$ collocation points given by

$$y_k = \cos\left(\frac{2k+1}{m-4}\frac{\pi}{2}\right) \quad k = 0, 1, 2, \ldots, m-5 \tag{8}$$

provides a set of $(m-4)$ equations in terms of the unknowns $c_j (j = 1, 2, \ldots, m)$, which can be written in matrix form as

$$[B][C^*] = [0], \tag{9}$$

where B and C^* are matrices of order $(m-4) \times m$ and $m \times 1$, respectively.

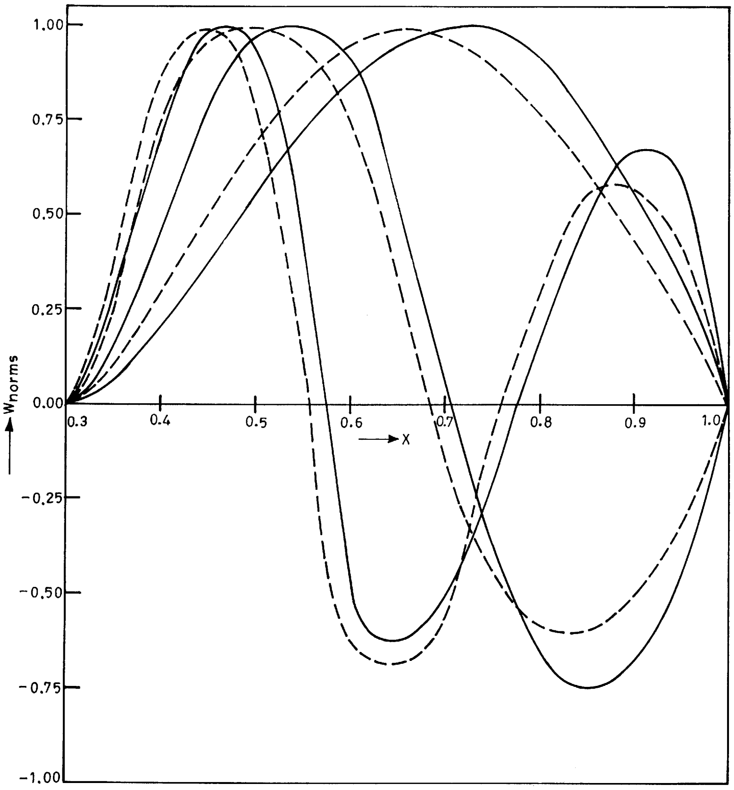

Fig. 8. Normalized displacements of C–S annular plate for the first three modes of vibration for $\varepsilon = 0.3$, $\mu = 0.5$, $p = 5.0$. $\alpha = -0.5$ ——; $\alpha = 0.5$ - - - - -.

4. Boundary conditions and frequency equations

By satisfying the relations $W = \mathrm{d}W/\mathrm{d}y = 0$, $W = \eta(\mathrm{d}^2W/\mathrm{d}y^2) + (v_\theta/x)(\mathrm{d}W/\mathrm{d}y) = 0$ and $\eta(\mathrm{d}^2W/\mathrm{d}y^2)+(v_\theta/x)(\mathrm{d}W/\mathrm{d}y)=\eta^2(\mathrm{d}^3W/\mathrm{d}y^3)+(\eta/x)(\mathrm{d}^2W/\mathrm{d}y^2)-(p/x^2)(\mathrm{d}W/\mathrm{d}y)=0$ for clamped, simply supported and free edge conditions, respectively, a set of four homogeneous equations are obtained for (i) C–C: both the inner and outer edges clamped, (ii) C–S: clamped at the inner edge and simply supported at the outer and (iii) C–F: clamped at the inner edge and free at the outer. These equations together with the field equation (9) give a complete set of m equations in m unknowns. For a C–C plate the frequency equation can be written as

$$\left[\frac{B}{B^{CC}}\right][C^*] = [0], \tag{10}$$

where B^{CC} is a matrix of order $4 \times m$.

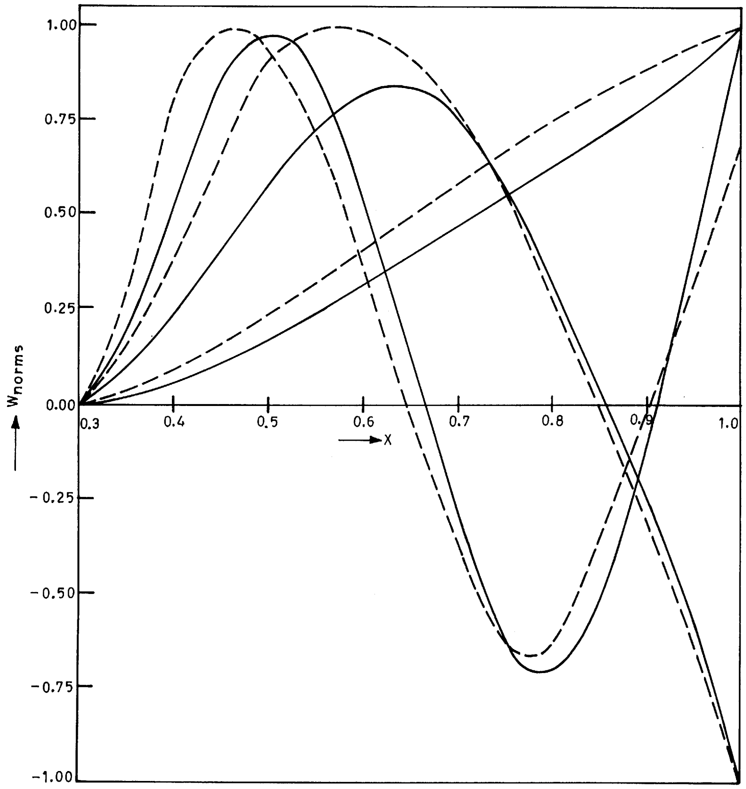

Fig. 9. Normalized displacements of C–F annular plate for the first three modes of vibration for $\varepsilon = 0.3$, $\mu = 0.5$, $p = 5.0$. $\alpha = -0.5$ ——; $\alpha = 0.5$ - - - - -.

For a non-trivial solution of Eq. (10), the frequency determinant must vanish and hence,

$$\left|\frac{B}{B^{CC}}\right| = 0. \tag{11}$$

Similarly for C–S and C–F plates frequency determinants can be written as

$$\left|\frac{B}{B^{CS}}\right| = 0, \tag{12}$$

$$\left|\frac{B}{B^{CF}}\right| = 0, \tag{13}$$

respectively.

5. Numerical results and discussions

The frequency equations (11)–(13) provide the values of the frequency parameter Ω for various values of plate parameters. In the work reported here, the first three natural frequencies of

Table 8
Comparison of frequency parameter Ω for, homogeneous ($\mu = 0$) annular plates of uniform ($\alpha = 0$) thickness for $\varepsilon = 0.3$, $v_\theta = 0.3$

	Mode	$p = 1.0$	$p = 1.0$	$p = 5.0$	$p = 5.0$
C–C	I	45.3462 45.2[b]	45.3371[a]	48.3540	48.3321[a]
	II	125.3621 125[b]	125.6191[a]	129.6030	129.8250[a]
	III	246.1563	246.6994[a]	250.9695	251.4816[a]
C–S	I	29.9777 29.9[b]	29.9689[a]	33.2694	33.2528[a]
	II	100.4228 100[b]	100.6065[a]	104.7699	104.9319[a]
	III	211.1291	211.5629[a]	216.3488	216.4574[a]
C–F	I	6.6604 6.66[b]	6.6542[a]	9.9115	9.9073[a]
	II	42.6141 42.6[b]	42.6156[a]	47.8284	47.8100[a]
	III	123.4662	123.5739[a]	128.7846	128.8986[a]

[a] Values taken from Verma [23].
[b] Values taken from Leissa [1]: Table 2.18 (C–C); Table 2.24 (C–S); Table 2.30 (C–F).

vibration have been computed for non-homogeneity parameter $\mu(= -0.5, -0.1, 0.0, 0.1, 0.5, 1.0)$, radii ratio $\varepsilon(= 0.3, 0.5)$, rigidity parameter $p(= 0.5, 1.0, 2.0, 5.0)$, taper constant $\alpha(= -0.5(0.2)0.5, 0.0)$ for $v_\theta = 0.3$ and for all the three boundary conditions. The numerical values showed a consistent improvement with the increase of the number of collocation points. In all the computations, the number of collocation points has been taken as $m = 15$, since further increase in m does not improve the results except at the fourth place of decimal (Fig. 1). The value of the thickness h_0 at the origin has been taken as 0.1.

The numerical results are presented in Tables (1–7) and Figs. 2–9. Fig. 2 shows the plots for Ω versus μ for $\varepsilon = 0.3$, $\alpha = \pm 0.5$ and two values of rigidity ratio $p = 1.0$, 5.0 for all the three boundary conditions for the fundamental mode of vibration. It is observed that frequency parameter decreases with the increasing values of non-homogeneity parameter μ in case of C–S and C–F plates for the same set of plate parameters. While it is just the reverse in case of C–C plate except for $\alpha < 0$. In this case the frequency parameter first decreases and then increases slightly with the increasing values of non-homogeneity parameter μ and thus give rise a local minima in the vicinity of $\mu = 0.5$. This may be attributed to the increased mass/radial stiffness of the plate with the increasing values of non-homogeneity parameter μ and clamped edge condition at the outer boundary which causes an additional constraint. A similar behaviour is observed from Figs. 3 and 4 when the plate is vibrating in the second and third modes, respectively except that the rate of change of Ω with μ increases for C–C plate when $\alpha \geqslant 0$ and decreases for C–S plate with the increase in the number of modes. This rate of decrease for C–F plate increases for both the modes; however it is more pronounced in the second mode as compared to the third mode.

Fig. 5 shows the effect of rigidity ratio p on the frequency parameter Ω for the first two modes of vibration for two values of radii ratio $\varepsilon = 0.3$, 0.5, $\mu = 0.5$ and $\alpha = 0.5$. The frequencies are found to increase as the plate becomes more and more stiff in the tangential direction ($p > 1$) as compared to radial direction ($p < 1$). Thus the effect of orthotropy is found to increase the frequencies keeping all other plate parameters fixed. This effect is more dominant in case of C–C plate as compared to C–S and C–F plates for both the modes. Fig. 6 shows the effect of taper parameter α on the frequency parameter Ω for $\varepsilon = 0.3$, 0.5, $\mu = 0.5$ and $p = 5.0$ for plates vibrating in fundamental/ second mode. The frequency parameter Ω is found to increase with increasing values of the taper parameter α. The rate of increase of Ω for $\alpha > 0$ is higher as compared to $\alpha < 0$ for all the three boundary conditions. This rate of increase reduces in the order of boundary conditions C–C, C–S, C–F but increases with the increase in the number of modes for same set of other plate parameters. Figs. 7–9 show the plots for normalized displacements for $\varepsilon = 0.3$, $\mu = 0.5$, $p = 5.0$ and $\alpha = \pm 0.5$ for the first three modes of vibration. The nodal circles are seen to shift towards the inner periphery as the outer edge becomes thicker and thicker for all the three boundary conditions. Table 8 shows a comparison of results for homogeneous ($\mu = 0.0$) isotropic ($p = 1$) and orthotropic ($p = 5$) plates of uniform thickness ($\alpha = 0$) for $\varepsilon = 0.3$ with those of Verma [23], obtained by quintic spline technique and exact solutions given by Leissa [1] for isotropic plates. A close agreement of the results shows the versatility of the Chebyshev collocation technique.

6. Conclusions

The effect of non-homogeneity together with that of material orthotropy on the natural frequencies of annular plates of exponentially varying thickness has been studied on the basis of the classical plate theory. It is observed that the frequency parameter decreases with the increasing values of non-homogeneity parameter in case of C–S and C–F plates while it is just the reverse in the case of C–C plate except for the thin plates ($\alpha < 0$) towards the outer edge. In this case there is a local minima in the vicinity of non-homogeneity parameter $\mu = 0.5$. The effect of orthotropy is found to increase the frequencies keeping all other plate parameters fixed. The increase/decrease in the radii ratio, i.e., hole size of the annular plate also increases/decreases the frequency parameter for all the three boundary conditions. The frequency parameter further increases with increasing values of the taper parameter. Thus a desired frequency can be obtained by a proper choice of the various plate parameters considered here which will be helpful to design engineers.

Acknowledgements

The authors are grateful to the learned reviewers for their constructive comments.

Appendix A. Nomenclature

a, b	outer and inner peripheral radii of the plate, respectively
r, θ	polar co-ordinates of a point in the mid plane of the plate
ρ	$\rho(r)$, mass density per unit volume
h	$h(r)$, thickness of the plate
h_o, ρ_o	non-dimensional thickness and density of the plate at the centre, respectively
w	$w(r,t)$, displacement function
$w_{,r}$	$\partial w/\partial r$
t	time
$w_{,t}$	$\partial w/\partial t$
E_r	$E_r(r)$, Young's modulus in radial direction
E_θ	$E_\theta(r)$, Young's modulus in tangential direction
D_r	$E_r h^3/12(1 - v_r v_\theta)$, flexural rigidity in radial direction
D_θ	$E_\theta h^3/12(1 - v_r v_\theta)$, flexural rigidity in tangential direction
v_r	Poisson's ratio defined as strain in tangential direction due to unit strain in radial direction
v_θ	Poisson's ratio defined as strain in radial direction due to unit strain in tangential direction
E_1, E_2	non-dimensional Young's moduli in radial and tangential directions at $x = 0$, respectively
p	E_2/E_1 , rigidity ratio
ε	b/a, ratio of inner and outer radii
x	r/a, non-dimensional independent variable in the range $(\varepsilon, 1)$
α	taper parameter
μ	non-homogeneity parameter
ω	frequency in radians per second
Ω^2	$12\ \rho_o a^2 \omega^2 (1 - v_r v_\theta)/E_1 h_o^2$, non-dimensional frequency parameter
η	$2/(1-\varepsilon)$
y	independent variable in the range $(-1,1)$, related to x by Eq. (4)
m	a positive integer, the number of collocation points
$c_1, c_2, \ldots, c_m$	unknown constants in Eq. (7)
$T_1, T_2, \ldots, T_m$	Chebyshev polynomials
$y_0, y_1, \ldots, y_{m-5}$	zeros of the Chebyshev polynomial T_{m-4} in Eq. (8)
T_k^j	jth integral of T_k
B, C^*, B^{CC}, B^{CS}, B^{CF}	denote matrices of different orders
C, S, F	denote clamped, simply supported and free peripheries, respectively

References

[1] A.W. Leissa, *Vibration of Plates*, NASA SP-160, 1969.
[2] R.K. Bose, Note on forced vibration of a thin nonhomogeneous circular plate with central hole, *Indian Journal of Physics* 41 (1967) 886–890.

[3] S.K. Biswas, Note on the torsional vibration of a finite circular cylinder of nonhomogeneous material by a particular type of twist on one of the plane surface, *Indian Journal of Physics* 43 (1969) 320–323.

[4] N.A. Rostovtsev, I.E. Khranevskaia, The plane problem for a nonhomogeneous elastic circular domain, *Journal of Applied Mechanics* 37 (1973) 95–102.

[5] S. Chakraverty, M. Petyt, Vibration of nonhomogeneous plates using two-dimensional orthogonal polynomials as shape functions in the Rayleigh–Ritz method, *Journal of Mechanical Engineering Science* 213 (1999) 707–714.

[6] J.S. Tomar, D.C. Gupta, N.C. Jain, Vibrations of nonhomogeneous plates of variable thickness, *Journal of the Acoustical Society of America* 72 (1982) 851–855.

[7] J.S. Tomar, D.C. Gupta, N.C. Jain, Axisymmetric vibrations of an isotropic elastic nonhomogeneous circular plate of linearly varying thickness, *Journal of Sound and Vibration* 85 (1982) 365–370.

[8] J.S. Tomar, D.C. Gupta, N.C. Jain, Free vibrations of an isotropic nonhomogeneous infinite plate of linearly varying thickness, *Meccanica* 18 (1983) 30–33.

[9] J.S. Tomar, D.C. Gupta, N.C. Jain, Free vibrations of an isotropic nonhomogeneous infinite plate of parabolically varying thickness, *Indian Journal of Pure and Applied Mathematics* 15 (1984) 211–220.

[10] I. Elishakoff, Axisymmetric vibration of inhomogeneous clamped circular plates: an unusual closed-form solution, *Journal of Sound and Vibration* 233 (2000) 727–738.

[11] I. Elishakoff, Axisymmetric vibration of inhomogeneous free circular plates: and unusual exact, closed-form solution, *Journal of Sound and Vibration* 234 (2000) 167–170.

[12] A.W. Leissa, Plate vibration research 1976-1980: complicating effects, *The Shock and Vibration Digest* 13 (1981) 19–36.

[13] A.W. Leissa, Recent studies in plate vibrations: 1981–85 part II complicating effects, *The Shock and Vibration Digest* 19 (1987) 10–24.

[14] C.W. Bert, F.W. Niedenfuhr, Stretching of a polar-orthotropic disk of varying thickness under arbitrary body forces, *American Institute of Aeronautics and Astronautics Journal* 1 (1963) 1385–1390.

[15] S.G. Lekhnitskii, *Anisotropic Plates*, Gordon and Breach Science Publishers, New York, 1968.

[16] N.M. Patsalides, T.J. Kim, Effect of nonhomogeneity in orthotropic curved plates, *American Institute of Aeronautics and Astronautics Journal* 17 (1979) 1276–1278.

[17] J.A. Masad, Free vibrations of a nonhomogeneous rectangular membrane, *Journal of Sound and Vibration* 195 (1996) 674–678.

[18] M. Jabareen, M. Eisenberger, Free vibrations of nonhomogeneous circular and annular membranes, *Journal of Sound and Vibration* 240 (2001) 409–429.

[19] M.K. Jain, S.R.K. Iyengar, R.K. Jain, *Numerical Methods for Scientific and Engineering Computation*, 2nd Edition, Wiley Eastern Limited, New Delhi, 1987.

[20] L. Fox, I.B. Parker, *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, Oxford, 1968.

[21] M.A. Snyder, *Chebyshev Methods in Numerical Approximations*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1969.

[22] R. Lal, U.S. Gupta, Axisymmetric vibrations of polar orthotropic annular plates of variable thickness, *Journal of Sound and Vibration* 83 (1982) 229–240.

[23] C.P. Verma, Some Problems in Vibration of Plates, Ph.D.Thesis, University of Roorkee, India, 1987.