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# Free vibration analysis of composite rectangular membranes with a bent interface

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**Abstract**

This paper is an extension of the author's previous work that dealt with a composite rectangular membrane with an oblique interface. In the present study, a practical method is presented for the free vibration analysis of a composite rectangular membrane with two homogeneous regions, of which the interface consists of two rectilinear parts and is named 'bent interface'. To the author's best knowledge, the vibration analysis of the composite membrane with this configuration is attempted for the first time in the paper. In order to extract the global system matrix of which the determinant gives natural frequencies, a special way of individually considering the two rectilinear parts of the bent interface and extracting local system matrices by means of applying the compatibility condition to each rectilinear part is revised. Case studies show that the natural frequencies and mode shapes obtained by the present method agree well with those given by exact solutions or FEM (ANSYS), even when a small number of base functions are used.


## 1. Introduction

The author proposed an effective method for the free vibration analysis of a composite rectangular membrane with two homogeneous regions, of which the interface is oblique against any of the four edges of the membrane [1]. In the previous work, an eigensolution for each homogeneous region was first obtained by considering fixed boundary conditions at boundary edges except the interface. Then, the frequency equation that gives natural frequencies was obtained from the compatibility conditions (conditions of continuity in displacement and slope) given at the interface. It may be said that the previous method is effective to solve a composite

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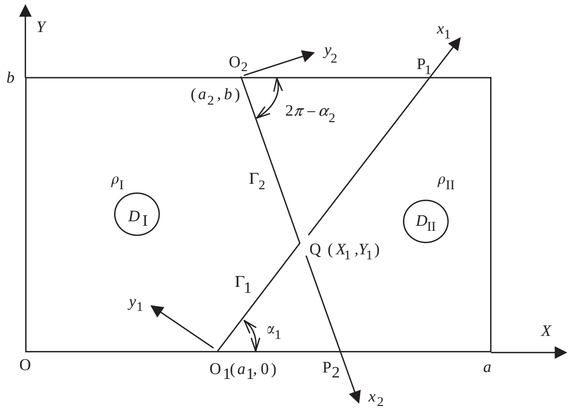


Fig. 1. Composite rectangular membrane with rectilinear interfaces $\Gamma_1$ and $\Gamma_2$ between two homogeneous regions $D_I$ and $D_{II}$.

membrane with *a rectilinear interface* in that it has simplicity in its theoretical development, unlike the finite element method [2] and the boundary element method [3,4]. However, the method may not be essentially extended immediately to a composite rectangular membrane with a bent interface, which is explained in Fig. 1.

This paper introduces a practical method for obtaining the natural frequencies and mode shapes of a composite rectangular membrane with two homogeneous regions, of which the interface is composed of two rectilinear parts corresponding to $\Gamma_1$ and $\Gamma_2$ in Fig. 1. The approximate solution of each homogeneous region for the free transverse vibration of the membrane is assumed by linear superposition of wave-type base functions, which satisfy the governing differential equation and some of four fixed boundary conditions given at the four edges of the membrane. A global system matrix, of which the determinant yields the natural frequencies, is obtained by assembling local system matrices, which are extracted by applying the compatibility condition to each rectilinear part. Prior to considering the compatibility condition, the displacement and slope shapes at each rectilinear interface are approximated by linearly superposing sine series functions. In particular, the sum of the number of the series functions used along interface $\Gamma_1$ and that used along interface $\Gamma_2$ is adjusted so as to be the same as the number of the base functions used for the approximate solution. By this adjustment, the global system matrix becomes a square matrix and, as a result, its determinant can be obtained.

The present method is simple in theoretical formulation and requires only a small amount of numerical calculation. Thanks to this feature, the method gives accurate results (natural frequencies and mode shapes) even when a small number of base functions are used. Although many engineering applications have dealt with a great variety of composite membranes of simple geometric shapes such as rectangular, circular and annular membranes [5–24], a survey of the open literature performed by the author reveals that no previous researcher has studied composite rectangular membranes with the bent interface considered in this paper.

Most investigators have mainly tackled the study of non-homogeneous membranes with stepped density [6–8,12–15,18] and continuously varying density [9–11,16,17,19–22,24,25]. Especially for non-homogeneous membranes with stepped density, only a simple case that interfaces between homogeneous regions are parallel to one of fixed edges has been dealt with.

Unlike the previous literature with this limitation, the paper considers a non-homogeneous rectangular membrane with two homogeneous regions of which the interface is not parallel to any of four fixed edges.

## 2. Theoretical formulation

### 2.1. Assumption of eigensolutions

A sketch of a composite rectangular membrane with a bent interface, global co-ordinate system $(X, Y)$ and local co-ordinate systems $(x_1, y_1)$ and $(x_2, y_2)$ is shown in Fig. 1. The composite rectangular membrane consists of two homogeneous regions $D_{\mathrm{I}}$ and $D_{\mathrm{II}}$, of which the bent interface is indicated by $\Gamma_1$ and $\Gamma_2$. In the same manner as in the author's previous paper [1], the approximate solutions of the two regions for the free transverse vibration are assumed by linear superposition of wave-type base functions: i.e.,

$$W_{\mathrm{I}}(X, Y) = \sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin[k_{\mathrm{I}} X] \sin[m\pi Y/b], \tag{1}$$

$$W_{\mathrm{II}}(X, Y) = \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin[k_{\mathrm{II}}(a - X)] \sin[n\pi Y/b], \tag{2}$$

where $k_{\mathrm{I}} = \sqrt{(\omega/c_{\mathrm{I}})^2 - (m\pi/b)^2}$ and $k_{\mathrm{II}} = \sqrt{(\omega/c_{\mathrm{II}})^2 - (n\pi/b)^2}$, expressed by the angular frequency $\omega = 2\pi f$, and the speed of wave propagation, $c_i = \sqrt{T/\rho_i}$, using the tension per unit length $T$ and the surface density given by $\rho_{\mathrm{I}}$ or $\rho_{\mathrm{II}}$. Eqs. (1) and (2) satisfy the governing differential equation (Helmholtz equation): i.e.,

$$\nabla^2 W_{\mathrm{I}} + k_{\mathrm{I}}^2 W_{\mathrm{I}} = 0, \quad \nabla^2 W_{\mathrm{II}} + k_{\mathrm{II}}^2 W_{\mathrm{II}} = 0, \tag{3,4}$$

and some of the fixed boundary conditions given at the four edges of the membrane as follows.

$$W_{\mathrm{I}}(X = 0) = W_{\mathrm{I}}(Y = 0) = W_{\mathrm{I}}(Y = b) = 0, \tag{5}$$

$$W_{\mathrm{II}}(X = a) = W_{\mathrm{II}}(Y = 0) = W_{\mathrm{II}}(Y = b) = 0. \tag{6}$$

### 2.2. Compatibility conditions and system matrix

In order for the approximate solutions (Eqs. (1) and (2)) to become eigensolutions for the free vibration of the composite rectangular membrane, it is required that $W_{\mathrm{I}}(X, Y)$ and $W_{\mathrm{II}}(X, Y)$ satisfy the compatibility conditions (the conditions of continuity in displacement and slope) at two interfaces $\Gamma_1$ and $\Gamma_2$ [26,27]: i.e.,

$$W_{\mathrm{I}}|_{\Gamma_1} = W_{\mathrm{II}}|_{\Gamma_1}, \partial W_{\mathrm{I}}/\partial n_1\big|_{\Gamma_1} = \partial W_{\mathrm{II}}/\partial n_1\big|_{\Gamma_1}, \tag{7,8}$$

$$W_{\mathrm{I}}|_{\Gamma_2} = W_{\mathrm{II}}|_{\Gamma_2}, \partial W_{\mathrm{I}}/\partial n_2\big|_{\Gamma_2} = \partial W_{\mathrm{II}}/\partial n_2\big|_{\Gamma_2}, \tag{9,10}$$

where $n_1$ and $n_2$ represent the normal directions from $\Gamma_1$ and $\Gamma_2$, respectively.

Before applying the compatibility conditions Eqs. (7)–(10) to the approximate solutions Eqs. (1) and (2), the local co-ordinate systems $(x_1, y_1)$ and $(x_2, y_2)$ are, respectively, defined for two interfaces $\Gamma_1$ and $\Gamma_2$ as shown in Fig. 1. Then, relationships between the global co-ordinate system and the local co-ordinate systems are given by

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} p_i & -q_i \\ q_i & p_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} + \begin{Bmatrix} a_i \\ b_i \end{Bmatrix}, \quad i = 1 \text{ or } 2, \tag{11}$$

where $p_i = \cos\alpha_i$ and $q_i = \sin\alpha_i$ ($\alpha_i$ denotes the angle between the $X$-axis and the $x_i$-axis as shown in Fig. 1); $(a_1, b_1) = (a_1, 0)$ and $(a_2, b_2) = (a_2, b)$, which correspond to the origins ($O_1$ and $O_2$) of local co-ordinate systems $(x_1, y_1)$ and $(x_2, y_2)$, respectively.

*2.2.1. Compatibility conditions at the rectilinear interfaces ($\Gamma_1$ and $\Gamma_2$)*

By the use of Eq. (11), the approximate solutions $W_{\mathrm{I}}(X, Y)$ and $W_{\mathrm{II}}(X, Y)$ are expressed with the local co-ordinate system $(x_i, y_i)$, respectively: i.e.,

$$W_{\mathrm{I}}(x_i, y_i) = \sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin[k_{\mathrm{I}}(p_i x_i - q_i y_i + a_i)] \sin[m\pi(q_i x_i + p_i y_i + b_i)/b], \tag{12}$$

$$W_{\mathrm{II}}(x_i, y_i) = \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin[k_{\mathrm{II}}(a - p_i x_i + q_i y_i - a_i)] \sin[n\pi(q_i x_i + p_i y_i + b_i)/b]. \tag{13}$$

Eqs. (7)-(10) are also expressed using the local co-ordinate system $(x_i, y_i)$: i.e.,

$$W_{\mathrm{I}}(x_i, y_i = 0) = W_{\mathrm{II}}(x_i, y_i = 0), \quad i = 1 \text{ or } 2, \tag{14}$$

$$\partial W_{\mathrm{I}}/\partial y_i(x_i, y_i = 0) = \partial W_{\mathrm{II}}/\partial y_i(x_i, y_i = 0), \quad i = 1 \text{ or } 2, \tag{15}$$

Substituting Eqs. (12) and (13) into Eqs. (14) and (15) gives

$$\begin{aligned} &\sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin[k_{\mathrm{I}}(p_i x_i + a_i)] \sin[m\pi(q_i x_i + b_i)/b] \\ &\quad = \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin[k_{\mathrm{II}}(a - p_i x_i - a_i)] \sin[n\pi(q_i x_i + b_i)/b] \\ &\quad i = 1 \text{ or } 2, \end{aligned} \tag{16}$$

$$\begin{aligned} &\sum_{m=1}^{N} A_m^{(\mathrm{I})} \frac{\partial}{\partial y_i} \{\sin[k_{\mathrm{I}}(p_i x_i + a_i)] \sin[m\pi(q_i x_i + b_i)/b]\} \\ &\quad = \sum_{n=1}^{N} A_n^{(\mathrm{II})} \frac{\partial}{\partial y_i} \{\sin[k_{\mathrm{II}}(a - p_i x_i - a_i)] \sin[n\pi(q_i x_i + b_i)/b]\} \\ &\quad i = 1 \text{ or } 2. \end{aligned} \tag{17}$$

In order to remove geometric variable $x_i$ involved in Eqs. (16) and (17), the $s$th basis $\sin s\pi x_i/\tilde{L}_i$ where $\tilde{L}_i$ denotes the length of segment $\overline{O_i P_i}$ is multiplied to both sides of Eqs. (16) and (17) and an integration is performed along the common interface $\Gamma_i$ of length $L_i$. Then, Eqs. (16) and (17) lead to, respectively,

$$\sum_{m=1}^{N} SM_{sm}^{(\mathrm{I})(i)} A_{\mathrm{m}}^{(\mathrm{I})} = \sum_{n=1}^{N} SM_{sn}^{(\mathrm{II})(i)} A_{n}^{(\mathrm{II})}, s = 1, 2, \ldots, N_i, i = 1 \text{ or } 2, \tag{18}$$

$$\sum_{m=1}^{N} VM_{sm}^{(\mathrm{I})(i)} A_{m}^{(\mathrm{I})} = \sum_{n=1}^{N} VM_{sn}^{(\mathrm{II})(i)} A_{n}^{(\mathrm{II})}, s = 1, 2, \ldots, N_i, i = 1 \text{ or } 2, \tag{19}$$

where $SM_{sm}^{(\mathrm{I})(i)}$, $SM_{sn}^{(\mathrm{I})(i)}$, $VM_{sm}^{(\mathrm{I})(i)}$ and $VM_{sn}^{(\mathrm{II})(i)}$ are given by

$$SM_{sm}^{(\mathrm{I})(i)} = \int_0^{L_i} \sin[k_{\mathrm{I}}(p_i x_i + a_i)] \sin[m\pi(q_i x_i + b_i)/b] \sin[s\pi x_i/\tilde{L}_i]\,\mathrm{d}x_i, \tag{20}$$

$$SM_{sn}^{(\mathrm{II})(i)} = \int_0^{L_i} \sin[k_{\mathrm{II}}(a - p_i x_i - a_i)] \sin[n\pi(q_i x_i + b_i)/b] \sin[s\pi x_i/\tilde{L}_i]\,\mathrm{d}x_i, \tag{21}$$

$$\begin{aligned} VM_{sm}^{(\mathrm{I})(i)} = \int_0^{L_i} & \{-q_i k_{\mathrm{I}} \cos[k_{\mathrm{I}}(p_i x_i + a_i)] \sin[m\pi(q_i x_i + b_i)/b] \\ & + (m\pi \rightleftarrows p_i/b) \sin[k_{\mathrm{I}}(p_i x_i + a_i)] \cos[m\pi(q_i x_i + b_i)/b]\} \sin[s\pi x_i/\tilde{L}_i]\,\mathrm{d}x_i, \end{aligned} \tag{22}$$

$$\begin{aligned} VM_{sn}^{(\mathrm{II})(i)} = \int_0^{L_i} & \{q_i k_{\mathrm{II}} \cos[k_{\mathrm{II}}(a - p_i x_i - a_i)] \sin[n\pi(q_i x_i + b_i)/b] \\ & + (n\pi p_i/b) \sin[k_{\mathrm{II}}(a - p_i x_i - a_i)] \cos[n\pi(q_i x_i + b_i)/b]\} \sin[s\pi x_i/\tilde{L}_i]\,\mathrm{d}x_i. \end{aligned} \tag{23}$$

Note that the integration procedures in Eqs. (20)–(23) are performed to eliminate the geometric variable $x_i$, which results from the fact that $\Gamma_i$ is oblique against both the $X$-axis and the $Y$-axis.

### 2.2.2. Extraction of a global system matrix from local system matrices

For simplicity, Eqs. (18) and (19) are rewritten in the matrix forms

$$\mathbf{SM}^{(\mathrm{I})(1)}\mathbf{A}^{(\mathrm{I})} = \mathbf{SM}^{(\mathrm{II})(1)}\mathbf{A}^{(\mathrm{II})}, \quad \mathbf{VM}^{(\mathrm{I})(1)}\mathbf{A}^{(\mathrm{I})} = \mathbf{VM}^{(\mathrm{II})(1)}\mathbf{A}^{(\mathrm{II})}, \quad \text{for } i = 1 \text{ (at } \Gamma_1), \tag{24, 25}$$

$$\mathbf{SM}^{(\mathrm{I})(2)}\mathbf{A}^{(\mathrm{I})} = \mathbf{SM}^{(\mathrm{II})(2)}\mathbf{A}^{(\mathrm{II})}, \quad \mathbf{VM}^{(\mathrm{I})(2)}\mathbf{A}^{(\mathrm{I})} = \mathbf{VM}^{(\mathrm{II})(2)}\mathbf{A}^{(\mathrm{II})}, \quad \text{for } i = 2 \text{ (at } \Gamma_2), \tag{26, 27}$$

where the sizes of $\mathbf{SM}^{(\mathrm{D})(i)}$ and $\mathbf{VM}^{(\mathrm{D})(i)}$ for $D = \mathrm{I}$ or II and $i = 1$ or 2 are commonly $N \times N_i$ and the size of $\mathbf{A}^{(D)}$ for $D = \mathrm{I}$ or II is $N \times 1$. Next, Eqs. (24) and (26), related to the condition of continuity in displacement, are assembled in the single matrix equation

$$\mathbf{SM}^{(\mathrm{I})}\mathbf{A}^{(\mathrm{I})} = \mathbf{SM}^{(\mathrm{II})}\mathbf{A}^{(\mathrm{II})}, \tag{28}$$

where $\mathbf{SM}^{(\mathrm{I})}$ and $\mathbf{SM}^{(\mathrm{II})}$ are

$$\mathbf{SM}^{(\mathrm{I})} = \begin{bmatrix} \mathbf{SM}^{(\mathrm{I})(1)} \\ \mathbf{SM}^{(\mathrm{I})(2)} \end{bmatrix}, \mathbf{SM}^{(\mathrm{II})} = \begin{bmatrix} \mathbf{SM}^{(\mathrm{II})(1)} \\ \mathbf{SM}^{(\mathrm{II})(2)} \end{bmatrix}, \tag{29, 30}$$

and their sizes are commonly $(N_1 + N_2) \times N$. Similarly, Eqs. (25) and (27), related to the condition of continuity in slope, are assembled in the single matrix equation

$$\mathbf{VM}^{(\mathrm{I})}\mathbf{A}^{(\mathrm{I})} = \mathbf{VM}^{(\mathrm{II})}\mathbf{A}^{(\mathrm{II})}, \tag{31}$$

where $\mathbf{VM}^{(\mathrm{I})}$ and $\mathbf{VM}^{(\mathrm{II})}$ are

$$\mathbf{VM}^{(\mathrm{I})} = \begin{bmatrix} \mathbf{VM}^{(\mathrm{I})(1)} \\ \mathbf{VM}^{(\mathrm{I})(2)} \end{bmatrix}, \quad \mathbf{VM}^{(\mathrm{II})} = \begin{bmatrix} \mathbf{VM}^{(\mathrm{II})(1)} \\ \mathbf{VM}^{(\mathrm{II})(2)} \end{bmatrix}, \tag{32, 33}$$

and their sizes are also commonly $(N_1 + N_2) \times N$.

Finally, a global system equation of the composite rectangular membrane is obtained by assembling Eqs. (28) and (31) into a single matrix equation as follows.

$$\mathbf{SM}(f)\mathbf{A} = \mathbf{0}, \tag{34}$$

where the global system matrix $\mathbf{SM}(f)$ of size $2N \times 2(N_1 + N_2)$ and the unknown coefficient vector $\mathbf{A}$ of size $2N \times 1$ are, respectively, given by

$$\mathbf{SM}(f) = \begin{bmatrix} \mathbf{SM}^{(\mathrm{I})} & -\mathbf{SM}^{(\mathrm{II})} \\ \mathbf{VM}^{(\mathrm{I})} & -\mathbf{VM}^{(\mathrm{II})} \end{bmatrix}, \mathbf{A} = \begin{Bmatrix} \mathbf{A}^{(\mathrm{I})} \\ \mathbf{A}^{(\mathrm{II})} \end{Bmatrix}. \tag{35, 36}$$

The natural frequencies of the composite rectangular membrane theoretically correspond to the roots of $\det[\mathbf{SM}(f)] = 0$, which is called the *frequency equation* in the paper. However, since $\det[\mathbf{SM}(f)] = 0$ cannot be analytically solved, values of $f$ satisfying $\det[\mathbf{SM}(f)] \approx 0$ are approximately found by increasing $f$ discretely in the range of interest.

On the other hand, $N_1$ and $N_2$ are adjusted so that $N_1 + N_2$ is identical to $N$ (as the result, $\mathbf{SM}(f)$ becomes a square matrix and its determinant can be calculated). In this adjustment, the reasonable manner of approaching the ratio of $N_1 : N_2$ to the length ratio of $\Gamma_1 : \Gamma_2$ is required to obtain more accurate natural frequencies and modes. If the natural frequencies are obtained from the frequency equation, the $j$th mode shape for the $j$th natural frequency $f_j$ can be obtained by plotting Eqs. (1) and (2) with the $j$th eigenvector $\mathbf{A}_{(j)}$ extracted from $\mathbf{SM}(f_j)\mathbf{A} = \mathbf{0}$.

## 3. Case studies

To verify the method presented in this paper, the free vibration analysis of a composite rectangular membrane, of which the configuration is given by $a = 1 \cdot 8$, $b = 1 \cdot 0$, $a_1 = 0 \cdot 7$ and $a_2 = 0.8$ in Fig. 1, is carried out. In the case studies, the surface density of homogeneous region $D_{\mathrm{I}}$ and the tension per unit length are, respectively, fixed as $\rho_{\mathrm{I}} = 1.293 \times 10^{-5}\,\mathrm{kg/m^2}$ and $T = 1.503$ N/m, but the surface density of homogeneous region $D_{\mathrm{II}}$ and the location of point $Q(X_1, Y_1)$ related to the skew angles ($\alpha_1$ and $\alpha_2$) are varied for various numerical tests. The first ten natural frequencies and mode shapes obtained by the present method are compared with those obtained by exact and numerical analyses to ensure the validity of the proposed method.

### 3.1. Homogeneous rectangular membrane

To show an excellent convergence of the proposed method to exact solutions, a simple case of $\rho_{\mathrm{II}} = \rho_{\mathrm{I}}$ and $Q(1 \cdot 1, 0 \cdot 4)$ is considered in the section. For various combinations of $N_1$ and $N_2$, logarithm values of $\det[\mathbf{SM}(f)]$ are plotted as a function of $f$ in Fig. 2 where the values of $f$ corresponding to the troughs represent the natural frequencies of the homogeneous rectangular membrane and the cut-off frequencies of the two homogeneous regions. The cut-off frequencies,

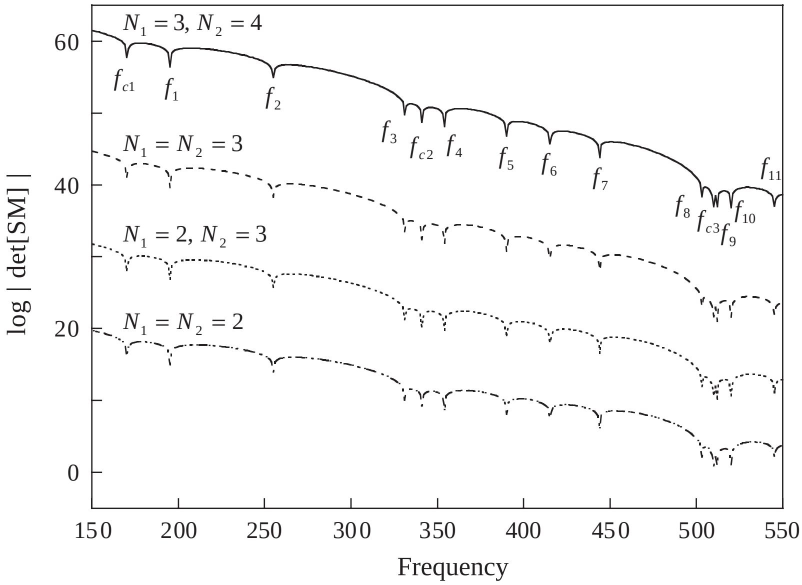


Fig. 2. Determinant of the system matrix versus frequency for the homogeneous rectangular membrane ($\rho_{\mathrm{I}} = \rho_{\mathrm{II}}$) when $N_1$ and $N_2$ are varied.

which are labelled by $f_{c1}, f_{c2}$ and $f_{c3}$ in the figure, were in detail demonstrated in the previous paper [1] and they coincide with the roots of $k_{\mathrm{I}}(f) = 0$ and $k_{\mathrm{II}}(f) = 0$ (in the current example, $k_{\mathrm{I}}(f) = k_{\mathrm{II}}(f) = 0$ due to $\rho_{\mathrm{I}} = \rho_{\mathrm{II}}$). Since the cut-off frequencies are easily obtained by the simple manner aforementioned, correct troughs corresponding to the natural frequencies can be clearly distinguished from spurious troughs corresponding the cut-off frequencies in the determinant curve.

In Table 1, the natural frequencies obtained by the present method are compared with the exact solutions and the FEM results. Only a small number of series functions, in the current instance $N_1 = N_2 = 2$, are enough to yield accurate natural frequencies converged to the exact solutions. On the other hand, the natural frequencies calculated by FEM (ANSYS) approach the exact solutions when a large number of elements are used.

### 3.2. Composite rectangular membranes

#### 3.2.1. Convergence and accuracy of the proposed method

In the section, a composite rectangular membrane for the case of $\rho_{\mathrm{II}} = 2\rho_{\mathrm{I}}$ and $Q(1 \cdot 1, 0 \cdot 4)$ is first solved by the proposed method. In Fig. 3 are shown logarithm values of det[$\mathbf{SM}(f)$] as a function of $f$ for some combinations of $N_1$ and $N_2$ to find the natural frequencies of the membrane. The cut-off frequencies of two homogeneous regions $D_{\mathrm{I}}$ and $D_{\mathrm{II}}$ are indicated by $f_{c1}^{(\mathrm{I})}$, $f_{c2}^{(\mathrm{I})}$, $f_{c1}^{(\mathrm{II})}$, $f_{c2}^{(\mathrm{II})}$ and $f_{c3}^{(\mathrm{II})}$, which correspond to the roots of $k_{\mathrm{I}}(f) = 0$ or $k_{\mathrm{II}}(f) = 0$. The natural frequencies found from the figure are summarized in Table 2 where it may be observed that the natural frequencies by the proposed method for $N_1 = 3$ and $N_2 = 4$ a little lower than those by FEM (ANSYS) for $N_{ele} = 1133$. This fact indicate that the proposed method yields accurate natural frequencies close to exact solutions in that the FEM results gradually approach lower

Table 1
Comparison of the natural frequencies of the homogeneous rectangular membrane obtained by the proposed method, the exact method, and FEM (ANSYS) when $N_1$ and $N_2$ are varied

| Natural frequencies | Proposed method | | | | Exact values (mode shapes) | FEM | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $N_1 = 2$, $N_2 = 2$ | $N_1 = 2$, $N_2 = 3$ | $N_1 = 3$, $N_2 = 3$ | $N_1 = 3$, $N_2 = 4$ | | $N_{ele} = 800$ | $N_{ele} = 450$ | $N_{ele} = 200$ | $N_{ele} = 50$ |
| $f_1$ | 195.01 | 195.01 | 195.01 | 195.01 | 195.01 (1,1) | 195.18 | 195.32 | 195.70 | 197.76 |
| $f_2$ | 254.83 | 254.83 | 254.83 | 254.83 | 254.83 (2,1) | 255.12 | 255.35 | 256.01 | 259.55 |
| $f_3$ | 331.34 | 331.34 | 331.34 | 331.34 | 331.34 (3,1) | 332.07 | 332.64 | 334.28 | 343.15 |
| $f_4$ | 353.85 | 353.85 | 353.85 | 353.85 | 353.85 (1,2) | 355.22 | 356.29 | 359.35 | 375.92 |
| $f_5$ | 390.02 | 390.02 | 390.02 | 390.02 | 390.02 (2,2) | 391.42 | 392.51 | 395.63 | 412.48 |
| $f_6$ | 415.41 | 415.41 | 415.41 | 415.41 | 415.41 (4,1) | 417.03 | 418.29 | 421.90 | 441.29 |
| $f_7$ | 443.81 | 443.81 | 443.81 | 443.81 | 443.81 (3,2) | 445.47 | 446.77 | 450.46 | 470.30 |
| $f_8$ | 503.28 | 503.28 | 503.28 | 503.28 | 503.28 (5,1) | 506.46 | 508.95 | 516.08 | 546.55 |
| $f_9$ | 509.66 | 509.66 | 509.66 | 509.66 | 509.66 (4,2) | 512.02 | 513.86 | 519.11 | 554.74 |
| $f_{10}$ | 520.11 | 520.11 | 520.11 | 520.11 | 520.11 (1,3) | 524.80 | 528.46 | 538.95 | 592.56 |

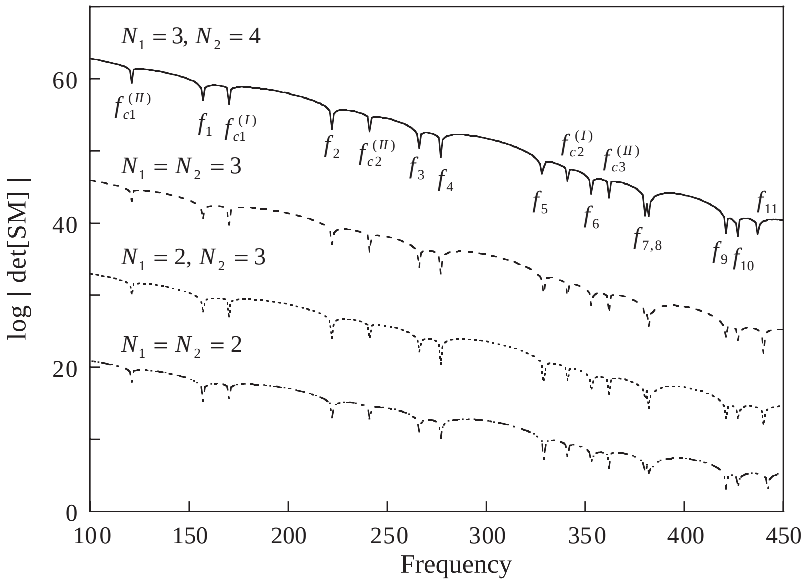


Fig. 3. Determinant of the system matrix versus frequency for the composite rectangular membrane ($\rho_{II} = 2\rho_{II}$) when $N_1$ and $N_2$ are varied.

values as the number of finite elements is increased (in general, FEM provides upper bounds for exact solutions).

Fig. 4 shows the first ten modes obtained by the proposed method for $N_1 = 2$and $N_2 = 3$. The mode shapes have been found to be in good agreement with those given by FEM (ANSYS). It may be said in Fig. 4 that inhomogeneity in density results in the shifting of nodal lines in the $x$ direction, and that most modes have nodal lines parallel to fixed boundaries, except the 8th and 9th modes. Also, note that the shape of the bent interface is not reflected in mode shapes.

Table 2
Comparison of the natural frequencies of the composite rectangular membrane ($\rho_{II} = 2\rho_I$) obtained by the proposed method and FEM (ANSYS) when $N_1$ and $N_2$ are varied

| Natural frequencies | Proposed method | | | | FEM | | | |
|---|---|---|---|---|---|---|---|---|
| | $N_1 = 2$, $N_2 = 2$ | $N_1 = 2$, $N_2 = 3$ | $N_1 = 3$, $N_2 = 3$ | $N_1 = 3$, $N_2 = 4$ | $N_{ele} = 1133$ | $N_{ele} = 724$ | $N_{ele} = 470$ | $N_{ele} = 280$ |
| $f_1$ | 156.9 | 156.7 | 156.7 | 156.7 (156.7) | 156.9 | 156.9 | 157.0 | 157.1 |
| $f_2$ | 222.5 | 222.3 | 222.3 | 222.3 (222.3) | 222.6 | 222.7 | 222.9 | 223.4 |
| $f_3$ | 266.4 | 266.3 | 266.1 | 265.9 (265.9) | 266.9 | 267.0 | 267.6 | 268.4 |
| $f_4$ | 277.5 | 277.2 | 277.1 | 277.1 (277.1) | 277.9 | 278.1 | 278.6 | 279.4 |
| $f_5$ | 329.1 | 329.0 | 328.5 | 327.6 (327.6) | 329.5 | 329.8 | 330.7 | 332.2 |
| $f_6$ | 353.5 | 353.2 | 353.2 | 352.9 (352.9) | 354.3 | 355.0 | 356.2 | 357.9 |
| $f_7$ | 381.5 | 381.6 | 381.6 | 381.6 (381.5) | 382.7 | 383.4 | 384.5 | 387.2 |
| $f_8$ | 382.7 | 382.7 | 382.6 | 382.1 (382.0) | 384.9 | 385.4 | 387.4 | 389.6 |
| $f_9$ | 421.1 | 420.8 | 420.7 | 420.7 (420.6) | 423.0 | 423.9 | 425.9 | 429.1 |
| $f_{10}$ | 427.3 | 427.3 | 427.3 | 426.0 (425.9) | 429.0 | 429.8 | 431.7 | 434.8 |

Note: parenthesized values denote natural frequencies obtained for $N_1 = 5$ and $N_2 = 6$.

### 3.2.2. *Accuracy of the proposed method when the ratio of $\rho_I : \rho_{II}$ is varied*

As another verification example, the natural frequencies and mode shapes of composite rectangular membranes with the same configuration as in Section 3.2.1 but different values in the surface density of region $D_{II}$ are obtained by the proposed method. For $\rho_{II} = \rho_I/2$, $\rho_{II} = 3\rho_I$ and $\rho_{II} = 4\rho_I$, the natural frequencies of the composite membranes are summarized in Table 3 where a comparison between the proposed method and the numerical method (FEM) indicates that the proposed method yields accurate results even when only a small number of series terms ($N_1 = 2$ and $N_2 = 3$) are used. Although the mode shapes of the composite membranes by the proposed method are not presented in the paper, they have been found to agree excellently with those by FEM.

### 3.2.3. *Composite membrane with a highly oblique interface part*

Finally, free vibration analysis is carried out for a composite rectangular membrane with a particular interface shape, for which the location of point $Q$ in Fig. 1 is moved into $Q(1 \cdot 2, 1/6)$ so that the interface $\Gamma_1$ more approaches the $X$-axis. Also in the current, particular case, the proposed method yields the accurate natural frequencies close to the FEM results (see Fig. 5 and Table 4). Furthermore, the mode shapes of the membrane obtained by the proposed method have been found to be in excellent agreement with those by FEM (see Fig. 6). In the current case that a special shape is given for the interface, it may be seen that nodal lines are not parallel to fixed boundaries, and that this tendency becomes clearer for higher modes.

## 4. Conclusion

In this paper, an effective method has been presented that can be applied to the free vibration analysis of composite rectangular membranes with a bent interface. The proposed method yields

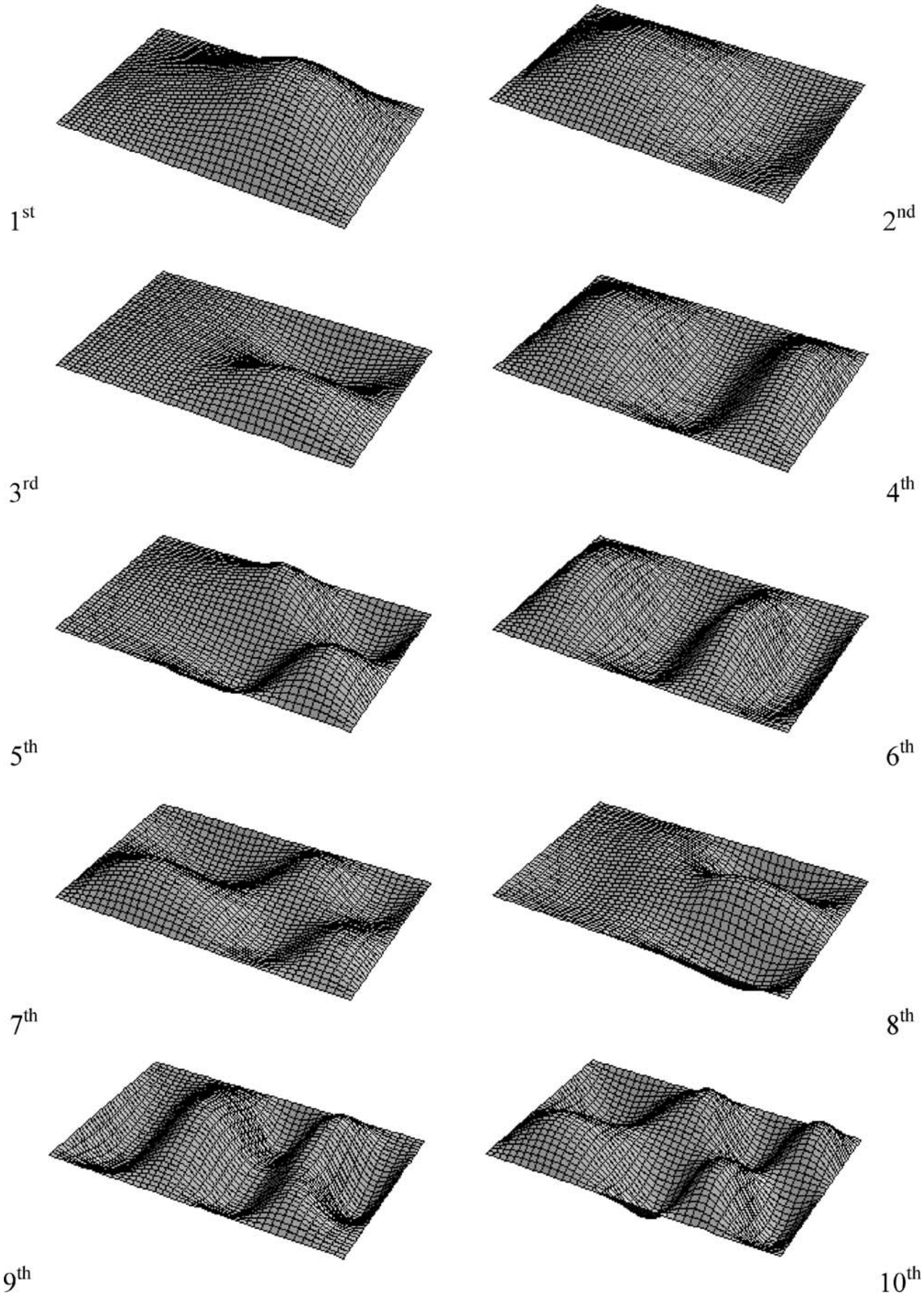


Fig. 4. Mode shapes of the composite rectangular membrane ($\rho_{II} = 2\rho_I$) obtained by the proposed method when $N_1 = 2$ and $N_2 = 3$.

Table 3
Comparison of the natural frequencies of the composite rectangular membrane obtained by the proposed method and FEM (ANSYS) when $\rho_{II}$ is varied

| Natural frequencies | Proposed method ($N_1 = 2$ and $N_2 = 3$) | | | FEM ($N_{ele} = 1133$) | | |
|---|---|---|---|---|---|---|
| | $\rho_{II}$ | | | $\rho_{II}$ | | |
| | $\rho_I/2$ | $3\rho_I$ | $4\rho_I$ | $\rho_I/2$ | $3\rho_I$ | $4\rho_I$ |
| $f_1$ | 210.2 | 131.4 | 115.0 | 210.4 | 131.6 | 115.2 |
| $f_2$ | 299.8 | 201.4 | 181.6 | 300.1 | 201.8 | 182.2 |
| $f_3$ | 371.9 | 219.0 | 190.3 | 373.4 | 219.5 | 190.7 |
| $f_4$ | 383.7 | 249.9 | 236.3 | 384.5 | 250.4 | 236.7 |
| $f_5$ | 451.2 | 275.6 | 240.7 | 453.6 | 276.0 | 241.0 |
| $f_6$ | 474.2 | 308.7 | 270.5 | 475.9 | 310.1 | 271.9 |
| $f_7$ | 534.5 | 317.8 | 283.9 | 537.3 | 319.4 | 285.4 |
| $f_8$ | 536.3 | 345.0 | 305.3 | 541.3 | 345.7 | 306.0 |
| $f_9$ | 581.0 | 363.4 | 316.7 | 584.7 | 365.0 | 318.1 |
| $f_{10}$ | 593.3 | 380.3 | 347.7 | 596.8 | 381.9 | 349.4 |

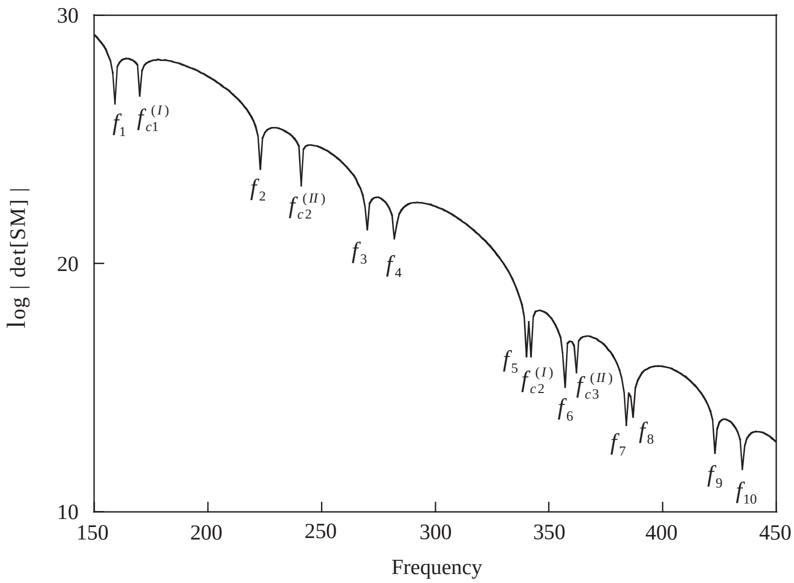


Fig. 5. Determinant of the system matrix versus frequency for the particular composite rectangular membrane ($\rho_{II} = 2\rho_I$) when $N_1 = 2$ and $N_2 = 3$.

Table 4
Comparison of the natural frequencies of the particular composite rectangular membrane ($\rho_{II} = 2\rho_I$) obtained by the proposed method ($N_1 = 2$ and $N_2 = 3$) and the FEM ($N_{ele} = 1217$)

| Natural frequencies | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Proposed | 158.6 | 222.9 | 269.7 | 282.3 | 339.9 | 356.5 | 383.6 | 386.8 | 423.2 | 435.2 |
| FEM | 158.8 | 223.1 | 270.1 | 282.9 | 340.2 | 357.2 | 383.9 | 389.9 | 425.7 | 436.9 |

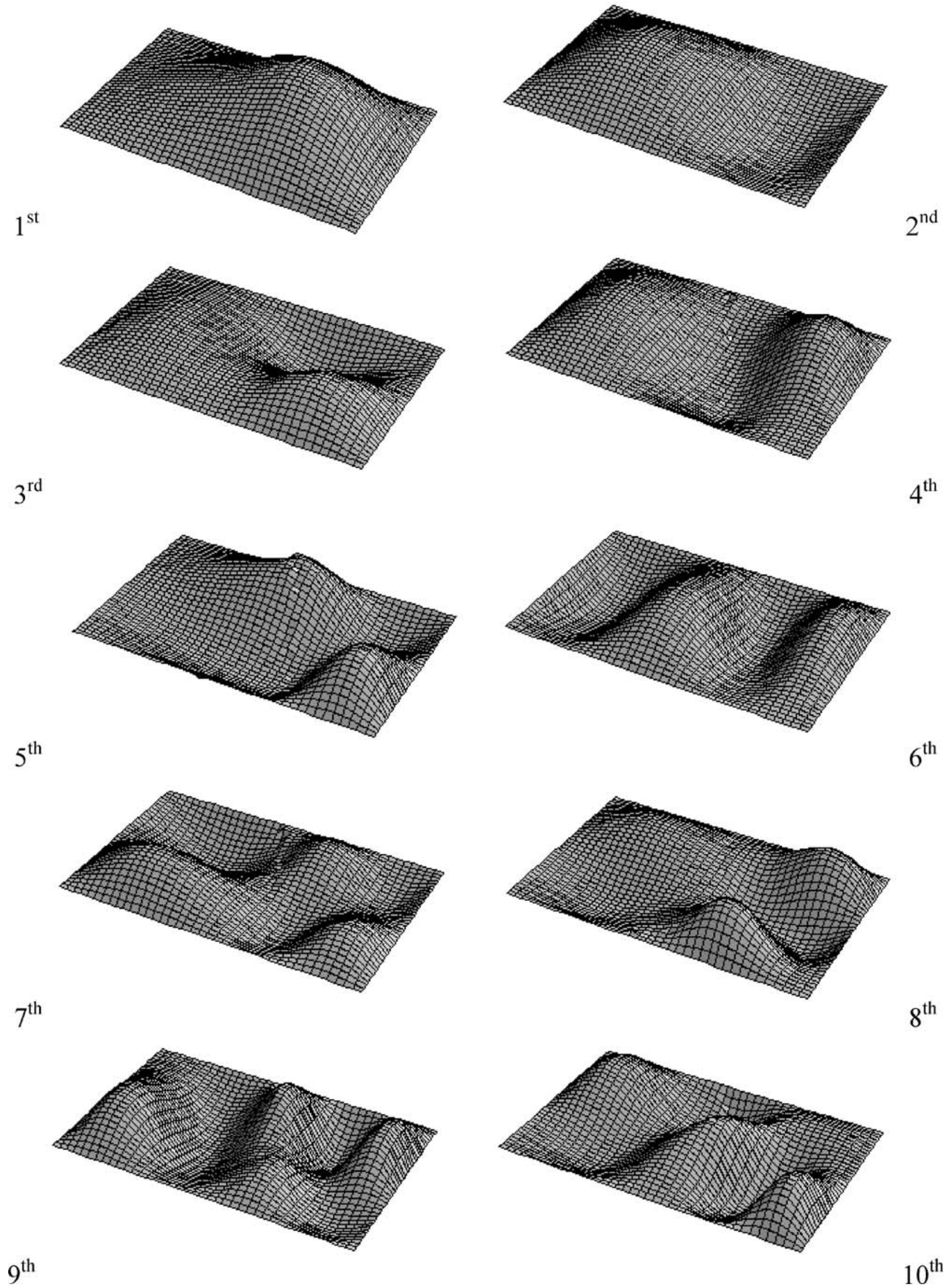


Fig. 6. Mode shapes of the composite rectangular membrane ($\rho_{II} = 2\rho_I$) with the particular interface shape when $N_1 = 2$ and $N_2 = 3$.

accurate natural frequencies and mode shapes compared with exact solutions or FEM (ANSYS), in spite of only a small amount of computation effort.

It is expected that the method presented in this work can be applied to analyze composite membranes with arbitrarily shaped interfaces by discretizing a curved interface into a great number of rectilinear interfaces (More concrete process is explained in Appendix A).

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## Appendix A

Consider a composite membrane with a multiple interface composed of $n$ rectilinear interfaces $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$. If the compatibility conditions are applied to each rectilinear interface, one can obtain $2n$ matrix equations:

$$\mathbf{SM}^{(\mathrm{I})(\mathrm{i})}\mathbf{A}^{(\mathrm{I})} = \mathbf{SM}^{(\mathrm{II})(\mathrm{i})}\mathbf{A}^{(\mathrm{II})}, \quad \mathbf{VM}^{(\mathrm{I})(\mathrm{i})}\mathbf{A}^{(\mathrm{I})} = \mathbf{VM}^{(\mathrm{II})(\mathrm{i})}\mathbf{A}^{(\mathrm{II})}, \quad \text{for } i = 1, 2, \ldots, n. \tag{A.1, A.2}$$

Note that Eqs. (A.1, A.2) correspond to Eqs. (24)–(27) in the composite membrane with the bent interface. Eqs. (A.1) may be rewritten as

$$\mathbf{SM}^{(\mathrm{I})}\mathbf{A}^{(\mathrm{I})} = \mathbf{SM}^{(\mathrm{II})}\mathbf{A}^{(\mathrm{II})}, \tag{A.3}$$

where $\mathbf{SM}^{(\mathrm{I})}$ and $\mathbf{SM}^{(\mathrm{II})}$ are

$$\mathbf{SM}^{(\mathrm{I})} = \begin{bmatrix} \mathbf{SM}^{(\mathrm{I})(1)} \\ \mathbf{SM}^{(\mathrm{I})(2)} \\ \vdots \\ \mathbf{SM}^{(\mathrm{I})(n)} \end{bmatrix}, \mathbf{SM}^{(\mathrm{II})} = \begin{bmatrix} \mathbf{SM}^{(\mathrm{II})(1)} \\ \mathbf{SM}^{(\mathrm{II})(2)} \\ \vdots \\ \mathbf{SM}^{(\mathrm{II})(n)} \end{bmatrix}. \tag{A.4, A.5}$$

Also, Eq. (A.2) may be rewritten as

$$\mathbf{VM}^{(\mathrm{I})}\mathbf{A}^{(\mathrm{I})} = \mathbf{VM}^{(\mathrm{II})}\mathbf{A}^{(\mathrm{II})}, \tag{A.6}$$

where $\mathbf{VM}^{(\mathrm{I})}$ and $\mathbf{VM}^{(\mathrm{II})}$ are given by

$$\mathbf{VM}^{(\mathrm{I})} = \begin{bmatrix} \mathbf{VM}^{(\mathrm{I})(1)} \\ \mathbf{VM}^{(\mathrm{I})(2)} \\ \vdots \\ \mathbf{VM}^{(\mathrm{I})(n)} \end{bmatrix}, \mathbf{VM}^{(\mathrm{II})} = \begin{bmatrix} \mathbf{VM}^{(\mathrm{II})(1)} \\ \mathbf{VM}^{(\mathrm{II})(2)} \\ \vdots \\ \mathbf{VM}^{(\mathrm{II})(n)} \end{bmatrix}. \tag{A.7, A.8}$$

Note that Eqs. (A.3)–(A.8) corresponds to Eqs. (28)–(33), respectively. Finally, a global system equation, which corresponds to Eq. (34), can be obtained from Eqs. (A.3)–(A.6).

Furthermore, the methodology explained in the above may be applied to a composite membrane with arbitrarily shaped interface $\Gamma_{arbi}$ as shown in Fig. A1 where $\Gamma_{arbi}$ has been

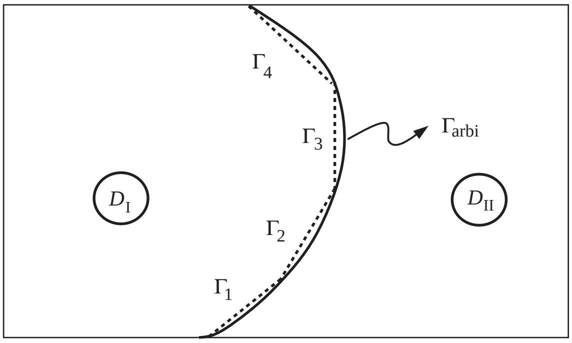


Fig. A1. Discretization of a curved interface into four rectilinear interfaces.

discretized with four rectilinear interfaces $\Gamma_1 - \Gamma_4$. Note that, if the number of rectilinear interfaces used is increased, more accurate results can be obtained.

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