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Journal of Sound and Vibration 272 (2004) 413–424
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Evaluation of elastic constants of specially orthotropic plates through vibration testing

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Received 7 January 2003; accepted 20 June 2003

1. Introduction

Structures made of advanced composites such as fibre-reinforced plastics often consist of a number of layers having unidirectional fibres. These layers, or laminates will generally be orthotropic. When orthotropic laminates are stacked to form a laminate, the resulting structure is generally anisotropic. The laminate will be orthotropic only for a certain stacking sequences. Composites offer high specific strength and stiffness. One of the main disadvantages of laminated composite materials is the existence of coupling responses such as stretching–shearing, stretching–bending, stretching–twisting, shearing–twisting and bending–twisting. The degree of these coupling effects depends very much on the stacking sequence of the lamination. The specially orthotropic condition occurs when no coupling responses are present and this is achieved by having all plates at 0° or 90° or symmetric cross-ply. The measurement of stiffness for composite materials is more difficult than for isotropic cases. When restricting the subject to symmetric plates, it is well known that the elastic behavior of such structures is characterized by a set of six in-plane rigidities and six flexural rigidities. The flexural deformations are dependent on the material elastic constants in accordance with classical lamination theory. These elastic constants include the longitudinal Young's modulus (E_{xx}), transverse Young's modulus (E_{yy}), major the Poisson ratio (ν_{xy}), minor the Poisson ratio (ν_{yx}), and the in-plane shear modulus (G_{xy}). Of the five constants named only four are independent since $\nu_{xy}/E_{xx} = \nu_{yx}/E_{yy}$ due to symmetry of the compliance matrix. The use of composite materials for engineering applications thus requires the determination of their properties for the analysis, manufacture and quality control of the materials. If vibrations are induced in a specially orthotropic plate, then its dynamic response will be a function of plate geometry, density, boundary conditions and the elastic constants. This

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doi:10.1016/S0022-460X(03)00657-6

implies the possibility of using plate vibrations theory to develop a non-destructive test to determine the elastic constants of a specially orthotropic plate [1–3].

In this paper, a technique through vibration testing for the determination of the properties of thin specially orthotropic plates is discussed. The precision of the elastic constants determination in the present study, is linked to the accuracy of the eigenvalue calculation. Knowing that only the natural frequencies experimentally determined are to be calculated, we need a fast means to approximate and hence identify the eigenvalues corresponding to given model indices. An expression is derived for the determination of the natural frequencies of thin specially orthotropic rectangular plates. A single term expression having characteristic beam functions of a free–free beam, is considered for the deflection function to derive the frequency expression through Rayleigh–Ritz method. Experimentally determined natural frequencies of free–free thin rectangular plates are used in conjunction with the derived expression to determine the properties of the thin specially orthotropic plates under consideration.

2. Frequency expression for a free–free thin specially orthotropic rectangular plates

The purpose of the present study is to examine the possibility of determining the elastic constants of an orthotropic material from the results of a single non-destructive test of free vibration of a thin specially orthotropic rectangular plate. The plate is supposed to be perfectly rectangular, with its length equal to a, its width equal to b and its thickness, h is constant and small compared to the other dimensions. Moreover, we suppose that the material is elastic and homogeneous and that damping can be neglected.

The partial differential equation governing the transverse vibration of a thin specially orthotropic rectangular plate is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{1}$$

The bending stiffnesses, D_{11}, D_{12} and D_{22} and torsional stiffness, D_{66}, are defined in terms of ply thickness and stiffness. Henceforth, all four terms are referred as bending stiffness, with the understanding that D_{66} is actually a torsional stiffness. The bending–stretching and bending–twisting coupling coefficients in Eq. (1) are absent for the case of a thin specially orthotropic rectangular plates. Eq. (1) cannot normally be solved to determine the natural frequencies and mode shapes of a rectangular plate with any combination of boundary conditions. An approximate technique must, therefore, be used for the calculation of the natural frequencies.

Rayleigh's quotient provides an estimate of the fundamental frequency, ω, in terms of the maximum potential energy, U_{max}, material density, ρ, assumed mode shape, $w(x, y)$, plate thickness, h, and plate dimensions a, and b. An expression for the maximum potential energy, U_{max}, derived for a rectangular plate in a state of transverse vibration following the classical lamination theory is

$$U_{max} = \frac{1}{2}\int_0^a \int_0^b \left\{ D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \mathrm{d}y\,\mathrm{d}x \right\}. \tag{2}$$

For a conservative system, the maximum potential energy equals the maximum kinetic energy

$$T_{max} = \tfrac{1}{2}\rho h \omega_{mn}^2 \int_0^a \int_0^b w^2 \,\mathrm{d}y\,\mathrm{d}x. \tag{3}$$

This relationship may be re-arranged to form Rayleigh's quotient as

$$\omega_{mn}^2 = \frac{2}{\rho h}\left\{\frac{U_{max}}{\int_0^a \int_0^b W^2 \,\mathrm{d}y\,\mathrm{d}x}\right\}. \tag{4}$$

The natural frequencies are obtained by assuming the deflected form of the vibrating plate as

$$w = A_{mn}\phi_m(x)\theta_n(y). \tag{5}$$

The functions $\phi_m(x)$ and $\theta_n(y)$ which have been chosen to represent the deflected form (Eq. (5)) of the vibrating plate are those, which represent the normal modes of vibration of uniform beams. They were chosen because the nodal patterns, which correspond to the natural frequencies of rectangular plates, take the form of lines, which are approximately parallel with the edges of the plate. The nodal patterns can be defined, therefore, by the notation m/n, in which m is the number of nodal lines in the x direction and n is the number of the nodal lines in the y direction.

Defining $\xi = x/a$, $\eta = y/b$, $\lambda = a/b$, and using Eq. (5) in Eq. (4), the natural frequency of mode m/n can be obtained from the following expression:

$$\rho h a^4\, \omega_{ij}^2 = \alpha_{ij}D_{11} + 2\lambda^2\beta_{ij}D_{12} + \lambda^4\gamma_{ij}D_{22} + 4\lambda^2\delta_{ij}D_{66}, \tag{6}$$

where

$$\alpha_{ij} = \frac{\int_0^1 \int_0^1 (\phi_i''\theta_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}{\int_0^1 \int_0^1 (\phi_i\theta_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}, \tag{7}$$

$$\beta_{ij} = \frac{\int_0^1 \int_0^1 (\phi_i''\theta_j\phi_i\ddot{\theta}_j) \,\mathrm{d}\eta\,\mathrm{d}\xi}{\int_0^1 \int_0^1 (\phi_i\theta_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}, \tag{8}$$

$$\gamma_{ij} = \frac{\int_0^1 \int_0^1 (\phi_i\ddot{\theta}_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}{\int_0^1 \int_0^1 (\phi_i\theta_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}, \tag{9}$$

$$\delta_{ij} = \frac{\int_0^1 \int_0^1 (\phi_i'\dot{\theta}_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}{\int_0^1 \int_0^1 (\phi_i\theta_j)^2 \,\mathrm{d}\eta\,\mathrm{d}\xi}. \tag{10}$$

Primes denote differentiation with respect to ξ and over dots denote differentiation with respect to η.

The deflection functions in Eq. (5) as the product of beam functions are chosen as the fundamental mode shapes of having the boundary conditions of the plate. This choice of functions then exactly satisfies all boundary conditions for the plate, except in the case of free edge, where the shear condition is approximately satisfied. Since the problem under consideration is towards the development of an expression for the determination of natural frequencies of free–free rectangular plates, ϕ_i and θ_i represent the same mode shape. Hence, we need to obtain any one of the functions.The mode shape $\theta(\eta)$ for the free–free condition is obtained by solving

$$\frac{d^4\theta_n}{d\eta^4} - l_n^4\theta_n = 0. \tag{11}$$

With boundary conditions

$$\frac{d^2\theta_n}{d\eta^2} = \frac{d^3\theta_n}{d\eta^3} = 0 \quad \text{at} \ \eta = 0, \tag{12}$$

$$\frac{d^2\theta_n}{d\eta^2} = \frac{d^3\theta_n}{d\eta^3} = 0 \quad \text{at} \ \eta = 1. \tag{13}$$

The general solution of Eq. (11) is of the form

$$\theta_n(\eta) = A\cosh l_n\,\eta + B\sinh l_n\eta + C\cosh l_n\eta + D\sinh l_n\eta. \tag{14}$$

Substituting Eq. (14) in the boundary conditions (12) and (13), we obtain

$$A - C = 0, \tag{15}$$

$$B - D = 0, \tag{16}$$

$$A(\cosh l_n - \cos l_n) + B(\sinh l_n - \sin l_n) = 0, \tag{17}$$

$$A(\sinh l_n + \sin l_n) + B(\cosh l_n - \cos l_n) = 0. \tag{18}$$

From Eqs. (17) and (18), we get a condition

$$1 - \cosh l_n \cos l_n = 0 \tag{19}$$

for obtaining the eigenvalues, l_n.

The first six eigenvalues (l_n, $n = 1, 2, 3, 4, 5, 6$) obtained by solving Eq. (19) through Newton–Raphson iterative scheme are 0, 4.73004, 7.8532, 10.9956, 14.13716 and 17.27876, respectively.

From Eqs. (15)–(18), the constants B, C and D are written in terms of A as

$$B = -r_nA, \quad C = A, \quad D = -r_nA, \tag{20−22}$$

where $r_n = (\sinh l_n + \sin l_n)/(\cosh l_n - \cos l_n)$.

Substituting B, C and D in terms of A in Eq. (14) and using

$$\int_0^1 \theta_n^2 \, d\eta = 1 \tag{23}$$

the constant A is determined.

The mode shape $\theta_n(\eta)$ for the free–free conditions is

$$\theta_n(\eta) = \cosh l_n\eta - \cos l_n\,\eta - r_n(\sinh l_n\eta + \sin l_n\eta). \tag{24}$$

Eq. (24) is used to obtain mode shape functions for all non-zero eigenvalues.

For the case of zero eigenvalue, the symmetric and antisymmetric mode shape functions are

$$\theta_1(\eta) = 1, \tag{25}$$

$$\theta_2(\eta) = \sqrt{3}(1 - 2\eta). \tag{26}$$

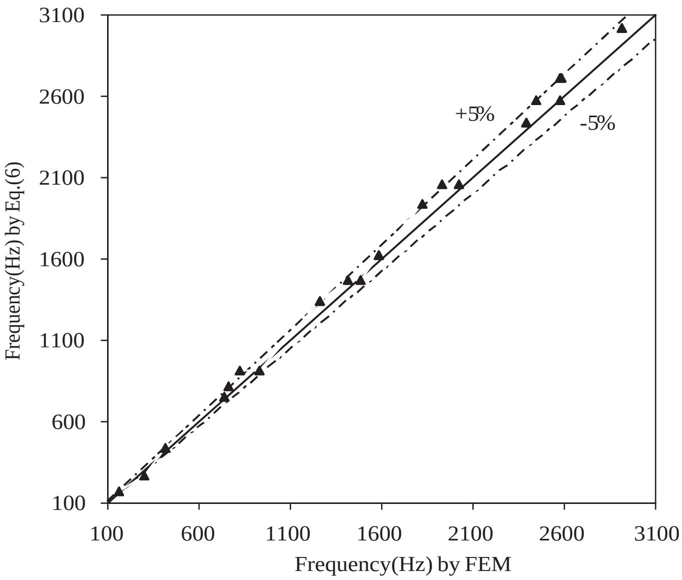

Fig. 1. Comparison of natural frequencies of a free–free aluminium alloy square plate.

The mode shape $\phi_m(\xi)$ for the present study is

$$\phi_m(\xi) = \theta_m(\xi). \tag{27}$$

By using the mode shape functions $\phi_i(\xi)$ and $\theta_j(\eta)$ in Eqs. (7)–(10), and integrating analytically, the values of α_{ij}, β_{ij}, γ_{ij} and δ_{ij} are obtained. With these values, the frequencies (ω_{ij}) are determined directly from the frequency expression (6) for the free–free specially orthotropic rectangular plate.

The adequacy of the analytical expression (6) for computing the natural frequencies, is examined for the case of a free–free square plate. A finite element solution is obtained by using an eight-noded quadrilateral isoparametric shell element available in the well-known finite element code, COSMOS. Finite element idealization for a free–free aluminium alloy square plate has been made with four hundred element to carry out free vibration analysis. Dimensions of the plate are $a = b = 254\,\text{mm}$ and $h = 3.16\,\text{mm}$. Material properties assumed in the analysis for aluminium alloy are $E_{11} = E_{22} = 72.4\,\text{GPa}$; $G_{12} = 28\,\text{GPa}$; $v_{12} = 0.33$ and $\rho = 2770\,\text{kg/m}^3$. Fig. 1 shows the comparison of natural frequencies of the analytical expression (6) with those obtained from the first 27 eigenvalues of the finite element analysis. It is seen that the estimate of the natural frequencies through an approximate expression (6) were found to be higher than those obtained from the finite element analysis.

It is a fact that natural frequencies obtained using the Rayleigh–Ritz method are always higher than the exact values due to the assumed plate mode shape, which inherently increase the rigidity of the plate. The deflection function, $w(x, y)$ is assumed in general as a linear series of admissible functions and adjusting the coefficients in the series so as to minimize Eq. (4). When this deflection function is substituted in Eq. (4), the right-hand side becomes a function of the coefficients of the admissible functions. Taking the partial derivative with respect to each coefficient and equating to zero minimize this. Thus we arrive a system of linear homogeneous equations in the unknown coefficients of the admissible functions. The natural frequencies, ω_{mn} are determined from the condition that the determinant of the system must vanish. Deobald and Gibson [1] have used 36-terms characteristic beam functions to postulate the deflected shape, and obtained the results comparable to the finite element solution for plate configurations with different boundary conditions. Hence, the accuracy of the Rayleigh–Ritz method depends on the selection of compatible shape functions, which satisfy the geometrical boundary conditions of the plate. A single-term expression (5) having free–free beam characteristic functions $\phi_m(x)$ and $\theta_n(y)$, is chosen here for the deflection function while deriving the frequency expression (6). For the specified elastic constants, the frequency expression (6) results higher values compared to those of finite element solutions. It is obvious that prediction of elastic constants for the measured frequency values from the frequency expression (6), results lower values. To improve the accuracy, the frequency expression (6) has to be modified by introducing influence coefficients, through finite element solutions.

3. Modified frequency expression

In general, the actual natural frequencies will lie below the Rayleigh–Ritz solution and above the finite element solution. The discrepancy between the finite element solution and the present single term Rayleigh–Ritz solution is mainly due to the assumed mode shape. Using the natural frequencies of finite element solution, the frequency expression (6) is modified by introducing influence coefficients. These influence coefficients are obtained by fitting the natural frequencies of the aluminium alloy square plate from the eigenvalues of the finite element solution, in the frequency expression (6) through least square curve fit. Now the modified frequency expression is

$$\rho h a^4 \omega_{ij}^2 = f_1 \alpha_{ij} D_{11} + 2 f_2 \lambda^2 \beta_{ij} D_{12} + f_3 \lambda^4 \gamma_{ij} D_{22} + 4 f_4 \lambda^2 \delta_{ij} D_{66}. \quad (28)$$

The influence coefficients in Eq. (28) for free–free rectangular plate are $f_1 = 0.98946$, $f_2 = 1.0838$; $f_3 = 0.90982$; and $f_4 = 0.85068$. Fig. 2 shows a reasonably good comparison of the present finite element results with those obtained from the modified frequency expression (28). The adequacy of the modified frequency expression (28) is examined further by considering the analytical natural frequencies of Ref. [1] on aluminium and graphite/epoxy square plates. The dimensions of square plates are $a = b = 254$ mm and $h = 3.16$ mm for aluminium plate whereas $h = 1.483$ mm for graphite/epoxy plates. The properties of aluminium plate are $E_{11} = E_{22} = 72.4$ GPa; $G_{12} = 28$ GPa; $\nu_{12} = 0.33$; $\rho = 2770$ kg/m^3. The properties of graphite/epoxy plate are $E_{11} = 127.9$ GPa; $E_{22} = 10.27$ GPa; $G_{12} = 7.32$ GPa; $\nu_{12} = 0.22$; $\rho = 1584$ kg/m^3. Figs. 3 and 4 show the comparison of natural frequencies of aluminium and graphite/epoxy square plates using the modified frequency expression (28) with those presented in Ref. [1] through a SAP IV finite element model and 36 terms Rayleigh–Ritz model. It is found that the results of frequency expression (28) for graphite/epoxy plate closely match with finite element results of Ref. [1]. Tables 1–3 show a reasonably good comparison of the results obtained from the modified frequency expression (28) with those of experimental results from aluminium alloy square plate [1], glass/epoxy rectangular plate [2], and carbon/epoxy square plate [3]. The natural frequencies obtained from the frequency expression (6), which does not include influence coefficients, were found to be comparatively

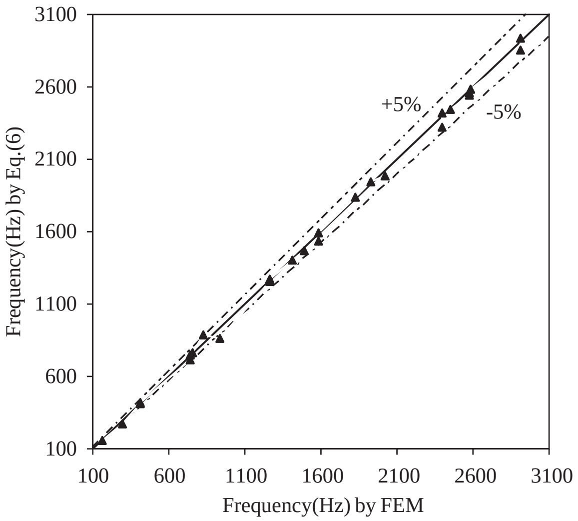

Fig. 2. Comparison of natural frequencies of the finite element solution with those obtained from the modified frequency expression (28).

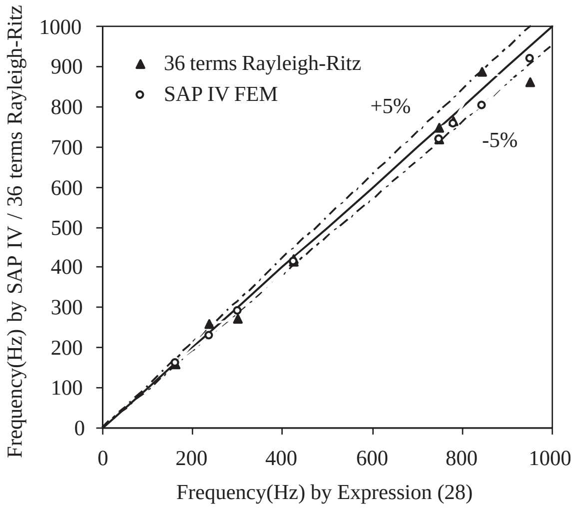

Fig. 3. Comparison of natural frequencies of aluminium plate using the modified frequency expression (28) with SAP IV FEM and 36 terms Rayleigh–Ritz method of Ref. [1].

higher than test results. The modified frequency expression (28) estimates reasonably accurate natural frequencies for free–free specially orthotropic rectangular plates.

It should be noted that the direct application of the calculus of variation to minimize Eq. (4) lead to the partial differential Eq. (1) for a vibrating plate. The characteristic beam functions used in the present study satisfies the geometric boundary conditions of a free–free rectangular plate. The bending and torsion stiffness (D_{11}, D_{12}, D_{22} and D_{66}) in Eq. (28) will take care of the isotropic as well as specially orthotropic plate materials.

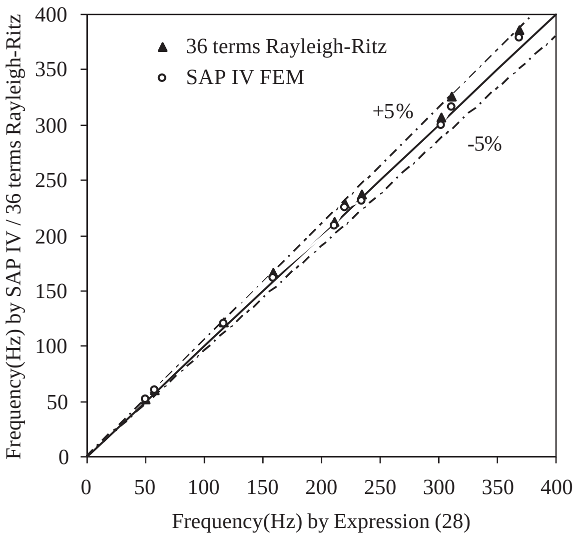

Fig. 4. Comparison of natural frequencies of graphite/epoxy using the modified frequency expression (28) with SAP IV FEM and 36 terms Rayleigh–Ritz method of Ref. [1].

Table 1
Natural frequencies (Hz) of a free–free aluminium alloy square plate $a = b = 254\,\mathrm{mm}$; $h = 3.16\,\mathrm{mm}$ $E_{11} = E_{22} = 72.4\,\mathrm{GPa}$; $G_{12} = 28\,\mathrm{GPa}$; $\nu_{12} = 0.33$; $\rho = 2770\,\mathrm{kg/m^3}$

Mode	Test results [1]	Frequency expression without influence coefficients		Frequency expression with influence coefficients	
		Eq. (6)	Relative error (%)	Eq. (28)	Relative error (%)
(2,2)	156.7	171.71	−9.6	158.37	−1.1
(1,3)	232.5	272.67	−17.3	260.09	−11.9
(3,1)	300.4	272.67	9.2	271.23	9.7
(2,3)	411.7	442.64	−7.5	413.61	−0.5
(3,2)	411.7	442.64	−7.5	420.70	−2.2
(1,4)	744.9	751.64	−0.9	716.95	3.8
(4,1)	744.9	751.64	−0.9	747.67	−0.4
(3,3)	755.7	815.39	−7.9	764.04	−1.1
(4,2)	821.8	912.47	−11.0	886.95	−7.9
(2,4)	936.5	912.47	2.6	861.21	8.0

4. Evaluation of elastic constants

The discrepancy between the analytical and experimental natural frequencies may be attributed to various factors. For the specified geometric and material properties, natural frequencies of a free–free orthotropic thin rectangular plate can be obtained using the modified frequency expression (28). An attempt is made here to evaluate the elastic constants from the measured

Table 2
Natural frequencies (Hz) of a free–free glass/epoxy rectangular plate a = 240.1 mm; $b = 180.4$ mm; $h = 1.84$ mm $E_{11} = E_{22} = 69.88$ GPa; $G_{12} = 28.7$ GPa; $\nu_{12} = 0.217$; $\rho = 2460\,\text{kg/m}^3$

Mode	Test results [2]	Frequency expression without influence coefficients		Frequency expression with influence coefficients	
		Eq. (6)	Relative error (%)	Eq. (28)	Relative error (%)
(2,2)	151.00	160.00	−6.0	147.57	2.3
(3,1)	174.25	179.13	−2.8	178.19	−2.3
(1,3)	317.75	317.13	0.1	302.67	4.7
(3,2)	352.25	371.00	−5.3	348.63	1.0
(2,3)	431.00	454.13	−5.4	425.91	1.2
(4,1)	497.25	493.79	0.7	491.18	1.2
(4,2)	666.75	690.06	−3.5	662.51	0.6
(1,4)	872.00	874.68	−0.3	834.31	4.3
(5,1)	964.75	968.02	−0.3	962.90	0.2
(2,4)	985.75	998.71	−1.3	945.38	4.1
(5,2)	1132.00	1155.72	−2.1	1125.29	0.6
(6,1)	1586.50	1600.19	−0.9	1591.73	−0.3
(1,5)	1718.50	1714.73	0.2	1635.59	4.8

Table 3
Natural frequencies (Hz) of a free–free carbon/epoxy square plate $a = b = 200$ mm; $h = 1.0$ mm $E_{11} = 120$ GPa; $E_{22} = 10$ GPa; $G_{12} = 4.9$ GPa; $\nu_{12} = 0.3$; $\rho = 1510\,\text{kg/m}^3$

Mode	Test results [3]	Frequency expression without influence coefficients		Frequency expression with influence coefficients	
		Eq. (6)	Relative error (%)	Eq. (28)	Relative error (%)
(2,2)	47.5	49.66	−4.5	45.80	3.6
(1,3)	65.7	66.38	−1.0	63.32	3.6
(2,3)	117.4	120.72	−2.8	112.51	4.2
(1,4)	180.4	182.98	−1.4	174.54	3.2
(3,1)	227.9	229.95	−0.9	228.74	−0.4
(2,4)	232.6	236.36	−1.6	222.50	4.3
(3,2)	247.1	251.09	−1.6	246.92	0.1
(3,3)	306.2	316.24	−3.3	304.73	0.5
(1,5)	354.9	358.72	−1.1	342.16	3.6

natural frequencies using the modified frequency expression (28). To determine the four elastic constants (namely, E_{11}, E_{22}, G_{12} and ν_{12}), a minimum of four measured frequencies is essential. The bending stiffnesses, D_{11}, D_{12}, D_{22} and D_{66} are obtained by substituting these four measured frequencies in Eq. (28). From these bending stiffnesses, the four elastic constants are obtained using the following equations:

$$E_{11} = \frac{12}{h^2}\left(D_{11} - \frac{D_{12}^2}{D_{22}}\right), \tag{29}$$

$$E_{22} = \frac{12}{h^2}\left(D_{22} - \frac{D_{12}^2}{D_{11}}\right), \tag{30}$$

$$\nu_{12} = \frac{D_{12}}{D_{22}}, \tag{31}$$

$$G_{12} = \frac{12}{h^2} D_{66}. \tag{32}$$

5. Results and discussion

The accuracy in the elastic constants depends heavily on the measured frequencies. Deobald and Gibson [1] have considered four sets of measured frequencies to obtain the elastic constants and obtained negative the Poisson ratio. These may be mainly due to consideration of erroneous measured frequency data. It is very difficult to know in advance about the accuracy in the measured frequencies. Under these circumstances, it is better to consider the measured frequencies as many as possible and determine the elastic constants. These constants are used in Eq. (28) and dropped the data whichever showed large discrepancy between analytical and experimental results. The bending stiffnesses are determined through least square curve fit by considering the measured frequency data and comparable with the analytical expression. The four elastic constants are obtained using these bending stiffnesses. Strictly speaking, $D_{12}(= \nu_{12}D_{22} = \nu_{21}D_{11})$ is not independent. If it is treated as independent bending stiffness, then there is a possibility of obtaining negative the Poisson ratio due to inaccurate measured frequencies.

The bending stiffnesses in Eq. (28) corresponding to the measured frequencies are evaluated through an iterative process, which was continued till standard error

$$SE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\{1 - \frac{\text{analysis result}}{\text{test result}}\right\}_i^2} \tag{33}$$

attains a minimum value. Here, N is the number of measured frequencies. Table 4 gives elastic constants obtained from the measured frequency data of Tables 1–3, which are comparable to those specified. The difference in the estimate of elastic constants using frequency expressions (6) and (28) is mainly due to influence coefficients. The discrepancy between the estimates of elastic constants and the specified values is mainly due to measured natural frequencies, which were slightly different from those estimated from the frequency expression. Table 5 gives the comparison on the estimate of elastic constants from the measured frequencies of Ref. [1] with the specified values. The present analysis results are also comparable with those specified elastic constants of Ref. [1].

In general, estimates of the elastic constants from the specified natural frequencies using Eq. (6) are lower than those obtained from the modified frequency expression (28). The influence coefficients in the modified frequency expression (28) depend on the boundary conditions of the plate. For the case of a free–free plate, the squared frequency values from the finite element analysis were found to be slightly lower than those obtained from the frequency expression (6), which is derived using a single term deflection function. Hence, the influence coefficients in

Table 4
Comparison of elastic constants obtained from the measured frequencies in Tables 1–3 using frequency expressions (6) and (28) to examine the impact of influence coefficients

Elastic constants	Materials					
	Aluminium		Glass/epoxy		Carbon/epoxy	
	Without influence coefficient Eq. (6)	With influence coefficient Eq. (28)	Without influence coefficient Eq. (6)	With influence coefficient Eq. (28)	Without influence coefficient Eq. (6)	With influence coefficient Eq. (28)
E_{11} (GPa)	72.16	74.50	67.65	69.17	117.11	118.70
E_{22} (GPa)	66.68	71.74	68.51	73.19	9.76	10.52
ν_{12}	0.33	0.31	0.23	0.21	0.28	0.26
G_{12} (GPa)	25.55	27.30	26.19	29.78	4.53	5.15

Table 5
Comparison of elastic constants obtained from the measured frequencies of Ref. [1] using the frequency expression (28) with the specified values of Ref. [1]

Elastic constants	Aluminium			Graphite/epoxy		
	Specified value Ref. [1]	Estimated		Specified value Ref. [1]	Estimated	
		Ref. [1]	Present study		Ref. [1]	Present study
E_{11} (GPa)	72.40	69.50	74.50	127.90	125.20	127.79
E_{22} (GPa)	72.40	69.90	71.74	10.27	10.30	10.79
ν_{12}	0.33	0.36	0.31	0.22	−0.24	0.19
G_{12} (GPa)	28.00	25.60	27.30	7.31	6.60	6.95

Eq. (28) for free–free plate configurations vary from 0.85 to 1.08, and the values of natural frequencies from Eqs. (6) and (28) will have a maximum of 5% difference. The large discrepancy in the results of analytical and experimental results is mainly due to the assumed elastic constants. It can be seen from the results presented in Table 5 that the use of influence coefficients in the frequency expression, improve the estimates of the elastic constants compared to those obtained in Ref. [1]. The modified frequency expression (28) is one of the fast means to identify the eigenvalues of the model indices close to the exact. The method of evaluation of elastic constants from the measured frequencies in the present study is different from those followed in Refs. [1–3]. In the present analysis, all the measured natural frequencies of the plate are considered while evaluating the elastic constants. In Ref. [1], several sets of the measured natural frequencies were made and obtained directly the four elastic constants for each set having four measured frequencies, and presented average values of the elastic constants. Inaccuracy in any one of the minimum required four measured natural frequencies in a set, may cause absurd estimates of elastic constants. It is preferable to use more measured natural frequencies than the minimum of four required, while evaluating the elastic constants.

6. Conclusion

This paper discusses the determination of elastic constants, through vibration testing, of thin specially orthotropic plates with free boundary conditions. The natural frequencies obtained from Eq. (6) using the Rayleigh–Ritz method with a single term expression having characteristic functions of a free–free beam for the deflection function of the plate, are always higher than the exact values. Influence coefficients are introduced in the frequency expression through least square curve fit of the natural frequencies of aluminium plate from a finite element solution. The modified frequency expression having influence coefficients is validated through the finite element solution of Ref. [1], related to graphite/epoxy plates. The four elastic constants Young's modulus (E_{xx}), transverse Young's modulus (E_{yy}), major the Poisson ratio (v_{xy}), and the in-plane shear modulus (G_{xy}) of a specially orthotropic material have been found through measured frequencies from a single vibration test. The modified frequency expression is useful for the evaluation of natural frequencies of free–free specially orthotropic rectangular thin plates.

The influence coefficients in the modified frequency expression (28) depend on the boundary conditions of the vibrating plate. For the case of free–free rectangular plates, the squared frequency values of the finite element solutions were found to be slightly lower than those obtained from Eq. (6). Thus, estimates of elastic constants from the natural frequencies using Eqs. (6) and (28) showed little variation. For vibrating plates with other supporting conditions, the influence coefficients will be different and appreciable variation in frequency values can be expected from Eqs. (6) and (28). However, we need fast means to approximate and identify the eigenvalues corresponding to given model indices, which is possible through the modified frequency expression (28).

Acknowledgements

The authors would like to thank the reviewers for their valuable suggestions to improve the clarity of the paper.

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