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The Hilbert phenomenon in chaotic motions

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Abstract

It is proved that chaotic dynamical systems have $\mathscr{D}^1$-singularity. Due to such singularity and its periodicity in Hausdorff metric spaces, a chaotic dynamical system induces a Hilbert iterated system, which was discovered at the end of the 19th century. This implies that the most puzzling discovery in nature in the 20th century can be put into deep correspondence with that in thoughts in the 19th century. All this is verified by the numerical experiments of chaotic vibrations of a beam and a tension-slack oscillator.

1. Introduction

Chaos is one of the most puzzling discoveries in the 20th century. Chaotic motion has attracted wide attention in modern science and technology [1–35], and has been found in various fields, such as atmospheric science, classical mechanics, civil engineering and so on. To understand such strange phenomenon, numerous researchers investigated it by means of different methods, from different points of view. In computer simulations Thompson and Ghaffari [7] observed the phenomenon of period doubling route to chaos of the impact oscillator in the marine structural dynamics. Shaw and Holmes [8] studied the stability, bifurcations and chaos of the system by examining the eigenvalues of the Jacobian matrix of the Poincaré map. Kim and Noah [17] developed a modified harmonic balance/Fourier transform to analyze the stability, bifurcations and chaos of a impact system. Lu [4,5] observed the periodic behavior of chaotic oscillators in Hausdorff metric spaces.

In this paper it is rigorously proved that any chaotic dynamical systems have $\mathscr{D}^1$-singularity. Although chaotic dynamical systems are non-periodic in Euclidean metric spaces, the researches [4,5] showed that they are periodic in the Hausdorff metric space. We find that the $\mathscr{D}^1$-singularity

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and the periodicity of chaotic dynamical systems in the Hausdorff metric space induce a Hilbert iterated system, which was discovered in 1891 [36]. It is well known that the Hilbert iterated system is one of the most famous examples in the critical investigation of geometry at the end of the 19th century, which played important roles in the development of many mathematical branches in the 20th century. Thus, the work presented here build a bridge between the most puzzling discovery in nature in the 20th century and that in thoughts in the 19th century. To verify the theory, we observe the Hilbert phenomena hidden in the chaotic vibrations of a buckled beam and a tension-slack oscillator. The results of the experiments confirm the predictions.

2. Non-autonomous dynamical systems and Hausdorff phase spaces

Because this research is based on the study of non-autonomous dynamical systems in Hausdorff metric spaces (HMS), here, it is necessary to give a brief review of some results in the work presented by Lu.

First, let us introduce the main idea of the work [4,5]. One of the most important properties of chaotic systems is that the responses of the deterministic systems to periodic excitation are non-periodic. This is in conflict with the traditional belief that there should exist periodic elements in the responses of a deterministic dynamical system to periodic external excitation. However, we have to recognize that the behavior of a dynamical system is always observed in a specific phase space. From physical point of view, a phase space is a logic one in which the object is observed in some manner. An object can be observed from different points of view, i.e., the same system can be investigated in different phase spaces. Since dynamical systems were framed by Issac Newton, they have been investigated in Euclidean metric space all along. The existence of chaotic systems shows that Euclidean metric spaces may no longer be suitable for the observation of the behavior of chaotic oscillators. The experiment results showed that chaotic dynamical systems are periodic in HMS.

Let (Λ, μ) be a metric space. $\phi(\tau, t_0, \cdot)$, $\tau, t_0, \in \mathbf{R}$, denotes a double parameter family of C^β maps of the metric space (Λ, μ) onto itself.

Definition 1. $\phi(\tau, t_0, \cdot)$ is called a (continuous) non-autonomous dynamical system (or a non-autonomous flow) in the metric space (Λ, μ), if it satisfies

$$\begin{cases} \phi(0, t_0, p) = p, & p \in \Lambda, \quad t_0 \in \mathbf{R}, \\ \phi(s + r, t_0, p) = \phi[s, t_0 + r, \phi(r, t_0, p)], & p \in \Lambda, \quad s, r, t_0 \in \mathbf{R}. \end{cases} \tag{1}$$

Particularly, if $\tau, s, r, t_0 \in \mathbf{N}$, ϕ is called a discrete, non-autonomous dynamical system in (Λ, μ). (Λ, μ) is called a phase space of ϕ (see Ref. [4]).

When the problem is independent of t_0, $\phi(\tau, t_0, \cdot)$ reduces to $\phi(\tau, \cdot)$, which is the subject of the modern theory of dynamical systems [28]. Both of the continuous autonomous and non-autonomous dynamical systems were suggested to be called flow by Refs. [4,5].

Most of dynamical problems in various fields, such as physics, mechanics, civil engineering and so on, are described by following non-autonomous differential equations in the n-dimensional

Euclidean metric space $(\mathbf{R}^n, d)$,

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{F}(\mathbf{X}, t), \tag{2}$$

where $\mathbf{X} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector, $\mathbf{F} = [f_1, f_2, \ldots, f_n]^{\mathrm{T}}$.

If ϕ is the solution of Eq. (2), it can be proved that ϕ satisfies that [5]

$$\begin{aligned} &\phi(0, t_0, p) = p, && p \in \mathbf{R}^n, \quad t_0 \in \mathbf{R}, \\ &\phi(s + r, t_0, p) = \phi[s, t_0 + r, \phi(r, t_0, p)], && p \in \mathbf{R}^n, \quad s, r, t_0 \in \mathbf{R}. \end{aligned} \tag{3}$$

This implies that Eq. (2) describes a flow in Euclidean metric space. To make a good observation of chaotic flow, it is necessary to find another metric space and prove that the solution ϕ of Eq. (2) also satisfies Eq. (1) in the new metric space.

Let $\mathbf{H}^n$ be the collection of all non-empty closed subsets of $\mathbf{R}^n$. From the point of view of the observers in $\mathbf{R}^n$, a point of $\mathbf{H}^n$ may be a set containing numerous points of $\mathbf{R}^n$. The distance between $p(\in \mathbf{R}^n)$ and $A(\in \mathbf{H}^n)$ is defined as

$$\varrho(p, A) = \inf\{d(p, r), r \in A\}. \tag{4}$$

The Hausdorff distance between two points $A, B \in \mathbf{H}^n$ is defined as

$$\rho(A, B) = \sup\{\sup[\varrho(p, A), p \in B], \sup[\varrho(q, B), q \in A]\}. \tag{5}$$

$(\mathbf{H}^n, \rho)$ is a complete metric space. This metric space was often used by F. Hausdorff (1868–1942). Thus, Lu [4] called it a Hausdorff metric space.

Theorem 2. *If ϕ is a flow in the Euclidean metric space $(\mathbf{R}^n, d)$, then, it is also a flow in the corresponding Hausdorff metric space $(\mathbf{H}^n, \rho)$ (see Ref.* [4]).

Proof. The definition of the Hausdorff metric space shows that, if $A \in \mathbf{H}^n$, then $A \subset \mathbf{R}^n$. Therefore, Eq. (3) leads to

$$\begin{aligned} \phi(0, t_0, A) &= \{\phi(0, t_0, p) : p \in A\} \\ &= \{p : p \in A\} = A, \quad t_0 \in \mathbf{R}, A \in \mathbf{H}^n, \end{aligned} \tag{6}$$

$$\begin{aligned} \phi(r + s, t_0, A) &= \{\phi(r + s, t_0, p) : p \in A\} \\ &= \{\phi[s, t_0 + r, \phi(r, t_0, p)] : p \in A\} \\ &= \phi[s, t_0 + r, \phi(r, t_0, A)], \quad s, r, t_0 \in \mathbf{R}, \quad A \in \mathbf{H}^n. \end{aligned} \tag{7}$$

Eqs. (1), (6) and (7) show that non-autonomous flows in Euclidean metric spaces are also ones in Hausdorff metric spaces. This conclusion is also valid for an autonomous flow. □

In summary, to explore the behavior of chaotic oscillators in a new space, we proceeded in three steps. First, we defined a double parameter family of maps satisfying a specific condition in a general metric space as a dynamical system (a flow) in the metric space.

Secondly, a real oscillator can be described by an ordinary differential equation in a n-dimensional Euclidean metric space. The n-dimensional differential equation can be mathematically summarized in a double parameter family of the maps in n-dimensional

Euclidean metric space. It can be proved that such maps satisfy the specific condition in the Euclidean metric space. This implies that a real oscillator determines a flow in Euclidean space.

At last, we found a non-Euclidean space—Hausdorff metric space—and proved that a flow in the Euclidean metric space also satisfies the specific condition in the Hausdorff metric space. This means that a real oscillator also determines a flow in the Hausdorff metric space. Although chaotic motion is non-period in the Euclidean metric space, but it has perfect periodicity in the Hausdorff metric space.

In the modern theory of dynamical systems, non-autonomous systems are always made autonomous by redefining time as a new dependent variable, and a dynamical system is defined as a single parametric family of maps. Refs. [4,5] suggested that a dynamical system should be defined as a double parameter family of maps. This modification which may seem a trivial matter at first glance is, in fact, a very important step in the exploration of chaos. It led us to the discovery of the periodicity of chaotic systems, and now also help us to find the Hilbert phenomenon in chaos.

3. The Hilbert phenomenon in chaotic motions

The discussions in the above section showed that a n-dimensional dynamical problem not only describes a map in $\mathbf{R}^n$, but also describes a map in $\mathbf{H}^n$. In this section we will reveal an important property of the map described by a chaotic oscillator, i.e., the Hilbert phenomenon hidden in chaos.

Before investigating the Hilbert phenomena in chaotic vibrations, we have to clear that natural phenomena may be divided into two categories: those that can be observed with certainty and those that can be observed with a probability. The phenomena which can be observed with certainty may also be divided into two categories: those that can be observed in any cases and those that can be observed with the probability equal to one. The former may be referred to as a "math-certainty phenomenon", while the latter may be referred to as a "physics-certainty phenomenon". From physical point of view, there is no difference between these two kinds of phenomena. However, one has to pay attention to the difference, if trying to reveal a physical phenomenon by mathematical methods. The Hilbert phenomenon existing in chaotic motion is a physics-certainty phenomenon, but not a math-certainty phenomenon.

We have to do some mathematical preparation to present a statistical experiment investigating the Hilbert phenomenon in chaos. In this paper C is said to be a simple curve, if C is C^r and homeomorphic to a line segment. Let $\mathscr{L}(C)$ denote the length of a simple curve C. Because ϕ satisfies Eq. (3), we have

$$p_0 = \phi(0, t_0, p_0) = \phi(-r, t_0 + r, p), \tag{8}$$

where $p = \phi(r, t_0, p_0)$, $p_0 \in \mathbf{R}^n$. This shows that $\{\phi(\tau, t_0, \cdot), \tau, t_0 \in \mathbf{R}\}$ is a double parameter family of C^r homeomorphisms of $\mathbf{R}^n$ onto itself. Therefore, $\phi(\tau, t_0, C)$ is homeomorphic to C for any given $\tau, t_0 \in \mathbf{R}$. This implies that $\mathscr{L}(\cdot)$ can also be applied to $\phi(\tau, t_0, C)$.

Let us consider a statistical experiment that one observes the limit

$$\lim_{\tau \to \infty} \frac{\mathscr{L}[\phi(\tau, t_0, C)]}{\mathscr{L}[C]}, \tag{9}$$

after choosing at random a simple curve C from U^n, $U^n \subset \mathbf{R}^n$. E_∞ denotes the event that the limit (9) is infinity. E_∞ is a sample point of the statistical experiment.

Definition 3. The dynamical system ϕ has $\mathscr{D}^1$-singularity on U^n, if (i) the probability of the event E_∞ occurring is one; (ii) for any $p, q \in C$, $\tau \in \mathbf{R}$,

$$d[\phi(\tau, t_0, p), \phi(\tau, t_0, q)] < \text{Constant}, \tag{10}$$

where $d(\cdot)$ is the Euclidean distance between two points.

Although $\mathscr{D}^1$-singularity is very strange, the following theorem demonstrates that any chaotic dynamical systems have such $\mathscr{D}^1$-singularity.

Theorem 4. *The dynamical system ϕ has $\mathscr{D}^1$-singularity on U^n, if it is chaotic on U^n.*

Proof. Because ϕ is chaotic on U^n, Eq. (10) is satisfied.

$T_p(\cdot)$ denotes the tangent space of a manifold at point p. $D\phi_p(\tau, t_0, \cdot)$ is the linear mapping induced by ϕ which maps $T_p(\cdot)$ into $T_{\phi(\tau,t_0,p)}(\cdot)$. $\|\cdot\|$ denotes the Euclidean norm of a vector. Let

$$\mathbf{L}_p(\phi) = \{q : \lambda_p(\vec{e}_{qp}) \leqslant 0,\ q \in \mathbf{R}^n\}, \quad p \in U^n, \tag{11}$$

where $\vec{e}_{qp} = \vec{e}_q - \vec{e}_p$, $\vec{e}_p = [x_1(p), x_2(p), \ldots, x_n(p)]^{\mathrm{T}}$, $x_i(p)$ is the co-ordinate of the point p, $\vec{e}_q = [x_1(q), x_2(q), \ldots, x_n(q)]^{\mathrm{T}}$, $x_i(q)$ is the co-ordinate of the point q, $\lambda_p(\vec{e}_{qp})$ is the Lyapunov exponent relative to p and $\vec{e}_{qp}$. Because ϕ is chaotic on U^n, it has at least one positive Lyapunov exponent. This shows that the dimension of $\mathbf{L}_p(\phi)$ is less than n.

If ϕ has only one positive Lyapunov exponent, then $\mathbf{L}_p(\phi)$ is $(n-1)$-dimensional. After ϕ and p are given, $\mathbf{L}_p(\phi)$ is fixed. Thus, ϕ induces a mapping

$$\mathscr{F}_\phi : \mathbf{R}^n \times \mathbf{R}^{n-2} \to \mathbf{R}. \tag{12}$$

Let

$$\begin{aligned}\mathscr{S}_\phi = \{(y_1, \ldots, y_{2n-1}) : y_i &= x_i(p),\ i = 1, \ldots, n,\\ y_{n+j} &= x_j(q),\ j = 1, \ldots, n-2,\\ y_{2n-1} &= \mathscr{F}_\phi(p, q),\ p \in U^n, q \in \mathbf{R}^{n-2}\}.\end{aligned} \tag{13}$$

$\mathscr{S}_\phi$ is a $(2n-2)$-dimensional sub-manifold of $\mathbf{R}^{2n-1}$. When $n = 2$, $\mathscr{F}_\phi$ is a mapping from $\mathbf{R}^2$ onto $\mathbf{R}$, and $\mathscr{S}_\phi$ is a surface in $\mathbf{R}^3$.

Consider a point p on $\mathscr{S}_\phi$. Let p_u be the projection from p to $\mathbf{R}^n$. $p_u \in U^n \cdot p$ describes a line in $\mathbf{R}^n$, which passes through the point p_u and is in the direction in which the Lyapunov exponent at p_u is less than or equal to zero. $y_i(p)$, $i = 1, 2, \ldots, n$, describe the position of p_u in U^n. $y_i(p)$, $i = n+1, n+2, \ldots, 2n-1$, describe the direction of the line.

Now let us consider a curve C in $\mathbf{R}^n$. $T_p(C)$ is fixed after the curve C and p are given. Thus, after picking up a curve C from $\mathbf{R}^n$, we obtain a mapping

$$\mathscr{F}_c : C \to \mathbf{R}^{n-1}. \tag{14}$$

Let

$$\begin{aligned}\mathscr{C}_c = \{(y_1, \ldots, y_{2n-1}) : y_i &= x_i(p),\ i = 1, \ldots, n,\\ y_{n+j} &= x_j[\mathscr{F}_c(p)],\ j = 1, \ldots, n-1, p \in C\}.\end{aligned} \tag{15}$$

$\mathscr{C}_c$ is a curve in $\mathbf{R}^{2n-1}$. A point p on $\mathscr{C}_c$ describes a $T_{p_u}(C)$, where p_u is the projection from p to U^n, $p_u \in C$.

If $\mathscr{C}_c \subset \mathscr{S}_\phi$, then for any point $p \in C$, $T_p(C)$ is in the direction in which the Lyapunov exponent of ϕ at p is less than or equal to zero.

Choose at random a point p from U^n. $\mathbf{C}_p$ is the collection of all curves in U^n, which pass through p. Choose at random a curve C from $\mathbf{C}_p$. Let

$$\gamma_p(C) = \sup[\gamma = \vec{a} \cdot \vec{b}/(\|\vec{a}\| \, \|\vec{b}\|), \vec{a} \in T_p(C), \vec{b} \in \mathbf{L}_p(\phi)]. \tag{16}$$

Because $\mathbf{L}_p(\phi)$ is $(n-1)$-dimensional, the probability that $\gamma_p(C) < 1$ is one. Thus, we have at least a point p on $\mathscr{C}_c$, such that the probability of the event

$$\varrho(p, \mathscr{S}_\phi) > 0, \tag{17}$$

occurring is one, after choosing at random a curve C from U^n. Therefore, we have at least a segment C_g in C, such that the event

$$\mathscr{C}_{c_g} \cap \mathscr{S}_\phi = \Phi, \tag{18}$$

occurs with certainly (i.e., with probability equal to one).

Eq. (18) shows that, for any given $\tau_0 \in \mathbf{R}$, there exists $\tau_m > \tau_0$ such that

$$\|D\phi_p(\tau_0, t_0, \vec{e})\| < \|D\phi_p(\tau_m, t_0, \vec{e})\|, \quad p \in C_g, \vec{e} \in T_p(C_g). \tag{19}$$

After dividing C_g into $n-1$ segments, we obtain n points p_i, $i = 1, 2, \ldots, n$, on C_g. We have

$$\begin{aligned} \mathscr{L}[\phi(\tau_m, t_0, C_g)] &= \lim_{n\to\infty} \sum_{i=1}^{n} \|\mathbf{X}(\phi(\tau_m, t_0, p_i)) - \mathbf{X}(\phi(\tau_m, t_0, p_{i-1}))\| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \|D\phi_{p_{i-1}}[\tau_m, t_0, (\mathbf{X}(p_i) - \mathbf{X}(p_{i-1}))]\|, \end{aligned} \tag{20}$$

where $\mathbf{X}(p) = [x_1(p), x_2(p), \ldots, x_n(p)]^{\mathrm{T}}$. Eqs. (19) and (20) show that, for any given $\tau_0 \in \mathbf{R}$, there exists $\tau_m > \tau_0$ such that

$$\begin{aligned} \mathscr{L}[\phi(\tau_m, t_0, C_g)] &> \lim_{n\to\infty} \sum_{i=1}^{n} \|D\phi_{p_{i-1}}[\tau_0, t_0, (\mathbf{X}(p_i) - \mathbf{X}(p_{i-1}))]\| \\ &= \mathscr{L}[\phi(\tau_0, t_0, C_g)]. \end{aligned} \tag{21}$$

Thus, if ϕ is chaotic, then the probability of the event E_∞ occurring is unity.

If the number of the positive Lyapunov exponents is larger than 1, then the dimension of $\mathbf{L}_p(\phi)$ is less than or equal to $n-2$. Eq. (18) is still true in this case. Thus, the above conclusion is also true.

In summary, if ϕ is chaotic, then it is $\mathscr{D}^1$-singular. □

Now let us give a intuitive description of the singularity of a flow. If ϕ has $\mathscr{D}^1$-singularity in U^n, then, after choosing at random a simple curve C from U^n, one will find with certainty that

(a) *for any* τ, $\phi(\tau, t_0, C)$ *is a curve without any break, adhesion or intersection, i.e., the curve* $\phi(\tau, t_0, C)$ *is homeomorphic to* C;
(b) *the curve* $\phi(\tau, t_0, C)$ *is bounded*;
(c) *the length of the curve* $\phi(\tau, t_0, C)$ *tends to infinity as* τ *increases*;

Definition 3 can be easily generalized. $V^k(\cdot)$ denotes the k-dimensional volume of a k-dimensional sub-manifold of $\mathbf{R}^n$. $V^2(\mathscr{M}^2)$ is the area of a surface $\mathscr{M}^2$. The dynamical system

ϕ is $\mathscr{D}^k$-singular, if (i) the probability of the event

$$\lim_{\tau \to \infty} \frac{V^k[\phi(\tau, t_0, \mathscr{M}^k)]}{V^k[\mathscr{M}^k]} = \infty, \tag{22}$$

occurring is one, where $\mathscr{M}^k$ is a k-dimensional, random sub-manifold of $\mathbf{R}^n$; (ii) for any $p, q \in \mathscr{M}^k$, $\tau \in \mathbf{R}$, Eq. (10) is satisfied.

If ϕ has $\mathscr{D}^2$-singularity, and if S is a simple surface chosen at random from U^n, then $\phi(\tau, t_0, S)$ is a bounded, smooth surface without any break, adhesion or intersection, whose area tends to infinity as τ increases.

(a)–(c) describe an incredible phenomenon. It reminds us of the famous mathematical phenomenon "created" by D. Hilbert. It is well known that the Hilbert curve [36] was one of the most famous examples in the critical investigation of geometry at the end of the 19th century. In 1891 Hilbert created a strange iterated system $H_i(\cdot)$. H_i has the following properties:

(a') *for any i*, $H_i(L)$ *is a curve without any break, adhesion or intersection, where L is a line segment, i.e.*, $H_i(\cdot)$ *is a homeomorphism on* $\mathbf{R}^2$;
(b') $H_i(L)$ *is bounded*;
(c') *The length of* $H_i(L)$ *will tend to infinity as i increases*;
(d') $\{H_i(L),\ i = 1, 2, 3, \ldots\}$ *is a Cauchy sequence in the Hausdorff metric space.*

The evolution of the Hilbert curve $H_i(L)$ is shown in Fig. 1.

Although chaotic motion is non-periodic in the Euclidean space, it is periodic in the Hausdorff metric space. T_h denotes the period of chaotic ϕ in the Hausdorff metric space. Let

$$\varphi_i(\cdot) = \phi(iT_h, t_0, \cdot). \tag{23}$$

φ_i is a iterated system induced by ϕ. It has the following properties:

(a'') *for any i*, $\varphi_i(\tau, t_0, C)$ *is a curve without any break, adhesion or intersection, i.e., the curve* $\varphi_i(C)$ *is homeomorphic to* C;
(b'') *the curve* $\varphi_i(C)$ *is bounded*;
(c'') *the length of the curve* $\varphi_i(C)$ *tends to infinity as i increases*;
(d'') $\{\varphi_i(C),\ i = 1, 2, 3, \ldots\}$ *is a Cauchy sequence in the Hausdorff metric space.*

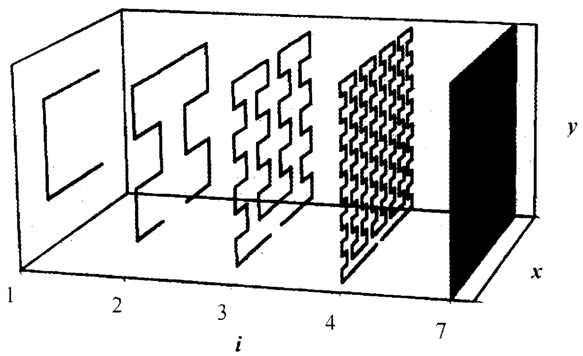

Fig. 1. The evolution of the Hilbert curve $H_i(L)$.

(a')–(d') and (a'')–(d'') show that φ_i is the Hilbert system hidden in chaotic motion. Please note that Hilbert system is "created" in the mind in the 19th century, while chaos is "discovered" in nature in the 20th century. The deep correspondence between them is really surprising.

4. Numerical experiments

To verify the predictions made above, we investigated many chaotic dynamical systems. The experiment results confirmed the predictions. Here we present the results of the observations of the Hilbert phenomenon in chaotic vibrations of a beam and a piecewise linear system.

Chaotic vibrations of beams are one of the most important examples of chaos in solids and structures. They attracted considerable interest in the past and continue to do so [9–12]. Consider a cantilevered beam shown in Fig. 2. In this example two magnets are moved toward the free end of a ferromagnetic cantilevered beam until the straight position becomes unstable. The single mode equation is a form of Duffing's equation which, in non-dimensional form, becomes

$$\frac{d^2x}{d\tau^2} + \zeta\frac{dx}{d\tau} - \frac{1}{2}x(1-x^2) = F_e \sin\omega\tau, \tag{24}$$

where $x(\tau)$ is the non-dimensional amplitude of the first bending mode of the beam.

Let $y = dx/d\tau$. Eq. (24) can be rewritten as

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{F}(\mathbf{X}, \tau), \tag{25}$$

where $\mathbf{X} = [x, y]^T \in \mathbf{R}^2$, $\mathbf{F}(\mathbf{X}, \tau) = [f_1, f_2]^T$, and

$$\begin{cases} f_1 = y, \\ f_2 = -\zeta y + \frac{1}{2}x(1-x^2) + F_e \sin\omega\tau. \end{cases} \tag{26}$$

The discussions in Section 2 show that Eq. (25) describes a non-autonomous flow in $\mathbf{R}^2$, i.e., its solution $\phi^b(\tau, t_0, \cdot)$ satisfies Eq. (3). Moreover, Theorem 2 shows that $\phi^b(\tau, t_0, \cdot)$ is not only a non-autonomous flow in the Euclidean space $\mathbf{R}^2$, but also one in the Hausdorff metric space $\mathbf{H}^2$. This implies that the vibrations of the beam shown in Fig. 2 describe a non-autonomous flow in $\mathbf{R}^2$ and $\mathbf{H}^2$.

The analysis of the Poincaré maps, phase trajectories, Lyapunov exponents of system (25) shows that ϕ^b is chaotic when $\zeta = 0.165$, $F_e = 0.21$ and $\omega = 1$. C_L denotes a line segment $\{(x, y) : x = y, 0 \leqslant x \leqslant 1\}$. $\phi^b(\tau, t_0, C_L)$ is the image of C_L under ϕ^b. If C_L is taken as a set of initial states, i.e, any point on C_L is taken as an initial condition of Eq. (24), then, $\phi^b(\tau, t_0, C_L)$ is the set of the states of system (24) at time $t_0 + \tau$. The discussions in the above section predicted that $\phi^b(\tau, t_0, C_L)$ has the

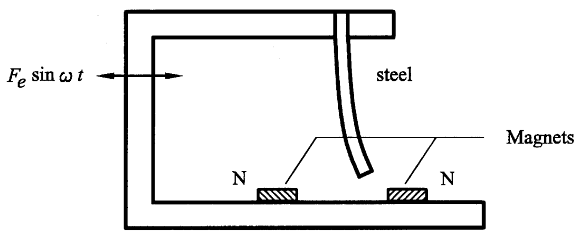

Fig. 2. A buckled beam and two magnets.

properties (a)–(c), i.e., it is a bounded curve without any break, adhesion or intersection, whose length will tend to infinity as τ increases. To verify the prediction, we observe the evolution of the curve. Figs. 3(a), (b) and 4(a) are respectively the images of the initial set C_L under ϕ^b at time $0.5T_e$, T_e, $3T_e$, where $T_e = 2\pi/\omega$ is the period of the external excitation, $t_0 = 0$. Physically speaking, a point in $\phi^b(3T_e, 0, C_L)$ represents the state of system (24) at time $3T_e$. Fig. 4(b) is the enlargement of a small window on $\phi^b(3T_e, 0, C_L)$. One can observe such interesting phenomenon for any given time interval τ, as long as his computer is powerful enough. As the time interval increases, the length of the curve tends to infinity. Figs. 5(a) and (b) show the curves $\phi^b(25.5T_e, 0, C_L)$ and $\phi^b(30T_e, 0, C_L)$. The curve with infinite length "covers" an area in the phase space.

Our experiments showed that when $\zeta = 0.165$, $F_e = 0.21$ and $\omega = 1$, the chaotic system (24) is periodic with T_e in the Hausdorff metric space, i.e., $T_h = T_e$. Let

$$\psi_i^b(\cdot) = \phi^b(iT_e, 0, \cdot). \tag{27}$$

That ϕ^b is periodic with T_e in the Hausdorff metric space shows that, for any $\varepsilon > 0$ there exists $\mathbf{S}^b \in \mathbf{H}^2$ and N satisfying

$$\rho[\mathbf{S}^b, \psi_i^b(C_L)] < \varepsilon, \tag{28}$$

where $\mathbf{S}^b$ is the strange invariant set [4]. This implies that $\psi_i^b(C_L)$, $i = 1, 2, 3, \ldots$, is a Cauchy sequence in the Hausdorff metric space. Fig. 6 shows the evolution process of the sequence

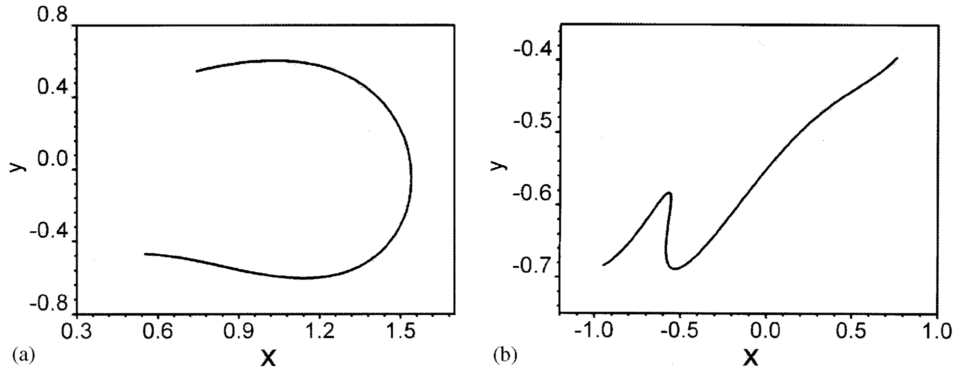

Fig. 3. The curve $\phi^b(\tau, 0, C_L)$ (a) $\tau = 0.5T_e$, (b) $\tau = T_e$.

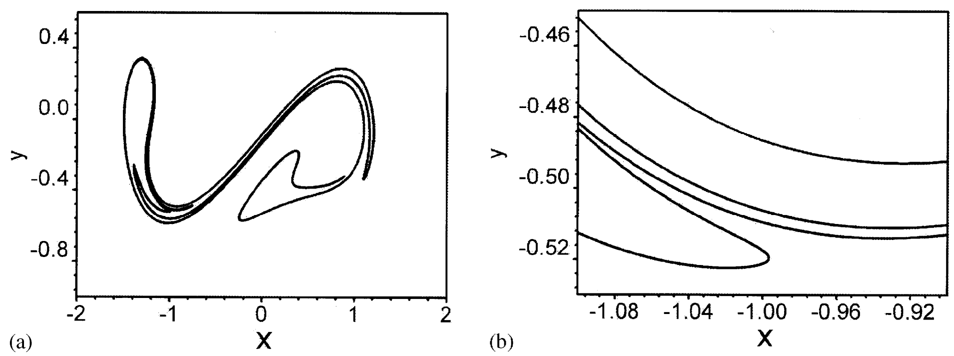

Fig. 4. The $\phi^b(\tau, 0, C_L)$ (a) $\tau = 3T_e$, (b) the enlargement of a small window on $\phi^b(3T_e, 0, C_L)$.

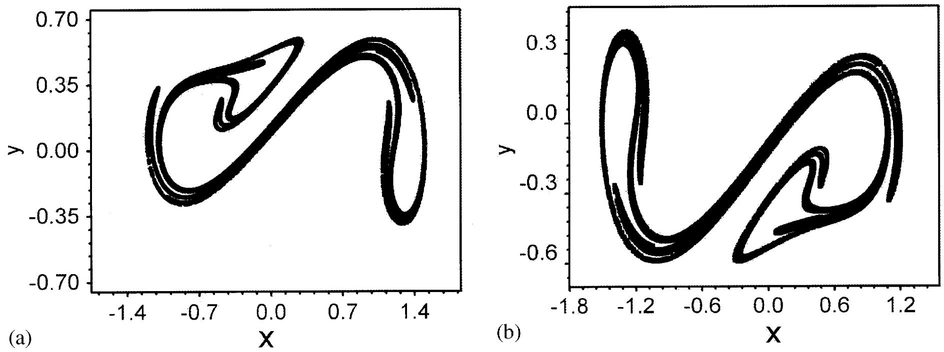

Fig. 5. The curve $\phi^b(\tau, 0, C_L)$ (a) $\tau = 25.5T_e$, (b) $\tau = 30T_e$.

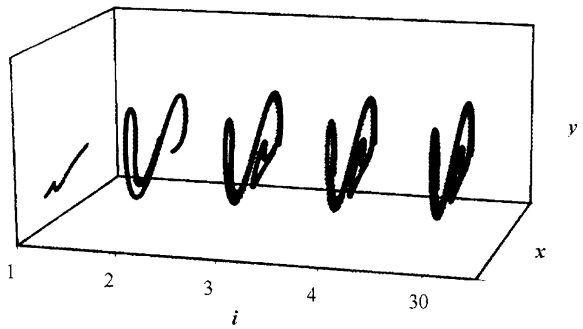

Fig. 6. The Hilbert phenomenon induced by the chaotic vibrations of the beam.

$\psi_i^b(C_L)$, $i = 1, 2, 3, \ldots$. All this shows that $\psi_i^b(C_L)$ has the properties (a'')–(d''). It is a Hilbert system induced by the chaotic vibrations of the beam shown in Fig. 2.

As a second example, let us observe the Hilbert phenomenon hidden in a chaotic piecewise linear system. Piecewise linear models are used as an approximation to dynamical problems in various engineering fields. Many researches showed that a piecewise linear oscillator may exhibit strongly chaotic motion [6–8,13–17]. Lu et al. [6] studied the bifurcation and chaos of a simple piecewise linear system called tension-slack oscillator. The periodic behavior of its chaotic motion in Hausdorff spaces was investigated by Lu et al. [5]. Here, we will observe the Hilbert phenomenon induced by the chaotic tension-slack oscillator.

Consider a typical tension-slack oscillator shown in Fig. 7. A mass m is attached to a spring of stiffness k and a linear dashpot with damping factor c, where $k = k_1$ when the spring is stretched, and $k = k_2$ when the spring is compressed. $k_1 \gg k_2$. When the system is externally excited by a harmonic base movement, the non-dimensional equation of motion may be written as

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{F}(\mathbf{X}, \tau), \tag{29}$$

where

$$\mathbf{F}(\mathbf{X}, \tau) = \left\{ \begin{array}{c} y \\ -2\zeta y - \theta x + F_e \sin \omega\tau + (1 - \theta) \end{array} \right\}, \tag{30}$$

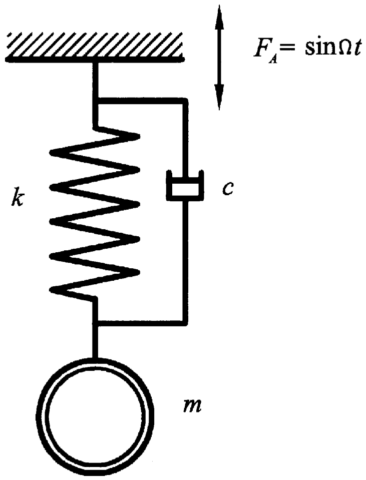

Fig. 7. A tension-slack oscillator.

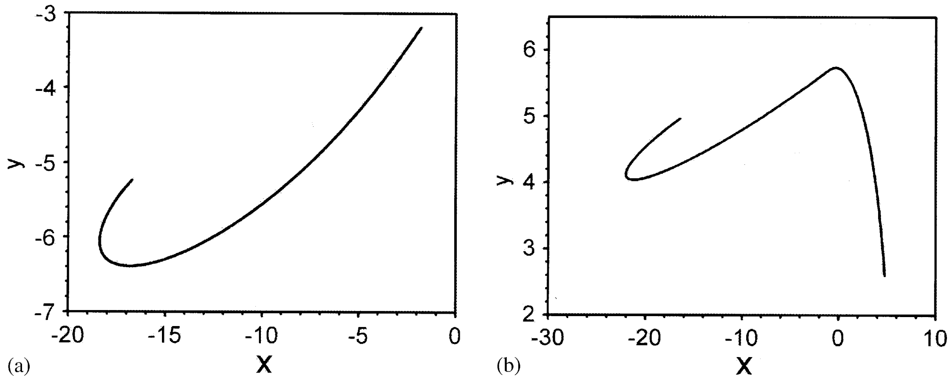

Fig. 8. The curve $\phi^b(\tau, 0, C_L)$ (a) $\tau = 3T_e$, (b) $\tau = 3.5T_e$.

$\mathbf{X} = [x, y]^{\mathrm{T}} \in \mathbf{R}^2$, $y = \mathrm{d}x/\mathrm{d}\tau$, $\theta = 1$ when $x \geqslant -1$, $\theta = k_2/k_1$ when $x < -1$, $\zeta = c/(2m\sqrt{k_1/m})$, $\omega = \Omega/\sqrt{k_1/m}$, $\Delta d = mg/k_1$, $x = z/\Delta d$, $F_e = \omega^2 F_A/\Delta d$, $\tau = \omega t$ is the non-dimensional time parameter, Ω is the frequency of external excitation, z is the additional dynamical displacement, F_A is the amplitude of external excitation.

System (29) describes a non-autonomous flow $\phi^s(\tau, t_0, \cdot)$ in $\mathbf{R}^2$ and $\mathbf{H}^2$. ϕ^s is chaotic when $\zeta = 0.025$, $k_2 = 0$, $F_e = 2$ and $\omega = 0.7$. Similar to the above discussions, $\phi^s(\tau, t_0, C_L)$ is also a bounded curve without any break, adhesion or intersection, whose length will tend to infinity as τ increases. Figs. 8(a), (b), 9(a), 10(a) and (b) are respectively the images of the initial set C_L under ϕ^s at time $3T_e$, $3.5T_e$, $7T_e$, $32.5T_e$, $35T_e$, where $T_e = 2\pi/\omega$ is the period of the external excitation, $t_0 = 0$. Fig. 9(b) is the enlargement of a small window on $\phi^s(7T_e, 0, C_L)$.

Let

$$\psi_i^s(\cdot) = \phi^s(iT_e, 0, \cdot). \tag{31}$$

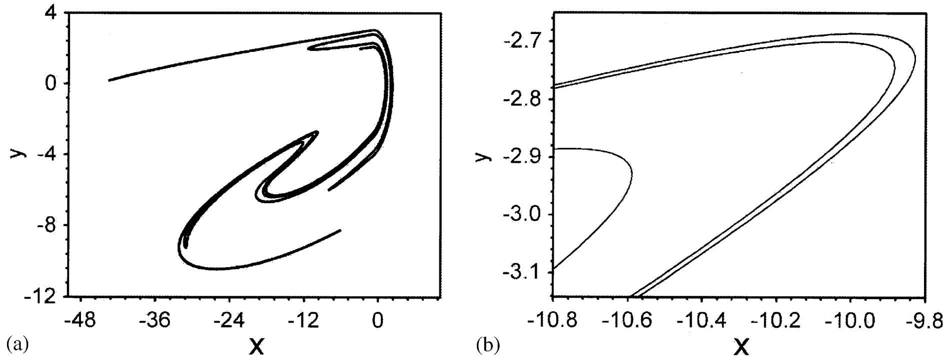

Fig. 9. The $\phi^b(\tau, 0, C_L)$ (a) $\tau = 7T_e$, (b) the enlargement of a small window on $\phi^b(7T_e, 0, C_L)$.

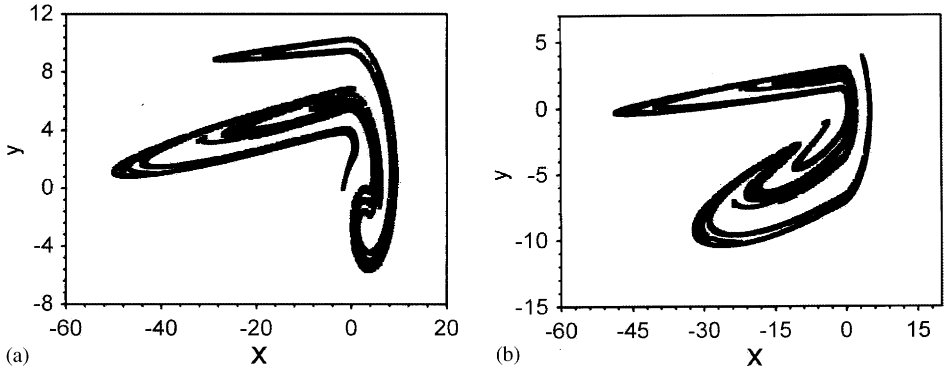

Fig. 10. The curve $\phi^b(\tau, 0, C_L)$ (a) $\tau = 32.5T_e$, (b) $\tau = 35T_e$.

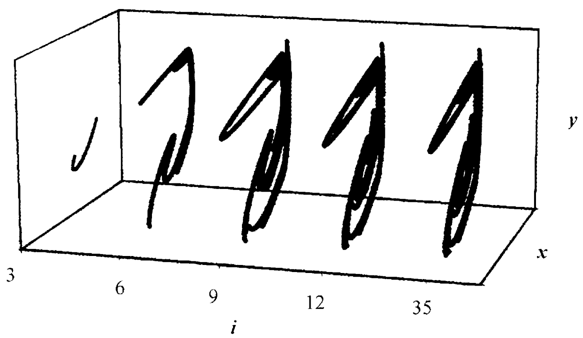

Fig. 11. The Hilbert phenomenon induced by the chaotic tension-slack oscillator.

The work [5] shows that when $\zeta = 0.025$, $k_2 = 0$, $F_e = 2$ and $\omega = 0.7$, the chaotic system (29) is periodic with T_e in the Hausdorff metric space. This means that $\psi_i^s(C_L)$, $i = 1, 2, 3, \ldots$, is a Cauchy sequence in the Hausdorff metric space. Fig. 11 shows the evolution process of the sequence $\psi_i^s(C_L)$, $i = 3, 6, 9, 12, 35$. All these show that $\psi_i^s(C_L)$ is a Hilbert system induced by the chaotic tension-slack oscillator.

5. Conclusions and discussions

In this paper we proved that any chaotic dynamical systems have $\mathscr{D}^1$-singularity. Such singularity and the periodicity of chaotic dynamical systems in Hausdorff metric spaces induce a very strange iterated system similar to the Hilbert mapping discovered in 1891 by imagination. This implies that there is a significant relationship between the most puzzling discovery in nature in the 20th century and in thoughts in the 19th century. It is well known that the Hilbert iterated system was one of the most important examples in the critical investigation of geometry at the end of the 19th century, which led to the development of many mathematical branches in the 20th century. Thus, the Hilbert phenomenon hidden in chaos is worth thinking over. In the near future we will investigate the fractal dimension of the Hilbert phenomenon induced by chaotic motion.

In addition, we believe that chaotic systems having $\mathscr{D}^2$-singularity exist in nature, although they have not been found so far.

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