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Steady state response of an infinite string on a non-linear visco-elastic foundation to moving point loads

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Abstract

The steady state response of a long string on a non-linear, visco-elastic foundation to uniformly moving constant point loads is studied. The phase plane is employed to construct a phase portrait that determines the string response. This portrait consists of trajectories of the autonomous system that represent the string displacement everywhere except at the loading points, and vertical segments that satisfy the boundary conditions in these points. It is shown that the string response depends crucially on the ratio between the load velocity and the wave speed in the string. If the load moves slower than the waves in the string, then the response is nearly symmetric with respect to the load and decays exponentially with the distance form the load. If the load velocity exceeds the wave speed, then the string exhibits a wave pattern, which extends well behind the load. In front of the load, in this case, the string is not disturbed.

1. Introduction

The increase in travelling speed of modern high-speed trains can lead to a significant dynamic amplification of vibrations of overhead power lines (catenaries). This amplification becomes especially pronounced when the train speed approaches the wave speed in a contact cable. The latter speed is about 250–350 km/h and is easily reachable for present day trains. Thus, the catenary system may undergo intense vibrations. Such vibrations would make the droppers that support the contact cable exhibit non-linear behaviour, manufested by a much higher resistance of the droppers to tension than to compression. To study the effect of this non-linearity on response

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of the catenary to a moving current collector (pantograph), a non-linear statement of the problem is necessary.

The vast majority of papers on the catenary dynamics deal with a linear statement of the problem. Within such statement, Jezequel [1], Vesnitskii and Metrikine [2], and Belotserkovskiy [3] studied the radiation of elastic waves into the catenary by a uniformly moving pantograph and analyzed conditions of resonance. Vesnitskii and Metrikine [2], and Wu and Brennan [4,5] investigated the stability of the transverse motion of the pantograph–catenary system. Manabe and Fujii [6] followed by Manabe [7], Aboshi and Manabe [8], and Arnold and Simeon [9] analyzed the contact force between the catenary and the moving pantograph. Since this force plays the major role in the efficiency of the current collection, Balestrino et al. [10] proposed an active control method for keeping the contact force constant. An extended reference list of other papers on linearized dynamics of the catenary–pantograph system can be found in Frýba [11].

Research on non-linear dynamics of the pantograph-catenary system and, in general, on the dynamics of a non-linear elastic system under a moving load has nearly always been accomplished numerically. Wu and Chen [12] analyzed the transient catenary–pantograph dynamics by using a direct time integration technique. Non-linear modelling of the dynamic response of suspended bridges to a moving load was presented by Hino et al. [13], Rakowski [14], and Lee [15]. An extension of these studies to the case of a random moving load was performed by Yoshimura et al. [16] and Bryja and Sniady [17].

According to the author's knowledge, the only analytical studies on the response of a non-linear elastic system to a moving load was presented by Yen and Sing [18] and Metrikine [19]. The former authors studied the response of a geometrically non-linear string to a moving load, focusing their attention on the transition through the wave speed. The study was accomplished by employing a perturbation technique. Metrikine [19] presented a geometric method of finding the stationary waves that can be generated by a moving load in a non-linear string.

In the present paper, a simple model for a one-level catenary is considered. The model is composed of a string and a distributed visco-elastic foundation, which supports the string. The string represents the contact cable, whilst the foundation models the reaction of the droppers. The stiffness of the foundation is assumed to be zero in compression and to have a non-linear character in tension. The string is subjected to gravitational loading.

The main objective of the study is to analyze the steady state response of the string to a set of two uniformly moving constant loads that represent the action of two pantographs of a train. In the steady state regime, the deflection field in the string remains stationary in the moving reference system that is fixed to the loads. This field is governed by an ordinary differential equation, as in the case of solitons, but with special discontinuous "boundary conditions" at the loading points.

The paper is structured in the following manner. First, a partial differential equation that governs the string motion is formulated along with the boundary conditions at the loading points and at infinity. Then, this equation is reduced to an ordinary differential equation by introducing a moving reference system that is fixed to the load. The latter equation, accompanied by accordingly reformulated boundary conditions, describes stationary (with respect to the loads) patterns in the string that may be generated by the loads. As a reference for the following non-linear analysis, first the string response is found analytically in the linearized case. Then, employing the phase plane, a non-linear response of the string to one load is determined thereby demonstrating the method of analysis. The paper concludes by applying the proposed method to the case of two moving loads.

2. Model

Consider an infinitely long taut string under tension T, supported by a continuously distributed visco-elastic foundation as shown in Fig. 1. The mass density per unit length of the string is μ, the stiffness per unit length and viscosity per unit length of the foundation are $k(w)$ and c_{dp}, respectively ($w = w(x,t)$ is the vertical displacement of the string). The string is subject to gravitational loading and two point loads of a constant magnitude P that move along the string with a constant velocity V, at a constant distance d from each other.

By neglecting the geometric non-linearity of the string, which is of secondary importance for the model in question, the equation of motion for a differential element of the string and the boundary conditions at the loading points and at infinity are given as [11]

$$\begin{aligned}
&\mu\frac{\partial^2 w}{\partial t^2} - T\frac{\partial^2 w}{\partial x^2} + c_{dp}\frac{\partial w}{\partial t} + k(w)w = -\mu g, \quad -\infty < x < \infty, \quad x \neq Vt, \quad x \neq Vt + d,\\
&w|_{x=Vt+0} = w|_{x=Vt-0}, \quad w|_{x=Vt+d+0} = w|_{x=Vt+d-0},\\
&\left(T - \mu V^2\right)\left(\left.\frac{\partial w}{\partial x}\right|_{x=Vt+0} - \left.\frac{\partial w}{\partial x}\right|_{x=Vt-0}\right) = -P,\\
&\left(T - \mu V^2\right)\left(\left.\frac{\partial w}{\partial x}\right|_{x=Vt+d+0} - \left.\frac{\partial w}{\partial x}\right|_{x=Vt+d-0}\right) = -P,\\
&\left|\lim_{|x-Vt|\to\infty} w(x,t)\right| = const < \infty.
\end{aligned} \tag{1}$$

The first equation of Eqs. (1) governs the vertical motion of the supported string under the gravitational loading. This equation is valid everywhere but in the loading points. The second line in Eqs. (1) presents the continuity condition for the string displacement at the loading points. The third and the fourth lines show the balance of vertical forces at the loading points. The last line states that the string deflection at an infinite distance from the moving loads should be equal to a finite constant, the value of which is determined by the gravitational loading and the stiffness $k(w)$.

Since the loads have a constant magnitude and move with a constant speed, after a sufficiently long time the string response may become stationary in the reference system that is fixed to the

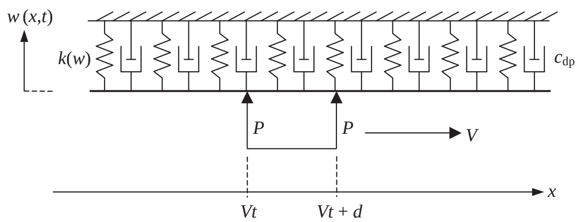

Fig. 1. Two loads moving on a visco-elastically supported string (linear).

loads. This paper aims to study possible patterns of this stationary response. To find these patterns, it is customary to introduce one variable $\xi = x - Vt$ instead of two independent variables x and t. Such an introduction reduces the original partial differential equation that describes the string vibrations to an ordinary differential equation. The idea of introducing the running variably ξ is widely used for finding solitary waves in non-linear systems. In this sense, it is possible to say that we are looking for *forced solitary waves* that are generated by the loads.

Passing to the variable ξ, we obtain from Eqs. (1) the following system of equations that describe the string response $w(\xi)$, which is "frozen" with respect to the loads:

$$
\begin{aligned}
&\left(c^2 - V^2\right)w'' + \frac{Vc_{dp}}{\mu}w' - \frac{k(w)}{\mu}w = g, \quad -\infty < \xi < \infty, \quad \xi \neq 0, \quad \xi \neq d,\\
&w|_{\xi=+0} = w|_{\xi=-0}, \quad w|_{\xi=d+0} = w|_{\xi=d-0},\\
&\left(c^2 - V^2\right)\left(w'|_{\xi=+0} - w'|_{\xi=-0}\right) = -P/\mu,\\
&\left(c^2 - V^2\right)\left(w'|_{\xi=d+0} - w'|_{\xi=d-0}\right) = -P/\mu,\\
&\left|\lim_{|\xi|\to\infty} w(\xi)\right| = const < \infty
\end{aligned}
\tag{2}
$$

with $c = \sqrt{T/\mu}$ the speed of transverse waves in the string and $w' = \mathrm{d}w/\mathrm{d}\xi$.

In the next section, an exact solution to Eqs. (2) will be obtained in the case of linear visco-elastic foundation, the stiffness of which is constant, that is $k(w) = k_0 = const$. This solution will serve as a reference for the non-linear analysis in the following section.

3. Exact solution in the linear case

In the case $k(w) = k_0 = const$, the system of Eqs. (2) is linear and its solution can be sought in the following form:

$$
w(\xi) = -\frac{\mu g}{k_0} + \sum_{n=1}^{2} A_n \exp(q_n \xi). \tag{3}
$$

By substituting Eq. (3) into the first equation of Eqs. (2), the following characteristic equation is obtained with respect to the eigenvalues q_n:

$$
\left(c^2 - V^2\right)q_n^2 + 2\gamma V q_n - \omega_0^2 = 0, \tag{4}
$$

with $\omega_0 = \sqrt{k_0/\mu}$ the cut-off frequency of the string on elastic foundation and $\gamma = c_{dp}/(2\mu)$ the characteristic damping.

The roots of Eq. (4) are given as

$$
q_1 = \frac{-\gamma V + \sqrt{\gamma^2 V^2 + \omega_0^2\left(c^2 - V^2\right)}}{c^2 - V^2}, \quad q_2 = \frac{-\gamma V - \sqrt{\gamma^2 V^2 + \omega_0^2\left(c^2 - V^2\right)}}{c^2 - V^2}. \tag{5}
$$

Eq. (5) shows that in the case of sub-critical damping ($\gamma < \omega_0$), depending on the velocity of the load, the eigenvalues q_n can be

(a) both real, one positive and one negative if $V < c$;

(b) both real and positive if $c < V < V^* = c\omega_0/\sqrt{\omega_0^2 - \gamma^2}$;

(c) complex conjugated, with a positive real part if $V > V^*$.

Case (a) is conventionally referred to as the sub-critical case in which the loads move more slowly than waves in the structure. Cases (b) and (c) are called super-critical since the load velocity is larger than the wave speed.

To find unknown amplitudes A_n, Eq. (3) should be substituted into the boundary conditions defined by lines 2–5 of Eqs. (2). Let us accomplish this substitution for the sub-critical and super-critical cases separately.

(a) $V < c$. In this case, to satisfy the boundary conditions at infinity, the string deflection must be sought for in the form (note that $Re(q_1) > 0$ and $Re(q_2) < 0$)

$$w(\xi) = -\frac{\mu g}{k_0} + \begin{cases} A_1 \exp(q_1\xi), & \xi < 0, \\ A_2 \exp(q_1\xi) + A_3 \exp(q_2\xi), & 0 < \xi < d, \\ A_4 \exp(q_2\xi), & \xi > d. \end{cases} \tag{6}$$

Constants $A_1 - A_4$ are found by substitution of Eq. (6) into the boundary conditions at the loading points. This yields

$$A_1 = \frac{F(1 + \exp(-q_1 d))}{q_1 - q_2}, \quad A_2 = \frac{F\exp(-q_1 d)}{q_1 - q_2},$$
$$A_3 = \frac{F}{q_1 - q_2}, \quad A_4 = \frac{F(1 + \exp(-q_2 d))}{q_1 - q_2}, \tag{7}$$

with $F = P/\mu/(c^2 - V^2)$.

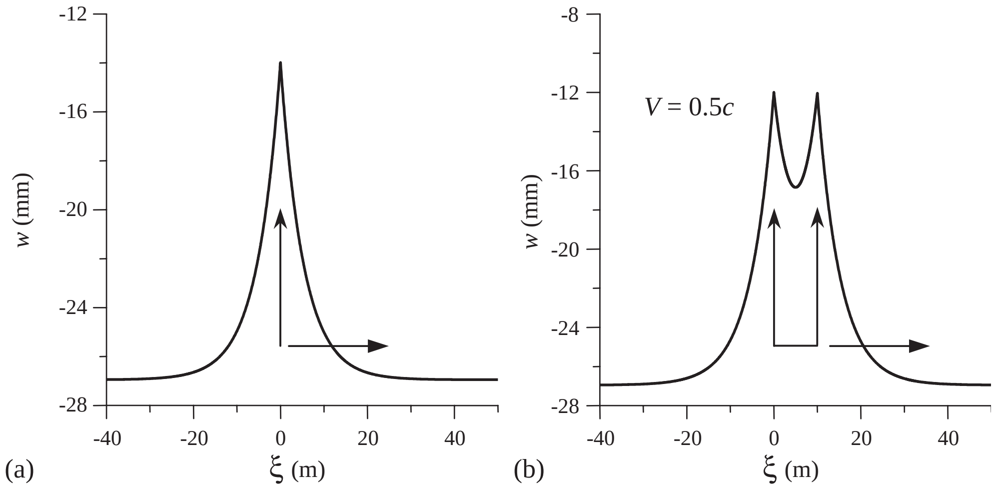

Fig. 2. Response of the string to *sub-critically* moving loads (linear).

The string pattern corresponding to this solution is shown in Fig. 2 for the following set of parameters:

$$\mu = 1.1 \text{ kg/m}, \quad T = 15 \text{ kN}, \quad k_0 = 0.4 \text{ kN/m}^2, \quad c_{dp} = 0.5 \text{ N s/m}^2,$$
$$P = 55 \text{ N}, \quad d = 10 \text{ m}, \tag{8}$$

and $V = 58\,\text{m/s} \approx 0.5c$. These parameters are representative for realistic catenaries provided that k_0 and c_{dp} are found by dividing the stiffness and the damping factor of a dropper by the inter-dropper distance.

Fig. 2(b) is plotted in accordance with expressions (6) and (7). Fig. 2(a) presents the string pattern for a single load at $\xi = 0$. In this case the string displacement would be given by

$$w(\xi) = -\frac{\mu g}{k_0} + \frac{P}{2\mu}\frac{1}{\sqrt{\gamma^2 V^2 + \omega_0^2(c^2 - V^2)}} \exp\left(\frac{-\gamma V \xi - |\xi|\sqrt{\gamma^2 V^2 + \omega_0^2(c^2 - V^2)}}{c^2 - V^2}\right). \tag{9}$$

Fig. 2 shows that in the sub-critical case the string deflection decays exponentially with the distance from the loads. If there were no viscosity in the string foundation, the string pattern would be symmetric with respect to the load(s).

(b) $V > c$. In the super-critical cases (b) and (c), both eigenvalues have a positive real part and, therefore, there is no way to satisfy the boundary condition at plus infinity unless in front of the first load (that is applied at $\xi = d$) the amplitudes A_n are equal to zero. Thus, the string pattern in this case must be sought for in the form

$$w(\xi) = -\frac{\mu g}{k_0} + \begin{cases} A_1 \exp(q_1\xi) + A_2 \exp(q_2\xi), & \xi < 0, \\ A_3 \exp(q_1\xi) + A_4 \exp(q_2\xi), & 0 < \xi < d, \\ 0, & \xi > d. \end{cases} \tag{10}$$

Substitution of Eq. (10) into the boundary conditions at the loading points yields

$$A_1 = \frac{F(1 + \exp(-q_1 d))}{q_1 - q_2}, \quad A_2 = -\frac{F(1 + \exp(-q_2 d))}{q_1 - q_2},$$
$$A_3 = \frac{F \exp(-q_1 d)}{q_1 - q_2}, \quad A_4 = -\frac{F \exp(-q_2 d)}{q_1 - q_2}. \tag{11}$$

If there were only one load applied at $\xi = 0$, then the string pattern would be given as

$$w(\xi) = -\frac{\mu g}{k_0} + \frac{P \exp\left(-\gamma V \xi/\left(c^2 - V^2\right)\right)}{\mu\sqrt{\gamma^2 V^2 + \omega_0^2(c^2 - V^2)}} \begin{cases} \sinh\left(\dfrac{\xi\sqrt{\gamma^2 V^2 + \omega_0^2(c^2 - V^2)}}{c^2 - V^2}\right), & \xi < 0, \\ 0, & \xi > 0. \end{cases} \tag{12}$$

For realistic parameters of the catenary system, case (b) is of minor importance because of a very small viscosity, V^* is just a fraction larger than the wave speed c. Therefore, in Fig. 3 the "truly" super-critical case is shown, in which the string exhibits a wave motion (in case (b) the string displacement decays exponentially behind the loads as in the sub-critical case). Fig. 3(a) and (b) show the string response to one load and two loads, respectively. Calculations were performed by employing the parameter set (8) and $V = 140\ \text{m/s} \approx 1.2c$.

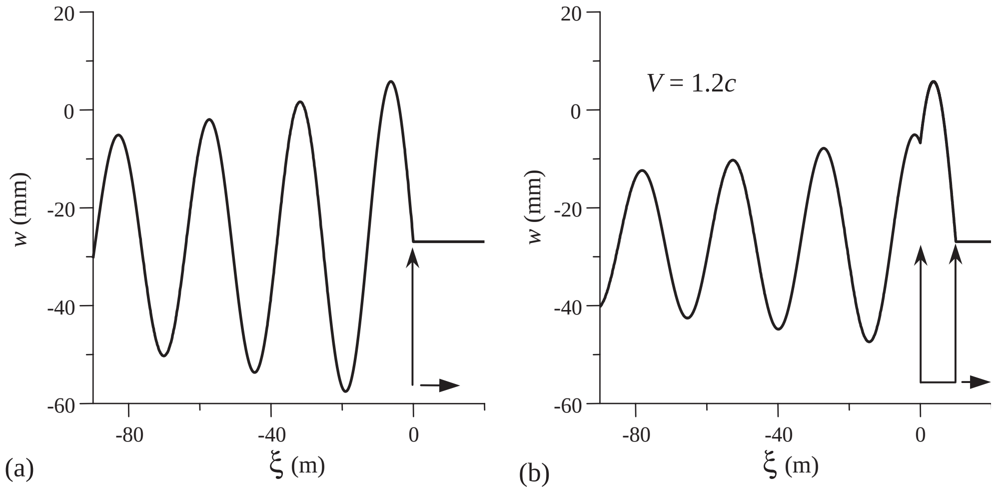

Fig. 3. Response of the string to *super-critically* moving loads (non-linear).

Figs. 2 and 3 clearly demonstrate the two crucial differences between the string response in the sub-critical and super-critical cases. First, the sub-critical response is nearly symmetric with respect to the loads (note that the asymmetry would grow if the viscosity were increased). On the contrary, the super-critical response has no such symmetry at all. In this case, in front of the loads the string exhibits a constant displacement caused by the gravitational loading alone. The loads do not disturb this region. This phenomenon is similar to the Mach effect [20] in acoustics and the Cherenkov effect [21] in electrodynamics. The second difference is concerned with the character of the string pattern. In the sub-critical case, the load-caused part of the string response is localised around the loads, decaying exponentially with the distance from them. In contrast to that, in the super-critical case, the loads generate waves in the string that propagate to a large distance from the loads.

In the following sections, the effect of non-linearity of the foundation on the stationary response of the string will be studied both in the sub-critical and super-critical case. In the next section, to explain the method of analysis, the response to a single load that is applied at $\xi = 0$ will be discussed.

4. Non-linear response to one load

With the assumption that only one load (applied at $\xi = 0$) moves on the string, Eq. (2) reduces to

$$
\begin{aligned}
&(c^2 - V^2)w'' + \frac{Vc_{dp}}{\mu}w' - \frac{k(w)}{\mu}w = g, \quad -\infty < \xi < \infty, \;\; \xi \neq 0,\\
&w|_{\xi=+0} = w|_{\xi=-0},\\
&(c^2 - V^2)\left(w'|_{\xi=+0} - w'|_{\xi=-0}\right) = -P/\mu,\\
&\left|\lim_{|\xi|\to\infty} w(\xi)\right| = const < \infty.
\end{aligned} \tag{13}
$$

Hereafter, we will consider the following non-linear expression for the foundation stiffness:

$$k(w) = \begin{cases} 0, & w > 0, \\ k_0 + k_2 w^2, & w < 0. \end{cases} \tag{14}$$

This expression describes reasonably well the elastic properties of the droppers, which support the contact cable; they do not resist compression but are quite stiff against tension.

The problem at hand can be analyzed very elegantly by using the phase plane $\{w, w'\}$. In accordance with the problem statement (13), the phase portrait that corresponds to the string response should be constructed from

(1) trajectories of the autonomous system that are defined by the first equation of Eqs. (13); relevant trajectories should be chosen so that the condition at infinity (the last equation in Eq. (13)) is satisfied;
(2) a vertical (parallel to w'-axis) segment of length $P/\mu/|c^2 - V^2|$; the length of the segment is defined by the balance of vertical forces in the loading point (third equation of Eq. (13)), while it verticality follows directly from the continuity condition (second equation of Eq. (13)).

Consider first the *sub-critical motion of the load* $V < c$. In this case, the autonomous system has one equilibrium state that is located at the point $\{w, w'\} = \{-\mu g/k_0, 0\}$. The characteristic exponents q_1 and q_2 for this equilibrium are given in Eq. (5). In the sub-critical case q_1 and q_2 are both real and have different signs, thus, the equilibrium state is a saddle (see Fig. 4). This saddle is the only equilibrium state of the autonomous system. Therefore, the only trajectories, which do not tend to infinity as $|\xi| \to \infty$ are the separatrices of the saddle. Consequently, the vertical segment that represents the load should connect two separators of the saddle. This segment should start from a separatrix that leaves the saddle and end at a separatrix that arrives at the saddle.

The resulting phase portrait is depicted in Fig. 4, which was calculated by using the parameter set (8) and $k_2/k_0 = 0.1\ 1/\mathrm{m}^2$. Figs. 4(a) and (b) are plotted for $V = 58\,\mathrm{m/s} \approx 0.5c$ and $V = 111\,\mathrm{m/s} \approx 0.95c$, respectively.

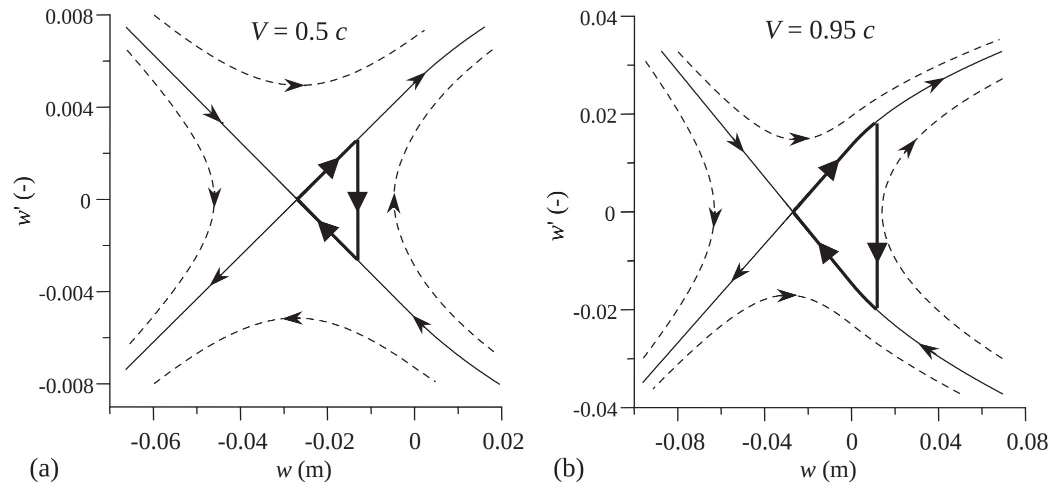

Fig. 4. Phase portrait in the *sub-critical* case. String is subjected to *one load* (non-linear).

In Fig. 4, the solid regular lines show the separatrices of the saddle, whereas other trajectories of the autonomous system are depicted as dashed lines. Bold lines distinguish the phase portrait of the string response. It consists of segments of two separatrices and a straight vertical segment. The segment of the upper separatrix corresponds to the string displacement behind the load. The vertical segment reflects the continuity of the string and a jump of the string slope at the loading point. The segment of the lower separatrix corresponds to the string displacement in front of the load. Note that the reason for the vertical segment to be inserted between the two separators that are located to the right from the saddle is that in the sub-critical case $w'|_{\xi=+0} - w'|_{\xi=-0} < 0$. This inequality implies that the motion along this segment should take place downward in the phase plane.

Fig. 4(a) shows that for relatively low velocity of the load $V = 0.5c$, the phase portrait of the string response has an almost perfectly triangular shape. This implies that the non-linearity of the foundation does not influence the string response, which remains exponential, as in the linear case. This response coincides very well with that presented in Fig. 2(a). For the chosen parameters of the system, the non-linearity starts to play a perceptible role when the load velocity approaches the wave speed in the string. As shown in Fig. 4(b), which is plotted for $V = 0.95c$, in this near-critical case the phase portrait of the string response is a deformed triangle. This deformation is a consequence of the non-linearity, which curves the separatrices. As it is to be expected, the non-linearity is more apparent in the vicinity of the load, where the deflection of the string is maximal. As for a relatively slow motion of the load, the string response in the near-critical case looks qualitatively similar to that depicted in Fig. 2(a).

Consider the *super-critical motion of the load* $V > c$. In this case, as in the previous one, the autonomous system has one equilibrium state in the point $\{w, w'\} = \{-\mu g/k_0, 0\}$. The type of this equilibrium, however, is different. Now it can be either unstable node or unstable focus. In accordance with Eq. (5), the former occurs if $c < V < V^*$, whereas the latter takes place if $V > V^*$. The phase plane analysis can be accomplished in both cases completely analogously. Therefore, we will present it on the hand of the "truly supercritical" motion of the load with $V > V^*$.

To construct a phase portrait for the string response we have to use a vertical segment to connect a trajectory that leaves the equilibrium point with a trajectory that arrives at this point. The complication is, however, *that all trajectories* leave the equilibrium and there is none to arrive at it. Thus, the only possibility of arriving at the equilibrium point is along the vertical segment that represents the load. Consequently, as shown in Fig. 5, the phase portrait (bold) in the super-critical case is composed of an unstable trajectory that leaves the focus and the vertical segment that returns the phase motion to the equilibrium point. The string pattern that is defined by such a portrait looks similar to that plotted in Fig. 3(a). The string behind the load corresponds to the unstable trajectory that leaves the focus. The vertical segment that goes directly to the equilibrium implies that the string deflection in front of the load is constant and equal to that at the loading point. This is in complete agreement with Fig. 3(a) which shows a horizontal string in front of the load in accordance with the Mach effect. Thus, in a sense, the phase portrait in Fig. 5 is a geometrical proof of the Mach effect in the non-linear case.

Fig. 5 shows that, as in the sub-critical case, the effect of the non-linearity becomes apparent in the near-critical regime that is shown in Fig. 5(a). As soon as the load velocity grows larger than the wave speed, the linear model becomes capable of accurate prediction of the string response.

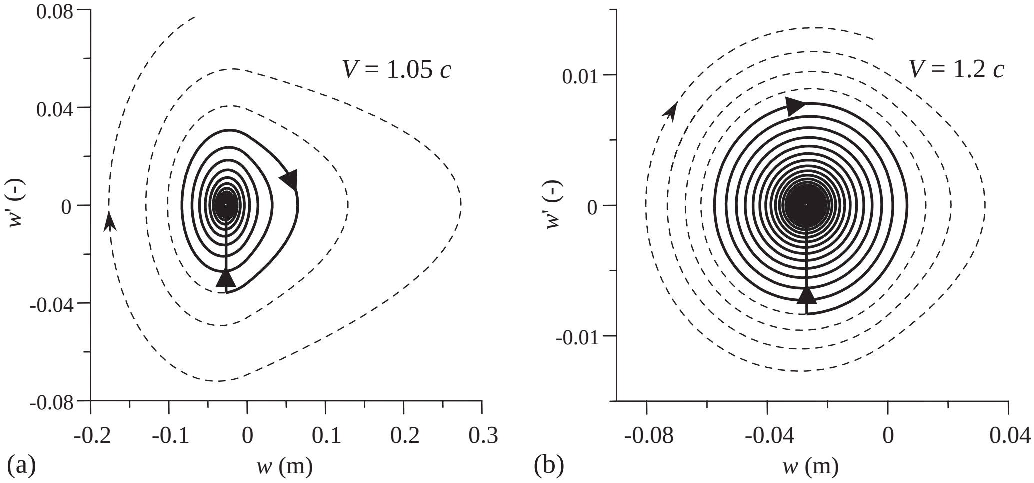

Fig. 5. Phase portrait in the *super-critical* case. String is subjected to *one load* (non-linear).

Thus, in this section, a method of analysis has been demonstrated that allows one to find the steady state response of a string on non-linear foundation to a single moving load. In the next section, this method will be generalised to the case of two moving loads.

5. Non-linear response to two loads

The phase-plane analysis that was applied in the previous section is capable of predicting the string response to an arbitrary number of loads, providing that all of them have a constant (not necessarily the same) magnitude and move with the same velocity. In this section such a capability is demonstrated by studying the string response to two loads, which is governed by Eq. (2).

Let us first consider the *sub-critical case* $V < c$. Obviously, the introduction of the second load does not change the phase portrait of the autonomous system so that it remains the same as shown in Fig. 4(a). This portrait contains one equilibrium point, which is the saddle that is located at the point $\{w, w'\} = \{-\mu g/k_0, 0\}$. Thus, as in the case of a single load, the phase portrait of the non-autonomous system should include two separatrices of the saddle: one leaving the saddle and the other arriving at it. A new element of the non-autonomous portrait is introduced by the second load. In contrast to the case of a single load, the phase portrait should now include not one but two vertical segments, each responsible for one load. Since the loads are assumed to have equal magnitudes, these segments should have the same length that, in accordance with Eq. (2), equals $P/\mu/|c^2 - V^2|$. To satisfy the boundary conditions at $|\xi| \to \infty$, the first segment that is responsible for the rear load ($\xi = 0$) should have its upper end attached to a separatrix that leaves the saddle, whereas the second segment (front load, $\xi = d$) should have its lower end at a separator that arrives at the saddle. The remaining ends of the segments should belong to one and the same trajectory, the phase motion along which corresponds to the distance between the loads.

The numerical program that constructs the phase portrait in the case of two loads (see Fig. 6) was written in the following manner. First, the two separatrices on the right side of the saddle

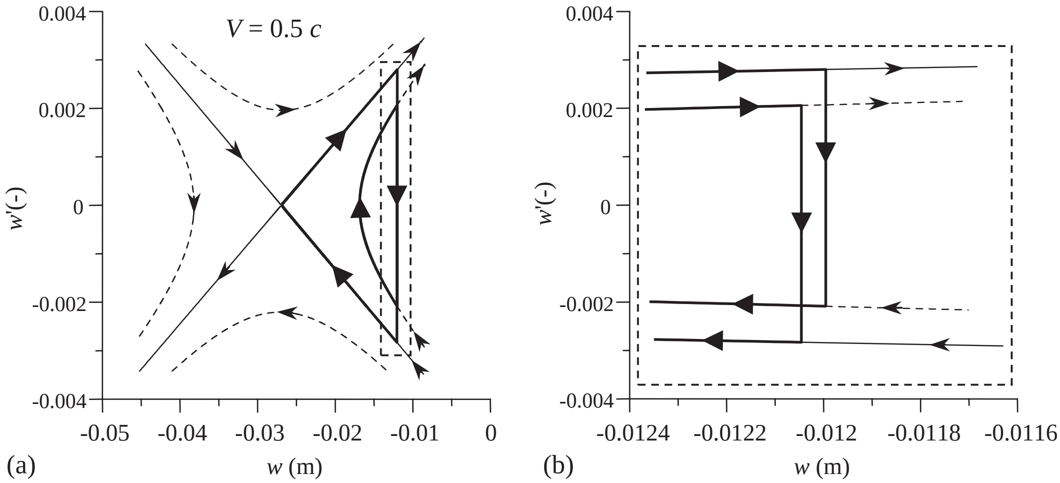

Fig. 6. Phase portrait in the *sub-critical* case. String is subjected to *two loads* (non-linear).

were found. Then, the first vertical segment of the length $P/\mu/(c^2 - V^2)$ was plotted with its upper end fixed to the upper separatrix. If the lower end of this segment happened below the lower separatrix, then the segment was moved rightward, until the lower end assumed a position above the lower separatrix. Next, starting from the lower end of the segment, a piece of trajectory of the autonomous system was plotted. The variable ξ along this trajectory was varied from 0 to d. The end of this trajectory was then fixed to the upper end of the second vertical segment. If the lower end of this segment happened below the lower separatrix, the first segment was moved rightward and this procedure was repeated until the lower end of the second segment happened on the lower separatrix, thereby completing the phase portrait.

The resulting phase portrait in the case of two sub-critically moving loads is depicted in Fig. 6 for $V = 0.5c$ (the other parameters are given by Eq. (8) and $k_2/k_0 = 0.1$ 1/m^2). Since the viscosity of the string foundation is very small, it is not possible to distinguish two vertical segments in Fig. 6(a) that presents a complete phase portrait. Therefore, the domain that surrounds these segments is magnified in Fig. 6(b). The phase portrait consists of two segments of the separatrices, two vertical segments, and a piece of trajectory of the autonomous system. The segment of the upper separatrix corresponds to the string displacement behind the rear load. The first vertical segment that starts at this separator reflects the continuity of the string at $\xi = 0$ and an abrupt change of its slope at this point. A piece of trajectory of the autonomous system that connects the vertical segments corresponds to the string displacement between the loads. The second vertical segment satisfies the boundary conditions at the front load and the segment of the lower separatrix corresponds to the string displacement in front of this load.

As in the case of a single load, the non-linearity of the string foundation is not significant for the chosen set of system parameters. Therefore, the string pattern that is shown in Fig. 3(a) corresponds to the phase portrait in Fig. 6(a) very well. To activate the non-linearity, the load should move with a velocity that is close to the wave speed in the string.

Consider the "truly" *super-critical case* $V > V^* > c$. The phase portrait that corresponds to this case is plotted in Fig. 7. It consists of two pieces of unstable trajectories, which are coming from

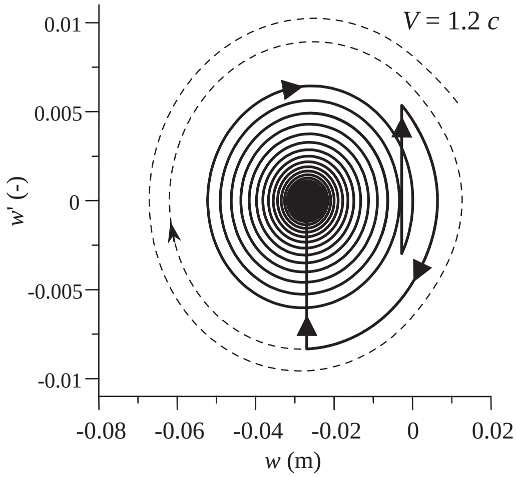

Fig. 7. Phase portrait in the *super-critical* case. String is subjected to *two loads* (non-linear).

the unstable focus and two vertical segments of the length $P/\mu/(V^2 - c^2)$. This portrait was obtained in the manner that is described below.

Since the only equilibrium state of the autonomous system in the "truly" super-critical case is an unstable focus, the only way to satisfy the boundary condition at $\xi \to +\infty$ is to arrive at the focal point along a vertical segment that is related to the front load ($\xi = d$). This makes it easy to construct the phase portrait for the non-autonomous system by decreasing ξ from plus to minus infinity. The reverse motion along the phase trajectories should start at the focal point and then move down along the vertical segment of the length $P/\mu/(V^2 - c^2)$, thereby completing the boundary conditions at the front load. Next, a phase trajectory of the autonomous system should be run from the lower end of this segment, decreasing the variable ξ from d to 0. This trajectory corresponds to the string response between the loads and, therefore, its end gives a start to the second vertical segment that, going downward, satisfies the boundary conditions at the rear load. The phase portrait is completed by a trajectory of the autonomous system that connects the lower end of the second segment and the focal point. This trajectory corresponds to the string displacement behind the rear load.

The phase portrait that is shown in Fig. 7 corresponds very well with the string pattern in Fig. 3(b). The reason to have nearly the same pattern in both the linear and non-linear cases is that the non-linearity is not activated as long as the load velocity is not close to the wave speed in the string.

6. Conclusions

In this paper, the steady state response of an infinitely long string on non-linear, visco-elastic foundation to uniformly moving constant point loads has been studied. A method of analysis has been proposed that employs the phase plane. In this plane, the horizontal axis is used for the string displacement, whereas the vertical axis represents the string slope. The independent variable for plotting the phase trajectories is the distance from the moving loads.

Using simple geometrical considerations, phase portraits for the string response have been constructed. These portraits consist of trajectories of the autonomous system that represent the string displacement everywhere except the loading points, and vertical segments that satisfy the boundary conditions at these points. It has been shown that the string response depends crucially on the ratio between the load velocity and the wave speed in the string. In the sub-critical case, in which the load moves slower than the waves in the string, the response is almost symmetric with respect to the load and decays exponentially with the distance form the load. In the super-critical case, in which the velocity of the load exceeds the wave speed, the string shows a wave response, which extends well behind the load. In front of the load, in this case, the string is not disturbed. Both the sub-critical and super-critical string patterns are similar to those, which would be obtained if the string foundation were linear.

It should be emphasised that the steady-state solutions that were obtained in this paper describe *possible shapes* the string might assume after a sufficiently long time, given certain initial conditions. To ensure that there indeed exist initial conditions, which lead to these shapes, the obtained solutions must be examined for stability, as it is done for solitons [21].

Another issue to be discussed is practical applicability of the considered model. There are two major features of the pantograph and catenary that were not accounted for in the model. First, the contact cable of a catenary system is always hung on discrete droppers, the distance between which is in the range of 7–9 m. Thus, modelling discrete droppers by a continuous foundation, one has to make sure that the resulting pattern in the catenary has a characteristic length that is much larger than the distance between droppers. The other issue, which is concerned with the discrete character of droppers, is that every passage of a pantograph through a suspension point, leads to a so-called transition radiation of waves in the catenary [2]. This radiation shows itself as a series of displacement pulses that the pantograph perturbs in the string. These pulses may form a repetitive pattern in the string but only a part of this pattern is stationary in the reference system that is fixed to the pantograph. The other part moves with respect to the pantograph. This part, however, could carry a negligible amount of energy, since resistance of droppers to compression is pretty small. If this is the case, then the continuous foundation is capable of representing the dropper's reaction.

The second drawback of the model is concerned with modelling the pantograph. In this paper it has been done with the help of constant loads. If there were only one pantograph, such a modelling would not be that far from reality, since variation of the contact force between the pantograph and catenary, which always exists, is not significant. In the case of two pantographs, however, by modelling the pantographs as loads, possessing no degrees of freedom, we miss the wave reflection between the pantographs. This phenomenon is known to be harmful for the current collection, therefore it is the modelling of the pantograph that has to be improved first in the course of development of the proposed model.

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