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Letter to the Editor

On the relationship between the fundamental matrices for different definitions of the state vector

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1. Introduction

Recently, the need arose, in the context of a research work, to know the relationship between the fundamental matrices of a linear mechanical system with n degrees-of-freedom (d.o.f.), the state vector of which is usually defined in two different forms. As is known, some authors define it as the $(2n \times 1)$ vector of the n generalized co-ordinates and the n generalized velocities [1–3], whereas some others define it in the reverse order [4,5]. The desired relationship was not found in the literature. Although it is acknowledged that the contribution of this study does not solve a very complex problem, it is nevertheless thought that the simple result established in the present letter can be helpful for those working in this area.

2. Theory

As is known, the free vibrations of a discrete linear mechanical system with n d.o.f. is governed in the physical space by the following matrix differential equation of order two:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ denote the $(n \times n)$ mass, damping and stiffness matrices, respectively. $\mathbf{q}(t)$ represents the $(n \times 1)$ vector of the generalized co-ordinates to describe the position of the mechanical system .This differential equation can equivalently be written in the so-called state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{2}$$

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where the $(2n \times 1)$ state vector $\mathbf{x}(t)$ and the $2n \times 2n$ system matrix $\mathbf{A}$ are defined as

$$\mathbf{x} = \left[\mathbf{q}^{\mathrm{T}} \cdots \dot{\mathbf{q}}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{3}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & \cdots & \cdots \\ -\mathbf{M}^{-1}\mathbf{K} & \vdots & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \tag{4}$$

$\mathbf{I}$ being the $(n \times n)$ unit matrix.

The general solution of the differential Eq. (2) can be written as [2]

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0, \tag{5}$$

where $\mathbf{x}_0$ denotes the initial state vector and the $(2n \times 2n)$ fundamental matrix is defined as

$$\mathbf{\Phi}(t) = \mathbf{X}\mathrm{e}^{\Lambda t}\mathbf{X}^{-1}, \tag{6}$$

where the modal matrix $\mathbf{X}$ and $\mathrm{e}^{\Lambda t}$ are defined as

$$\mathbf{X} = [\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{2n}], \quad \mathrm{e}^{\Lambda t} = \mathbf{diag}(\mathrm{e}^{\lambda_j t}) \quad (j = 1, \ldots, 2n). \tag{7}$$

Here λ_j and $\tilde{\mathbf{x}}_j$ denote the jth eigenpair corresponding to the eigenvalue problem regarding (2).

Now, let it be assumed that the state vector is defined as

$$\mathbf{x}' = \left[\dot{\mathbf{q}}^{\mathrm{T}} \vdots\, \mathbf{q}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{8}$$

such that the state-space equation reads now

$$\dot{\mathbf{x}}' = \mathbf{A}'\mathbf{x}'. \tag{9}$$

The counterparts of Eqs. (4)–(6) are

$$\mathbf{A}' = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{D} & \vdots & -\mathbf{M}^{-1}\mathbf{K} \\ \cdots & \cdots & \cdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix}, \tag{10}$$

$$\mathbf{x}'(t) = \mathbf{\Phi}'(t)\mathbf{x}'_0, \tag{11}$$

$$\mathbf{\Phi}'(t) = \mathbf{X}'\mathrm{e}^{\Lambda t}\mathbf{X}'^{-1}. \tag{12}$$

where the modal matrix $\mathbf{X}'$ consists of the new $2n$ eigenvectors $\tilde{\mathbf{x}}'_j$ of the eigenvalue problem, corresponding to (9).

The following $(2n \times 2n)$ matrix be introduced

$$\mathbf{Q} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & \cdots & \cdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix}. \tag{13}$$

It can easily be shown that $\mathbf{Q}$ is orthogonal and is equal to its own inverse. It can further be shown that the modal matrices of both representations are interrelated by

$$\mathbf{X}' = \mathbf{Q}\mathbf{X}. \tag{14}$$

If this is substituted into Eq. (12),

$$\mathbf{\Phi}'(t) = \mathbf{Q}\mathbf{X}\mathrm{e}^{\Lambda t}\mathbf{X}^{-1}\mathbf{Q} \tag{15}$$

is obtained. Considering Eq. (6), the last formula reduces to

$$\mathbf{\Phi}'(t) = \mathbf{Q}\mathbf{\Phi}(t)\mathbf{Q}, \tag{16}$$

which represents the desired result.

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