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Letter to the Editor

A superposition of exponential–trigonometric functions: generalized solution for several forced vibrations problems of a one-degree-of-freedom system

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1. Introduction

Consider a one-degree-of-freedom vibrating system subjected to a forced excitation of the type

$$F(t) = \sum_{i=1}^{N} F_i e^{\alpha_i t} \cos \omega_i t. \tag{1}$$

Accordingly the governing differential system is

$$M\ddot{x} + kx = \sum_{i=1}^{N} F_i e^{\alpha_i t} \cos \omega_i t, \quad x(0) = x_o, \quad \dot{x}(0) = v_0. \tag{2a–c}$$

Assuming now that $\alpha_i = \omega_i = 0$ and taking $i = 1$, Eq. (2a) reduces to

$$M\ddot{x} + kx = F_1, \tag{3}$$

which constitutes the situation where the system is subjected to a constant force of magnitude F_1.

On the other hand, if $\omega_i = 0$ one has the situation where the system is excited by exponentially varying forces. Possibly the most important situation is the one where

$$\alpha_1 = \beta_i < 0 \tag{4}$$

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where the force values decrease exponentially with the time variable and constitutes an acceptable approximation for blast loads [1]. The expression

$$F(t) = \sum_{i=1}^{N} F_i e^{\beta_i t} \tag{5}$$

may be adjusted to accomodate to experimentally obtained force values during a blast situation.

Finally, if $\alpha_i = 0$, one has (adding the term $F_0/2$)

$$F(t) = \tfrac{1}{2} F_0 + \sum_{i=1}^{N} F_i \cos \omega_i t, \quad \omega_i = \frac{i\pi}{T}, \tag{6}$$

which constitutes the Fourier series for the case where the forcing excitation is a periodic force and the F_i's are determined using the well-known Fourier formula for the cosine series expansion,

$$F_i = \frac{2}{T} \int_0^T F(t) \cos \frac{i\pi}{T} t \, \mathrm{d}t. \tag{7}$$

Accordingly differential system (2) is a rather general one for three types of basic excitations.

For the sake of simplicity it will be assumed that

$$x(0) = \dot{x}(0) = 0. \tag{8}$$

On the other hand the consideration of viscous damping does not add any formal complications to the treatment presented herewith.

No claim of originality is made by the authors but it is hoped that university students and practitioners will find the results useful in their work. On the other hand, the present Note constitutes an extension of a recent educational article [2].

2. Mathematical development

Expressing Eq. (2a) in the form

$$\ddot{x} + \omega_n^2 x = \sum_{i=1}^{N} F_i' e^{\alpha_i t} \cos \omega_i t, \tag{9}$$

where $\omega_n^2 = k/M$ and $F_i' = F_i/M$, assuming as particular solution of Eq. (9) the expression

$$x_p = \sum_{i=1}^{N} A_i e^{\alpha_i t} \cos \omega_i t + B_i e^{\alpha_i t} \sin \omega_i t \tag{10}$$

and substituting in Eq. (9), one obtains equating coefficients of like terms,

$$A_i = \frac{F_i}{M} \frac{\alpha_i^2 - \omega_i^2 + \omega_n^2}{(\alpha_i^2 - \omega_i^2 + \omega_n^2)^2 + 4\alpha_i^2 \omega_i^2}, \tag{11a}$$

$$B_i = \frac{F_i}{M} \frac{2\alpha_i \omega_i}{(\alpha_i^2 - \omega_i^2 + \omega_n^2)^2 + 4\alpha_i^2 \omega_i^2}. \tag{11b}$$

Accordingly the general solution of Eq. (9) is

$$x(t) = C_1\cos\omega_n t + C_2\sin\omega_n t + \frac{1}{M}\sum_{i=1}^{N}\frac{F_i}{(\alpha_i^2-\omega_i^2+\omega_n^2)^2+4\alpha_i^2\omega_i^2} \times [(\alpha_i^2-\omega_i^2+\omega_n^2)e^{\alpha_i t}\cos\omega_i t + (2\alpha_i\omega_i)e^{\alpha_i t}\sin\omega_i t]. \quad (12)$$

Since

$$x(0) = C_1 + \frac{1}{M}\sum_{i=1}^{N}\frac{F_i(\alpha_i^2-\omega_i^2+\omega_n^2)}{(\alpha_i^2-\omega_i^2+\omega_n^2)^2+4\alpha_i^2\omega_i^2} \quad (13)$$

one obtains

$$C_1 = -\frac{1}{M}\sum_{i=1}^{N}\frac{F_i(\alpha_i^2-\omega_i^2+\omega_n^2)}{(\alpha_i^2-\omega_i^2+\omega_n^2)^2+4\alpha_i^2\omega_i^2}.$$

Applying the second initial condition results in

$$C_2 = -\frac{1}{\omega_n M}\sum_{i=1}^{N}\frac{F_i}{(\alpha_i^2-\omega_i^2+\omega_n^2)^2+4\alpha_i^2\omega_i^2}[\alpha_i(\alpha_i^2-\omega_i^2+\omega_n^2)+2\omega_i^2\alpha_i]. \quad (14)$$

Eqs. (12)–(14) describe the response of the system under the generalized excitation.

For instance, if $\alpha_i = \omega_i = 0$ and $F_i = 0$ for $i > 1$ one obtains the well-known expression

$$x(t) = \frac{F_1}{k}(1-\cos\omega_n t). \quad (15)$$

On the other hand, if $\omega_i = 0$, the resulting expression yields

$$x(t) = -\frac{1}{M}\cos\omega_n t\sum_{i=1}^{N}\frac{F_i(\alpha_i^2+\omega_n^2)}{(\alpha_i^2+\omega_n^2)} - \frac{1}{\omega_n M}\sin\omega_n t\sum_{i=1}^{N}\frac{F_i\alpha_i(\alpha_i^2+\omega_n^2)}{\alpha_i^2+\omega_n^2} + \frac{1}{M}\sum_{i=1}^{N}\frac{F_i(\alpha_i^2+\omega_n^2)}{(\alpha_i^2+\omega_n^2)}e^{\alpha_i t}. \quad (16)$$

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