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Journal of Sound and Vibration 272 (2004) 853–868
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Imposing nodes at arbitrary locations for general elastic structures during harmonic excitations

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Received 9 December 2002; accepted 12 April 2003

Abstract

Spring–mass systems are frequently used as vibration absorbers to minimize excess vibration in structural systems. In this paper, sprung masses are used to impose the points of zero vibration for general elastic structures during forced harmonic excitations. For convenience, such points are referred to as nodes. When the oscillator attachment locations and the node locations coincide (or are collocated), it is always possible to select the spring–mass parameters such that multiple nodes are induced at any desired locations along the structure for any excitation frequency. When the oscillators and the node locations are not collocated, however, it is only possible to induce nodes at certain locations along the elastic structure for a given driving frequency. Moreover, when the desired node locations are closely spaced, it is possible to specify a region of nearly zero amplitudes for a particular driving frequency, effectively quenching vibration in that region. A procedure to guide the proper selection of the spring–mass parameters in order to induce multiple nodes is outlined in detail, and numerical experiments are performed to verify the utility of the proposed scheme of imposing nodes at multiple locations during harmonic excitations.

1. Introduction

Using vibration absorbers to eliminate excess vibration has been studied by many different authors over the years, and hence only a few selected references are given here. Jacquot [1] developed a technique to give the optimal dynamic vibration absorber parameters for the elimination of undesirable vibration in sinusoidally forced Euler–Bernoulli beams. However, he employed only a single mode expansion for the beam in an assumed-modes approach, which severely limits the applicability of his formulation. Özgüven and Çandir [2] presented a general

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00495-4

method for determining the optimum parameters of dynamic vibration absorbers attached to a beam to suppress any two resonances. The assumed-modes method was employed to calculate the response of the system to a concentrated harmonic excitation, and the optimization was performed to minimize the maximum response in the mode of interest. Manikanahally and Crocker [3] formulated a procedure that can be used to suppress any number of significant modes using vibration absorbers. For each absorber with a selected mass, the stiffness and damping parameters were optimized so as to minimize the dynamic response corresponding to the resonance frequency at which they are tuned to operate. The approach was successfully applied to a space structure modelled as a mass-loaded free–free beam when it is subjected to a single localized harmonic excitation. Keltie and Cheng [4] investigated the effects of point masses on the structural response of a finite beam. They developed an approach that can be used to determine the mass locations required to reduce the vibration level at any arbitrary location on a structure. In their optimization techniques, they considered only the mass locations as the design parameters. In a recent paper, Alsaif and Foda [5] proposed a method based on the dynamic Green function to determine the optimum values of masses and/or springs and their locations on a beam in order to confine the vibration at an arbitrary location. The masses are rigidly attached to the beam, and the springs are grounded at one end. While the method they used is exact, direct and elegant, it can only be applied when the Green function for the system can be derived.

Cha and Pierre [6] used a chain of oscillators as a means to passively impose a single node for the normal modes of any arbitrarily supported elastic structure. The desired node can either coincide with the oscillator chain or it can be located elsewhere. A procedure to guide the proper selection of the oscillator chain parameters for the purpose of inducing a single node for multiple normal modes was outlined in detail. In Ref. [7], the present author developed an approach that used a series of sprung masses to induce multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. By selecting the appropriate sprung masses, their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing the locations of the nodes to be specified anywhere along the structure and for any normal mode.

The focus of Refs. [6,7] was on imposing nodes for the normal modes of an elastic structure. In this paper, elastically mounted masses are used to induce a single or multiple nodes anywhere along an elastic structure that is harmonically excited with a localized force. This is beneficial because it would allow sensitive instruments to be placed near or at nodes where there are little or no vibration. In addition, the proposed scheme allows certain points along the structure to remain stationary without using any rigid supports.

2. Theory

2.1. Governing equations

Consider an arbitrarily supported elastic structure to which S-sprung masses are attached as shown in Fig. 1. A localized harmonic force

$$f(t) = F e^{j\omega t} \tag{1}$$

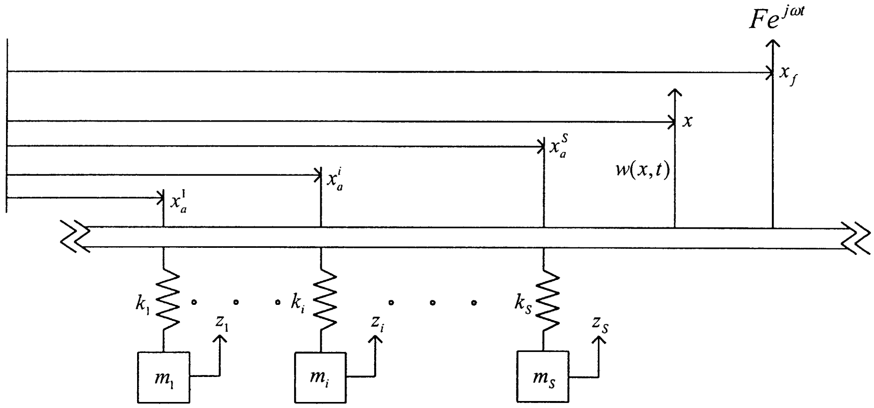

Fig. 1. An arbitrarily supported elastic structure that is subjected to a localized harmonic excitation and carrying any number of sprung masses.

is applied to the structure at x_f, where F represents the forcing amplitude, ω denotes the driving or excitation frequency, and $\mathrm{j} = \sqrt{-1}$. Using the assumed-modes method, the physical deflection of the structure at any point x is given by

$$w(x,t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{2}$$

where the $\phi_i(x)$ are the eigenfunctions of the linear structure (the elastic structure without any sprung masses) that form the basis functions for this approximate solution, the $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic and potential energies of the combined system, defined as the elastic structure carrying the elastically mounted masses, are given by

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} m_i\dot{z}_i^2(t) \tag{3}$$

and

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} k_i[z_i(t) - w(x_a^i, t)]^2, \tag{4}$$

where the M_i and K_i are the generalized masses and stiffnesses of the elastic structure, m_i and k_i are the mass and spring stiffness of the ith oscillator, $z_i(t)$ is its displacement, S is the total number of sprung masses attached to the elastic structure, an overdot denotes a derivative with respect to time, x_a^i represents the attachment location of the ith oscillator, and $w(x_a^i, t)$ represents the lateral displacement of the beam at x_a^i. Finally, the generalized force associated with the generalized co-ordinate $\eta_i(t)$ is

$$F_i(t) = f(t)\phi_i(x_f). \tag{5}$$

Applying Lagrange's equations and assuming simple harmonic motion with the same response frequency as the driving frequency,

$$\eta_i(t) = \bar{\eta}_i e^{j\omega t}, \quad z_i(t) = \bar{z}_i e^{j\omega t}, \tag{6}$$

the generalized co-ordinates $\underline{\bar{\eta}}$ and the mass amplitudes $\underline{\bar{z}}$ for the system of Fig. 1 correspond to the solution of the matrix equation

$$\begin{bmatrix} [\mathscr{K}] - \omega^2[\mathscr{M}] & [R] \\ [R]^{\mathrm{T}} & [k] - \omega^2[m] \end{bmatrix} \begin{bmatrix} \underline{\bar{\eta}} \\ \underline{\bar{z}} \end{bmatrix} = \begin{bmatrix} F\underline{\phi}(x_f) \\ \underline{0} \end{bmatrix}, \tag{7}$$

where $\underline{\bar{\eta}} = [\bar{\eta}_1\ \bar{\eta}_2\ \ldots\ \bar{\eta}_N]^{\mathrm{T}}$, $\underline{\bar{z}} = [\bar{z}_1\ \bar{z}_2\ \ldots\ \bar{z}_S]^{\mathrm{T}}$, and the $S \times S$ matrices $[m]$ and $[k]$ are both diagonal, whose ith elements are given by m_i and k_i, respectively. The $N \times N$ $[\mathscr{M}]$ and $[\mathscr{K}]$ matrices of Eq. (7) are

$$[\mathscr{M}] = [M^d], \qquad [\mathscr{K}] = [K^d] + \sum_{i=1}^{S} k_i \underline{\phi}(x_a^i)\underline{\phi}^{\mathrm{T}}(x_a^i), \tag{8}$$

where $[M^d]$ and $[K^d]$ are diagonal matrices whose ith elements are M_i and K_i, vectors $\underline{\phi}(x_a^i)$ and $\underline{\phi}(x_f)$ consist of the eigenfunctions of the elastic structure evaluated at x_a^i and x_f, respectively,

$$\begin{aligned} \underline{\phi}(x_a^i) &= [\phi_1(x_a^i) \quad \phi_2(x_a^i) \quad \ldots \quad \phi_N(x_a^i)]^{\mathrm{T}}, \\ \underline{\phi}(x_f) &= [\phi_1(x_f) \quad \phi_2(x_f) \quad \ldots \quad \phi_N(x_f)]^{\mathrm{T}}, \end{aligned} \tag{9}$$

and the $N \times S$ matrix $[R]$ is given by

$$[R] = [-k_1\underline{\phi}(x_a^1) \quad \ldots \quad -k_i\underline{\phi}(x_a^i) \quad \ldots \quad -k_S\underline{\phi}(x_a^S)]. \tag{10}$$

Note that $[\mathscr{M}]$ is a diagonal matrix and $[\mathscr{K}]$ is a diagonal matrix modified by S rank one matrices.

To induce nodes at any desired locations, x_n^r, along the elastic structure requires that

$$w(x_n^r, t) = \sum_{i=1}^{N} \phi_i(x_n^r)\eta_i(t) = \underline{\phi}^T(x_n^r)\underline{\eta} = \underline{\phi}^{\mathrm{T}}(x_n^r)\underline{\bar{\eta}} e^{j\omega t} = 0, \quad r = 1, \ldots, S. \tag{11}$$

Once the elastic structure and its boundary conditions are specified, the attachment locations x_a^i are given, and the excitation frequency ω and the excitation location x_f are known, Eqs. (7) and (11) can be used together to solve for the required oscillator parameters, the m_i and the k_i, in order to impose nodes at x_n^r.

2.2. Oscillators and node locations are collocated

Consider the case where the attachment and the node locations coincide. For this case, the oscillators and the nodes are said to be *collocated*. From Eq. (7), note that if

$$k_r = m_r\omega^2, \quad r = 1, \ldots, S, \tag{12}$$

then

$$[R]^{\mathrm{T}}\underline{\bar{\eta}} = \underline{0}. \tag{13}$$

Because the oscillators and the node locations are collocated, $x_a^r = x_n^r$, in which case the rth row of Eq. (13) yields

$$-k_r \phi^{\mathrm{T}}(x_a^r)\underline{\bar{\eta}} = -k_r \phi^{\mathrm{T}}(x_n^r)\underline{\bar{\eta}} = 0, \quad r = 1, \ldots, S, \tag{14}$$

which clearly satisfies Eq. (11). For a given excitation frequency, as long as the oscillator parameters satisfy Eq. (12), nodes will be induced at the attachment locations. Finally, the selection of the sprung masses is not unique. The actual choice is governed by the tolerable vibration amplitudes of the oscillator masses.

2.3. *Oscillators and node locations are not collocated*

Consider an elastic structure subjected to a localized harmonic input. For a certain application, a node or multiple nodes are desired along the elastic structure for a given excitation frequency. However, due to various physical constraints, oscillators cannot be attached at the desired node locations, but instead at some other points. For this case, the attachment and the node locations are said to be *not collocated*. When the attachment and the node locations are not collocated, Eq. (7), of size $(N+S)\times(N+S)$, can be reduced by simple algebraic manipulation. Using Eq. (7), the $\bar{z}_i$ are found to be

$$\bar{z}_i = \frac{k_i \phi^{\mathrm{T}}(x_a^i)}{k_i - \omega^2 m_i}\underline{\bar{\eta}}, \quad i = 1, \ldots, S. \tag{15}$$

Substituting the expressions of Eq. (15) into Eq. (7), the following matrix equation, of size $N \times N$, is obtained:

$$\left\{[K^d] + \sum_{i=1}^{S} \sigma_i \phi(x_a^i)\phi^{\mathrm{T}}(x_a^i) - \omega^2 [M^d]\right\}\underline{\bar{\eta}} = F\phi(x_f), \tag{16}$$

where

$$\sigma_i = \frac{k_i m_i \omega^2}{m_i \omega^2 - k_i}. \tag{17}$$

Incidentally, once the m_i, k_i and x_a^i have been selected, the natural frequencies of the modified structure may be found from the zeros of the characteristic determinant of the coefficient matrix of $\underline{\bar{\eta}}$ of Eq. (16). Assuming that the excitation frequency does not coincide with any natural frequencies of the modified system, the coefficient matrix of Eq. (16) can be inverted to give

$$\underline{\bar{\eta}} = \left\{[K^d] + \sum_{i=1}^{S} \sigma_i \phi(x_a^i)\phi^{\mathrm{T}}(x_a^i) - \omega^2 [M^d]\right\}^{-1} F\phi(x_f), \tag{18}$$

which allows Eq. (11), the constraint equations that dictate the location of nodes, to be rewritten as

$$\phi^{\mathrm{T}}(x_n^r)\left\{[K^d] + \sum_{i=1}^{S} \sigma_i \phi(x_a^i)\phi^{\mathrm{T}}(x_a^i) - \omega^2 [M^d]\right\}^{-1} F\phi(x_f) = 0, \quad r = 1, \ldots, S. \tag{19}$$

Eq. (19) can be used to solve for the required sprung masses in order to induce a single or multiple nodes at x_n^r.

2.3.1. One oscillator and one node

When only one node location at x_n is specified, and it is not collocated with the attachment location at x_a, the desired spring–mass parameters can be readily obtained. For $S = 1$, Eq. (19) simplifies to

$$\phi^{\mathrm{T}}(x_n)\left\{[K^d] + \frac{km\omega^2}{m\omega^2 - k}\,\phi(x_a)\phi^{\mathrm{T}}(x_a) - \omega^2[M^d]\right\}^{-1} F\phi(x_f) = 0. \tag{20}$$

Because the second term of Eq. (20) consists of a matrix modified by a rank one matrix, its inverse can be readily obtained by applying the Sherman–Morrison formula [8]. Assuming the excitation frequency ω, the oscillator stiffness parameter k, the attachment location x_a, the node location x_n, and the excitation location x_f are all specified, a closed-form expression for the required oscillator mass m in order to impose a node at x_f can be readily obtained as follows (see Appendix A for detailed derivations):

$$m = \frac{c_1 k}{\omega^2(c_1 + c_1 c_3 k - c_2 k)}, \tag{21}$$

where

$$c_1 = \sum_{i=1}^{N} \frac{\phi_i(x_n)\phi_i(x_f)}{K_i - M_i\omega^2}, \tag{22}$$

$$c_2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\phi_i(x_a)\phi_j(x_a)\phi_i(x_n)\phi_j(x_f)}{(K_i - M_i\omega^2)(K_j - M_j\omega^2)} \tag{23}$$

and

$$c_3 = \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{K_i - M_i\omega^2}. \tag{24}$$

It should be noted that unlike the collocated case where a node can always be induced anywhere along the structure for any excitation frequency, when the oscillator and the node locations are not collocated, it is only possible to impose a node at certain points along the structure. Mathematically, if Eq. (21) returns a mass value that is negative, this implies that a node cannot be enforced at the desired location for the given set of ω, k, x_a and x_f. In this case, one can change either the oscillator stiffness, k, or the attachment location, x_a, to obtain a physically meaningful, i.e., positive, value of m so that a node at x_n can be induced for the given x_f and ω.

2.3.2. Multiple oscillators and multiple nodes

Now consider the case of multiple oscillators and nodes. When the attachment locations and the node locations are not collocated, the solution scheme becomes more complicated and computational intensive. Nevertheless, the procedure to determine the required spring–mass parameters is still rather straightforward conceptually.

For multiple nodes, Eq. (19) yields a set of S equations, one for each node location. Assuming the excitation frequency ω, the oscillator stiffnesses k_i, the attachment locations x_a^i, and the excitation location x_f, all are specified, these S equations lead to a set of non-linear algebraic equations in the masses, m_i, which can be solved simultaneously so that the specified x_n^r are nodes. Because the S equations are generally totally independent, the solution to these simultaneous non-linear algebraic equations can often be difficult to obtain. The S equations define the zero contours of S independent functions, and the solution of the problem corresponds to the points of intersection of these contour curves. The S curves may have many points of intersection, or they may have none at all. Unlike a set of linear equations, where S independent equations are guaranteed to have S unique solutions, there is no corresponding statement for non-linear functions that specifies how many, if any, intersection points there are. To solve such problems, an estimation of the location of a solution can be first established by means of simultaneous contour plots of the functions. Once its approximate location is known (assuming there is a solution), very efficient numerical methods can then be used to converge to the desired result.

The MATLAB routine *fsolve* is employed in this paper to obtain the solution of a system of non-linear algebraic equations using a quasi-Newton method. For a set of initial guesses, if *fsolve* does not converge to a solution, then *fsolve* is ran again with a different set of starting values until a solution is obtained. The proposed technique of solving for the masses in order to impose nodes at x_n^r is very robust. In all of the cases considered by this author, *fsolve* successfully converged to a set of theoretically feasible solutions. Finally, if there is no set of m_i that satisfy the set of S non-linear algebraic equations, then one can change either one or more of the oscillator stiffnesses, k_i, or one or more of the attachment locations, x_a^i, to obtain the required lumped masses m_i so that nodes at x_n^i can be induced for the given x_f and ω.

3. Results

Because the assumed-modes method was used to formulate the equations of motion, the proposed procedures can be easily implemented to impose a single node or multiple nodes for any arbitrarily supported elastic structure during harmonic excitations. Without any loss of generality, a simply supported and a fixed-free uniform Euler–Bernoulli beam will be considered.

When the Euler–Bernoulli beam is uniform and simply supported, its normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions are given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{i\pi x}{L} \tag{25}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \quad \text{and} \quad K_i = (i\pi)^4 EI/(\rho L^4), \tag{26}$$

where E is Young's modulus, I is the moment of inertia of the cross-section of the beam. When the Euler–Bernoulli beam is uniform and fixed-free, its normalized eigenfunctions are

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L}(\sin \beta_i x - \sinh \beta_i x)\right) \tag{27}$$

Table 1
The first six natural frequencies of a uniform cantilever Euler–Bernoulli beam carrying one undamped oscillator, of mass $m = 0.1\rho L$ and stiffness $k = 0.75\,EI/L^3$, at $0.88L$. The natural frequencies are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$

Natural frequency	Exact	Assumed modes ($N = 15$)
ω_1	0.238798E+01	0.238798E+01
ω_2	0.402981E+01	0.402981E+01
ω_3	0.220473E+02	0.220473E+02
ω_4	0.616974E+02	0.616974E+02
ω_5	0.120903E+03	0.120903E+03
ω_6	0.199861E+03	0.199861E+03

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \quad \text{and} \quad K_i = (\beta_i L)^4 EI/(\rho L^4). \tag{28}$$

where $\beta_i L$ satisfies the transcendental equation

$$\cos \beta_i L \cosh \beta_i L = -1. \tag{29}$$

The assumed-modes method was used in the analysis. To validate this approximate approach, the natural frequencies of Fig. 1 are computed exactly, and the results are compared with those obtained using the approximate scheme, for the case of a uniform cantilever beam carrying one undamped oscillator. Using the assumed-modes method, the approximate natural frequencies correspond to the zeros of the characteristic determinant of the coefficient matrix of $\bar{\eta}$ of Eq. (16). Table 1 lists the first six natural frequencies of the system, for $m = 0.1\rho L$, $k = 0.75 EI/L^3$ and $x_a = 0.88L$. Note the excellent agreement between the exact and assumed-modes results for $N = 15$ (the number of component modes used in the assumed-modes expansion).

To illustrate the proposed approach of imposing a single node or multiple nodes during harmonic excitations, cases where the node and attachment locations are collocated and cases where they are not collocated will be thoroughly analyzed. In the following numerical examples, $N = 15$ to ensure the convergence of all the numerical results. In addition, the MATLAB routine *fsolve* will be used to find the required masses in order to impose nodes when the attachment and the node locations are not collocated. Finally, a few words regarding the excitation location x_f and the attachment location x_a are warranted. If $x_a = x_f$, then the elastic structure can be made to remain motionless as long as the oscillator parameters satisfy $k = m\omega^2$. In this case, the oscillator behaves as a simple undamped vibration absorber. Thus, in the subsequent analysis, only the more interesting cases of $x_a \neq x_f$ will be investigated.

3.1. Oscillators and node locations are collocated

Consider a uniform simply supported Euler–Bernoulli beam of length L. For a given application, it is wished that a node be imposed at $x_n = 0.31L$, for a concentrated harmonic force of amplitude F, an excitation frequency of $\omega = 42\sqrt{EI/(\rho L^4)}$, and an excitation location of $x_f = 0.87L$. Fig. 2 shows the steady state lateral displacement of the beam. The solid curve corresponds to the deformed shape of the beam with an oscillator attached at $x_a = 0.31L$, whose

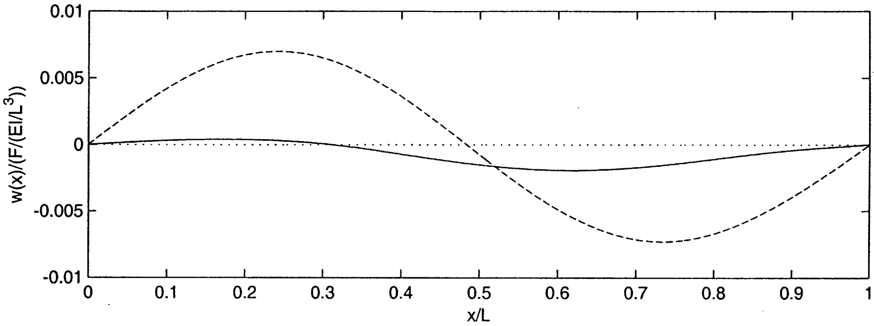

Fig. 2. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The horizontal line represents the configuration of the undeformed beam. The system parameters are $\omega = 42\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$ and $x_a = 0.31L$. The attachment and node locations are collocated.

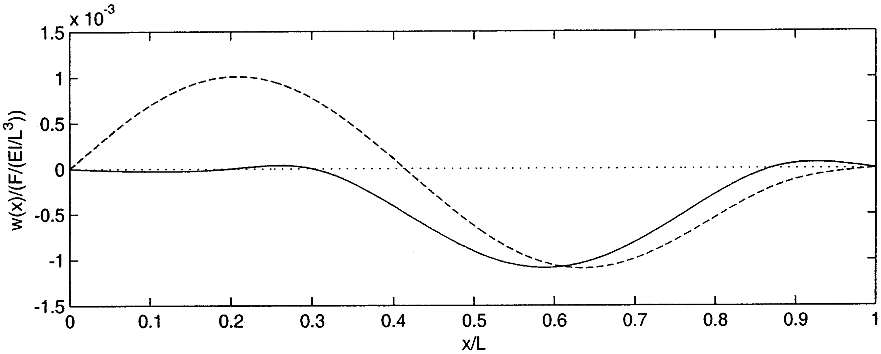

Fig. 3. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.2L$ and $x_a^2 = 0.3L$. The attachment and node locations are collocated.

spring–mass parameters are chosen such that $k = m\omega^2$. The dotted line corresponds to the deformed shape of the beam with no oscillator, and the horizontal line represents the configuration of the undeformed beam. Because the excitation frequency is in the vicinity of the second natural frequency of a uniform simply supported beam, the deformed shape of the beam with no oscillator resembles its second mode shape. Note that by attaching an oscillator with a properly chosen set of system parameters, its attachment location remains stationary, and its steady state response is substantially suppressed compared to the beam with no spring–mass attachment.

Consider again a simply supported beam. It is now desired that two nodes be imposed, at $x_n^1 = 0.2L$ and $x_n^2 = 0.3L$, for $\omega = 57\sqrt{EI/(\rho L^4)}$ and $x_f = 0.87L$. Fig. 3 shows the steady state lateral displacement of the beam. Note that by attaching two oscillators (whose system parameters are given by $k_r = m_r\omega^2$) at the desired node locations, the attachment points become nodes. Moreover, observe that the beam in the region between 0 and $0.3L$ has nearly zero amplitudes. Thus, by placing the oscillators at appropriate locations, it is possible to specify a region of nearly

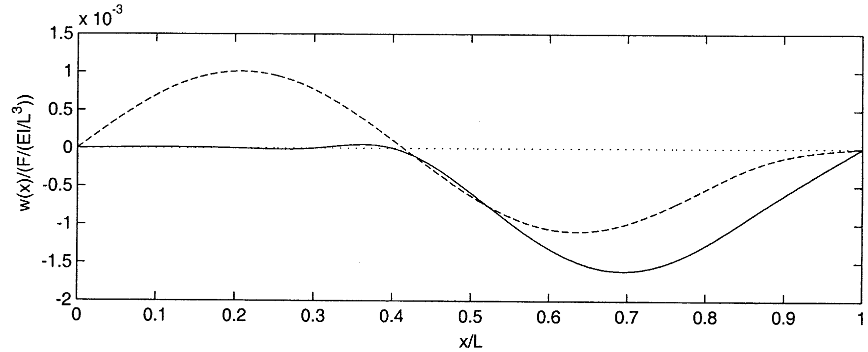

Fig. 4. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.2L$ $x_a^2 = 0.3L$ and $x_a^3 = 0.4L$. The attachment and node locations are collocated.

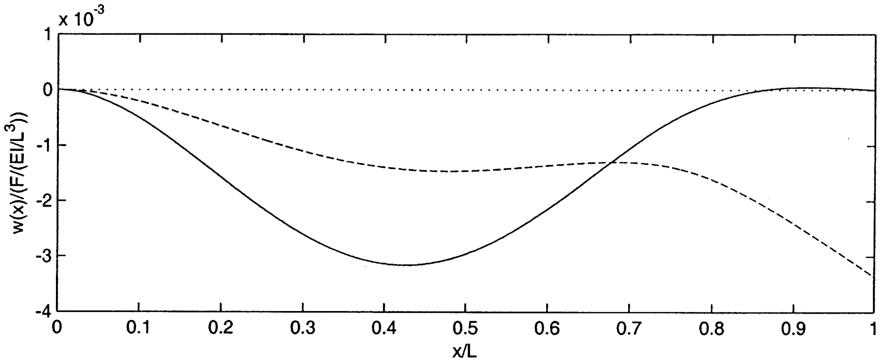

Fig. 5. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachment. The system parameters are $\omega = 31\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$ and $x_a = 1.0L$. The attachment and node locations are collocated.

zero displacements for a particular driving frequency, effectively quenching vibration in that segment of the beam.

Fig. 4 shows the steady state lateral displacement of a simply supported beam under harmonic excitation, with the same ω and x_f as those of Fig. 3, except now with three oscillators attached at $x_a^1 = 0.2L$, $x_a^2 = 0.3L$ and $x_a^3 = 0.4L$. For this case, note that the beam amplitude in the region between 0 and $0.4L$ remains nearly stationary, despite the fact that the same beam with no absorbers experiences substantial deflection within that region. This is clearly beneficial because it would allow sensitive instruments to be placed in the region where there is little or no vibration. Compared with the results of Fig. 3, note that by attaching an additional oscillator at $0.4L$, the region of nearly zero displacements is increased by approximately a third.

Consider now a uniform fixed-free Euler–Bernoulli beam. A localized harmonic force of frequency $\omega = 31\sqrt{EI/(\rho L^4)}$ is applied at $x_f = 0.77L$ along the beam. Fig. 5 shows the steady state lateral displacement of the cantilevered beam. Note that when the beam with no oscillator is subjected to the prescribed harmonic excitation, its free end exhibits the largest lateral displacement. The tip of the same beam with an oscillator (whose spring–mass parameters satisfy $k = m\omega^2$)

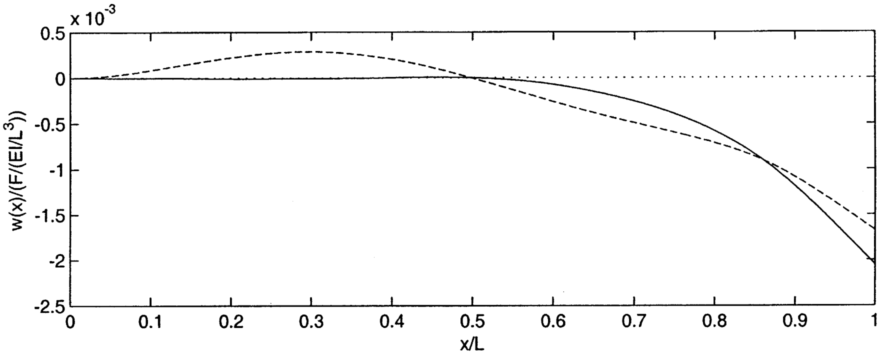

Fig. 6. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.4L$ and $x_a^2 = 0.5L$. The attachment and node locations are collocated.

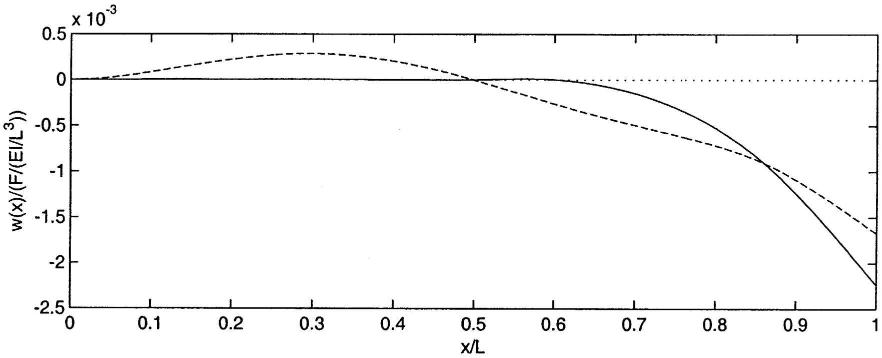

Fig. 7. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.4L$, $x_a^2 = 0.5L$ and $x_a^3 = 0.6L$. The attachment and node locations are collocated.

attached at $x_a = 1.0L$, however, becomes a node, even though its tip is not constrained. This has practical implications because by attaching a properly tuned oscillator anywhere along the beam, the attachment location can be made to remain completely stationary without using any rigid supports.

Fig. 6 illustrates the steady state deformed shape of a uniform cantilever beam subjected to a localized harmonic force applied at $x_f = 0.87L$, with a forcing frequency of $\omega = 57\sqrt{EI/(\rho L^4)}$. Note that by attaching two properly tuned oscillators at $x_a^1 = 0.4L$ and $x_a^2 = 0.5L$, nodes are induced at the same locations, which consequently leads to very little vibration in the region between 0 and $0.5L$. When an additional oscillator is attached at $0.6L$, the region of nearly zero displacements is extended to $0.6L$, as shown in Fig. 7.

3.2. Oscillators and node locations are not collocated

Consider a simply supported beam, with a concentrated harmonic force applied at $x_f = 0.77L$, with a forcing frequency of $\omega = 57\sqrt{EI/(\rho L^4)}$. For a given application, it is desired to have a

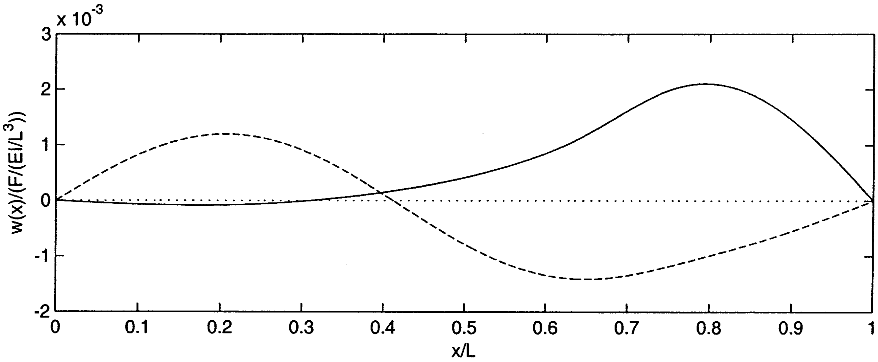

Fig. 8. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$, $x_a = 0.65L$, $x_n = 0.31L$, $k = 20EI/L^3$ and $m = 6.221 \times 10^{-3}\rho L$. The attachment and node locations are not collocated.

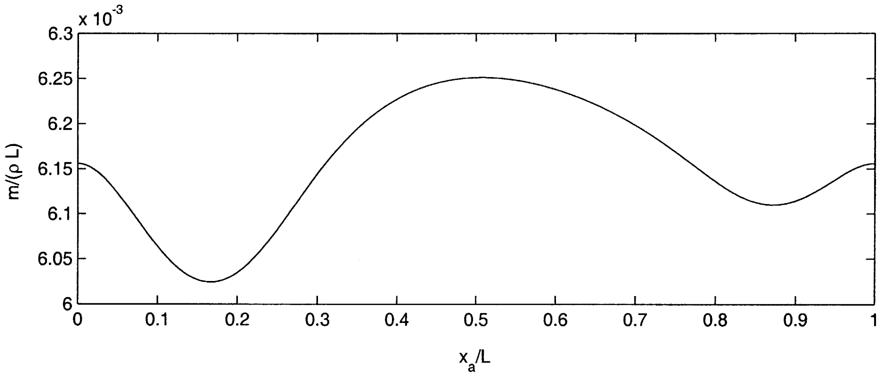

Fig. 9. The design plot of the required mass parameter $m/(\rho L)$ versus the attachment location x_a/L, for the same ω, x_f, x_n and k values of the simply supported beam of Fig. 8.

node at $x_n = 0.31L$. However, due to space constraint, an oscillator cannot be attached at that location but at some other point, say $x_a = 0.65L$. In this case, Eqs. (21)–(24) can be used to obtained the required spring–mass parameters in order to induce a node. For $k = 20EI/L^3$, Eq. (21) gives $m = 6.221 \times 10^{-3}\rho L$. Fig. 8 shows the deformed shape of the uniform simply supported beam. Note that deformed shape of the beam carrying the oscillator has a node at exactly $0.31L$, and the region between 0 and $0.31L$ experiences substantially less vibration comparing to the beam with no oscillator.

Fig. 9 shows the required mass parameter as a function of the attachment location, x_a, for the system and the x_f, x_n, k and ω values of Fig. 8. Because the beam is simply supported, the possible attachment locations are defined in the region $0<x_a<L$. Knowing the desired attachment location, this figure can be used as a design plot to select the required mass parameter in order to induce a node at x_n. If the attachment location is moved to $0.72L$ (closer to x_f), then for $k = 20EI/L^3$, Eq. (21) yields $m = 6.187 \times 10^{-3}\rho L$. For this case, the vibration of the beam with the oscillator attachment is substantially suppressed in the region between $0.4L$ and $1.0L$ (see Fig. 10) compared to the case where $x_a = 0.65L$ (see Fig. 8).

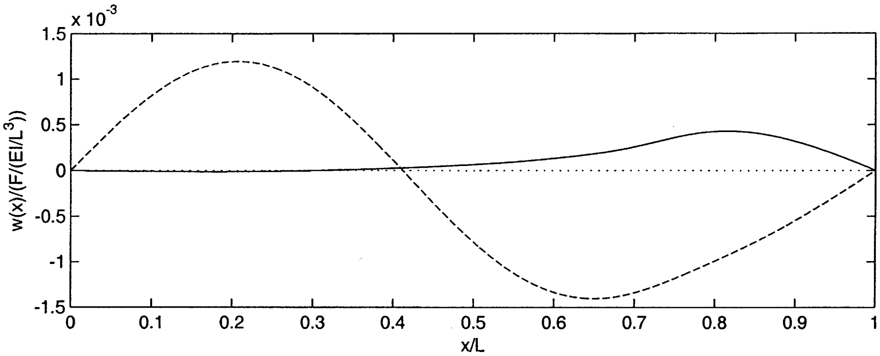

Fig. 10. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachment. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$, $x_a = 0.72L$, $x_n = 0.31L$, $k = 20EI/L^3$ and $m = 6.187 \times 10^{-3}\rho L$. The attachment and node locations are not collocated.

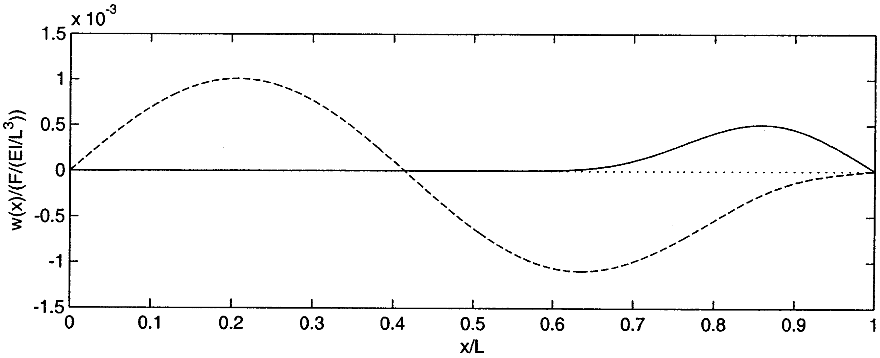

Fig. 11. The steady state deformed shapes of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 57\sqrt{EI/(\rho L^4)}$, $x_f = 0.87L$, $x_a^1 = 0.56L$, $x_a^2 = 0.71L$, $x_n^1 = 0.2L$, $x_n^2 = 0.3L$, $k_1 = 55EI/L^3$, $k_2 = 75EI/L^3$, $m_1 = 1.692 \times 10^{-2}\rho L$ and $m_2 = 2.333 \times 10^{-2}\rho L$. The attachment and node locations are not collocated.

A simply supported beam is excited by a concentrated harmonic force at $x_f = 0.87L$, with $\omega = 57\sqrt{EI/(\rho L^4)}$. For a specific application, a region of zero displacements is desired between $0.2L$ and $0.3L$. To accomplish this, two oscillators are attached at $x_a^1 = 0.56L$ and $x_a^2 = 0.71L$ for the purpose of inducing nodes at $x_n^1 = 0.2L$ and $x_n^2 = 0.3L$. For $k_1 = 55EI/L^3$ and $k_2 = 75EI/L^3$, solving the two equations of Eq. (19) simultaneously using MATLAB routine *fsolve* gives $m_1 = 1.692 \times 10^{-2}\rho L$ and $m_2 = 2.333 \times 10^{-2}\rho L$. Fig. 11 shows the steady state response of the simply supported beam carrying two oscillators (with the above set of system parameters) at the specified attachment locations. Note that the region of the beam between 0 and $0.6L$ is practically motionless, thus satisfying the design objectives.

Fig. 12 shows the steady state lateral displacement of a uniform fixed-free beam when it is being excited harmonically at $x_f = 0.77L$ with $\omega = 21\sqrt{EI/(\rho L^4)}$. A node is desired at $x_n = 1.0L$, and the oscillator is attached at $x_a = 0.62L$. For $k = 20EI/L^3$, Eq. (21) gives $m = 2.999 \times 10^{-2}\rho L$. Note that by attaching a spring–mass system with a set of properly chosen parameters, a node can

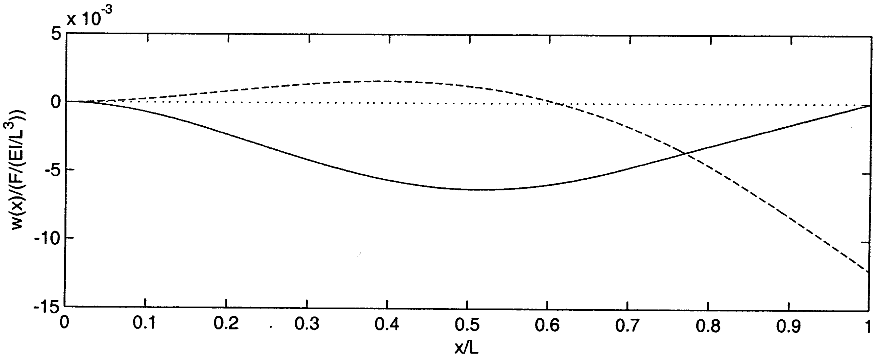

Fig. 12. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachment. The system parameters are $\omega = 21\sqrt{EI/(\rho L^4)}$, $x_f = 0.77L$, $x_a = 0.62L$, $x_n = 1.0L$, $k = 20EI/L^3$ and $m = 2.999 \times 10^{-2}\rho L$. The attachment and node locations are not collocated.

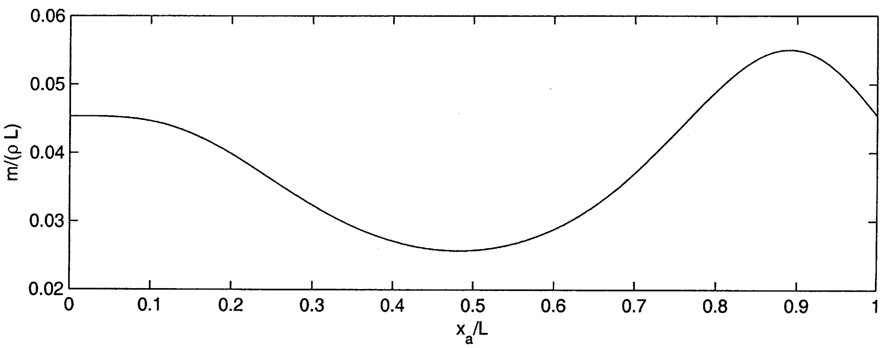

Fig. 13. The design plot of the required mass parameter $m/(\rho L)$ versus the attachment location x_a/L, for the same ω, x_f, x_n and k values of the cantilever beam of Fig. 12.

be induced at the tip of a fixed-free beam. Fig. 13 shows the design plot of m versus x_a for the system of Fig. 12 and the prescribed x_f, x_n, k and ω. Because the beam is fixed at one end, the range of possible attachment locations is given by $0 < x_a \leqslant L$.

Fig. 14 shows the steady state deformed shape of a uniform cantilever beam excited harmonically at $x_f = 1.0L$ with $\omega = 75\sqrt{EI/(\rho L^4)}$. Nodes are desired at $0.6L$ and $1.0L$, and oscillators are attached at $x_a^1 = 0.64L$ and $x_a^2 = 0.83L$. For $k_1 = 75EI/L^3$ and $k_2 = 50EI/L^3$, Eq. (19) returns $m_1 = 1.343 \times 10^{-2}\rho L$ and $m_2 = 8.099 \times 10^{-3}\rho L$. Note that for the chosen set of system parameters, the displacements at $0.6L$ and $1.0L$ indeed become zero, even though the localized force is applied at $x_f = 1.0L$.

A simple and efficient approach has been developed to solve the inverse problem of imposing nodes at multiple locations along any arbitrarily supported elastic structure that is subjected to a localized harmonic excitation. This has practical benefits because it allows certain points along the structure to remain stationary without using any rigid supports, and it enables certain regions of

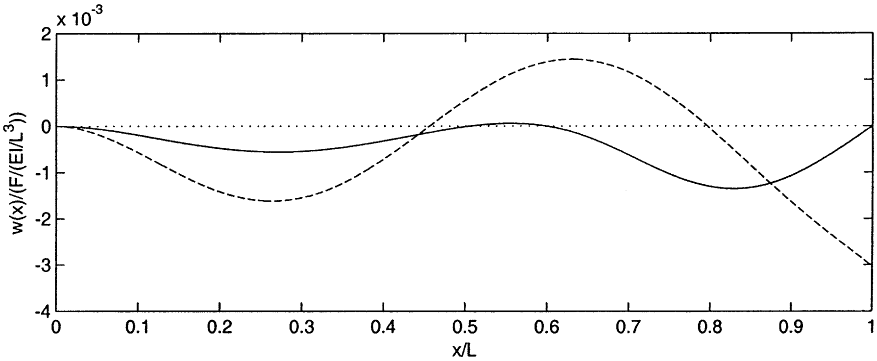

Fig. 14. The steady state deformed shapes of a uniform cantilever Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. The system parameters are $\omega = 75\sqrt{EI/(\rho L^4)}$, $x_f = 1.0L$, $x_a^1 = 0.64L$, $x_a^2 = 0.83L$, $x_n^1 = 0.6L$, $x_n^2 = 1.0L$, $k_1 = 75EI/L^3$, $k_2 = 50EI/L^3$, $m_1 = 1.343 \times 10^{-2}\rho L$ and $m_2 = 8.099 \times 10^{-3}\rho L$. The attachment and node locations are not collocated.

the structure to undergo very small deflections, thereby suppressing vibration in those sections. Finally, another important design specification is governed by the vibration of the absorber masses. If the vibration amplitudes of these masses approach very high values, then theoretically feasible solutions would be impossible to apply in practice, and it would be necessary to introduce dampers to the vibration absorbers. This interesting problem of imposing the additional constraint of maximum vibration amplitude of the masses will be left for a future research project.

4. Conclusions

Elastically mounted masses can be used to impose a single or multiple nodes on any elastic structure during harmonic excitations. When the parameters of the sprung masses are properly chosen, nodes can always be induced at the attachment locations for any excitation frequency and excitation location. When the attachment and the node locations are not collocated, it is only possible to induce a node or multiple nodes at certain locations along the structure. In addition, if the node locations are properly selected, a region of nearly zero amplitudes can be imposed along the elastic structure for a given localized harmonic force without using any rigid supports, effectively quenching vibration in that segment of the structure. A detailed procedure to assist in the selection of the attached spring–mass systems was outlined, and numerical experiments were performed to validate the utility of the proposed scheme of imposing a single or multiple nodes during harmonic excitations for the collocated and non-collocated cases.

Appendix A. Required oscillator mass

Consider two matrices $[A]$ and $[B]$ that are related as shown:

$$[A] = [B] + \underline{u}\,\underline{v}^{\mathrm{T}}.$$

If the inverse of $[B]$ is known, then the inverse of $[A]$ can be obtained by using the following Sherman–Morrison formula [8]:

$$[A]^{-1} = [B]^{-1} - \frac{[B]^{-1}\underline{u}\,\underline{v}^{\mathrm{T}}[B]^{-1}}{1 + \underline{v}^{\mathrm{T}}[B]^{-1}\underline{u}}.$$

The above result can be used to invert expression (20). Letting

$$[B] = [K^d] - \omega^2[M^d], \qquad \underline{u} = \alpha\underline{\phi}(x_a),$$

where

$$\alpha = \frac{km\omega^2}{m\omega^2 - k}$$

and

$$\underline{v}^{\mathrm{T}} = \underline{\phi}^{\mathrm{T}}(x_a),$$

the required inverse can be readily obtained in closed form. Expanding the triple product of Eq. (20) yields

$$c_1 - \frac{\alpha}{1 + c_3\alpha}c_2 = 0,$$

where and c_1 to c_3 are given by Eqs. (22)–(24), respectively. Multiplying the above equation through by $1 + c_3\alpha$ and solving for m, Eq. (21) is obtained.

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