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Minimax optimization of multi-degree-of-freedom tuned-mass dampers

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Abstract

Many methods have been developed for the design of a single-degree-of-freedom (SDOF) absorber to damp SDOF vibration. Yet there are few studies for the case where both the absorber and the main system have multiple degrees of freedom. In this paper, an efficient numerical approach based on the descent-subgradient method is proposed to maximize the minimal damping of modes in a prescribed frequency range for general viscous or hysteretic multi-degree-of-freedom (MDOF) tuned-mass systems. Examples are given to illustrate the efficiency of the minimax method and the damping potential of MDOF tuned-mass dampers (TMDs). The performance of minimax, H2, and H_∞ optimal TMDs are compared. Finally, the results of an experiment in which a 2-DOF TMD is optimized to damp the first two flexural modes of a free–free beam are presented.

1. Introduction

A tuned-mass damper (TMD), or dynamic vibration absorber (DVA), is an efficient passive vibration suppression device comprising a mass, springs, and (usually) viscous or hysteretic dampers. Since proposed in 1909, they have been widely used in machinery, buildings, and structures. The first theoretical investigation of TMD design was carried out by Ormondroyd and Den Hartog in 1928, and the details can be found in the text by Den Hartog [1]. By balancing the two fixed points in the frequency response, Den Hartog found the optimal tuning ratio f and damping ratio ζ for an auxiliary single-degree-of-freedom (SDOF) mass attached to an undamped

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SDOF system to be

$$f = \frac{1}{1+\mu}, \quad \zeta = \sqrt{\frac{3\mu}{8(1+\mu)}}, \tag{1}$$

where μ is the mass ratio. Later, SDOF systems with some inherent damping were also examined and various tuning rules were proposed in the time and frequency domains [2,3], including H2 and H_∞ optimal tunings [4,5].

Warburton [6] investigated the optimum tuning parameters for a SDOF TMD to minimize the response of a 2-DOF main system by treating the main system as an equivalent SDOF system; he also examined the effect of the distribution of natural frequencies on the optimum parameters. Ding [7] studied the vibration magnitude through a transition matrix and gave some general guidance for design of SDOF TMDs to absorb narrowband vibration of multiple-degree-of-freedom (MDOF) systems. Vakakis and Paipetis [8] investigated the effect of SDOF TMDs on the first mode of a MDOF primary system. Sadek et al. [9] found that a system comprising a SDOF structure and an optimized TMD has two modes with equal damping ratios greater than the average damping in the decoupled structure and TMD; for a SDOF TMD attached to a MDOF structure, they used an equivalent mass ratio based on unit modal participation.

To damp more than one mode of a MDOF primary system, usually several SDOF TMDs are used, and if the modal frequencies are well separated, the design approach based on equivalent SDOF systems works well. To achieve better performance, several direct design methods have been proposed. Snowdon et al. [10] developed a vibration absorber in the form of two crossed beams with masses at their ends. By minimizing the displacement response over certain frequencies with a time-efficient gradient-based optimization algorithm, Kitis et al. [11] designed two SDOF TMDs simultaneously to damp the first two modes of a cantilever beam. Using the simplex algorithm to minimize the peak of the frequency response over a certain frequency range, Rice [12] also obtained the optimal parameters (position, stiffness, and damping) of two SDOF TMDs for a cantilever beam. Rade and Steffen [13] employed a substructure-coupling technique, and by minimizing the performance obtained from a weighted transfer matrix at a set of discrete frequencies, obtained the optimal parameters of two SDOF TMDs attached to a free–free beam. Abe and Fujino [14] studied the design of several small TMDs tuned to a single resonance of a structure. Some sub-optimal LQG/H2 methods have also been used to design SDOF TMDs for MDOF main systems [15,16]. Using genetic algorithms to minimize the H2 norm, Arfiadi and Hadi [17,18] designed an H2 optimal SDOF TMD for MDOF building structures.

Whenever we add a body to a structure, it will have six degrees of freedom relative to the structure. By taking full advantage of the inertia of the body, we can damp as many as six modes, or make the system more robust or compact. However, there are few studies and available approaches on the use of more than one DOF of a body to damp either a SDOF or MDOF structure, except for the simple case where the motions are decoupled in space [19]. Dahlbe [20] showed that a two-segment cantilever beam can be more effective than a SDOF TMD for suppression of SDOF vibration. Beams or plates have been used as absorbers to suppress vibration in one mode of a beam or plate based on Den Hartog's method and the concept of an equivalent modal mass [21–23]. Yamaguchi [24] damped two modes of a clamped–clamped beam by attaching a double-cantilever beam with a spring–dashpot connection. To the authors' knowledge, these are the only studies where both the damper and main system have more than one coupled mode. The methods of Kitis et al. [11] and Hadi and Arfiadi [17] can be extended to design MDOF TMDs for MDOF primary systems. Zuo and Nayfeh [25,26] have also applied decentralized control techniques (H2 and H_∞ optimization) to design MDOF TMDs and other passive mechanical systems.

In this paper, we use the descent-subgradient method to maximize the minimal damping in a prescribed frequency range for a general structure to which is attached an MDOF TMD or multiple SDOF TMDs. Taking the location and inertias of the absorbers as well as the locations of the springs and dampers connecting the absorbers to the structure as fixed, we cast TMD design as an optimization problem in the framework of decentralized static output control in state space. Next, we introduce some key concepts of minimax optimization and propose an optimization method based on the descent-subgradient method. Design examples are given to compare the performance obtained from minimax optimization to those obtained from H2 and H_∞ optimization and to compare the performance obtained using an MDOF TMD to that obtained using multiple SDOF TMDs. Finally, we present the results of an experiment in which a 2-DOF TMD is designed to damp the first two flexural modes of a free–free beam.

2. Formulation

If the geometry and inertia of a passive mechanical system are fixed, optimal selection of the stiffness and damping elements of the mechanical system can be cast as a static decentralized control problem in state space by treating the springs and dashpots as local position and velocity feedback elements [26]. To optimize a TMD of a fixed configuration, our task is to select the parameters of the stiffness and damping (which can be either viscous or hysteretic), to maximize some measure of the performance. Replacing the effect of the connections between the absorber mass and the original system with the control force vector u, we can write the equation of motion in matrix form

$$M_q\ddot{q} + C_q\dot{q} + K_q q = B_q u, \tag{2}$$

where M_q, C_q, and K_q are, respectively, the mass, damping and stiffness matrices of the system with the connections between the absorber mass and the original system removed. Using geometric information, we can write the relative displacement output vector between connection points as

$$p = C_p q.$$

By defining the state vector x as $[q', \dot{q}']'$ and the output vector y as the relative displacement and velocity $[p_1, \dot{p}_1, p_2, \dot{p}_2, \ldots]'$ (where primes indicate matrix transposition), we obtain the state-space description

$$\dot{x} = Ax + Bu, \qquad y = Cx. \tag{3}$$

The matrices A, B, and C are given by

$$A = \begin{bmatrix} 0 & I \\ -M_q^{-1}K_q & -M_q^{-1}C_q \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ M_q^{-1}B_q \end{bmatrix}, \qquad C = T\begin{bmatrix} C_p & 0 \\ 0 & C_p \end{bmatrix}, \tag{4}$$

where I is identity matrix and T is a matrix to reorder $[q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots]'$ into $[q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots]'$.

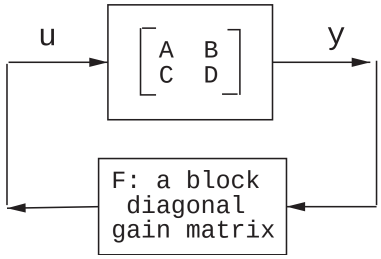

Fig. 1. Block diagram showing a TMD system formulated as a feedback controller.

As can be determined from Fig. 1, the "closed-loop" system is described by

$$\dot{x} = (A + BFC)x, \tag{5}$$

where F is a static block-diagonal feedback matrix whose elements are the design parameters: the stiffness and damping of the connections between the tuned mass and the original system. For a viscously damped system, the feedback matrix usually takes the form

$$F = \begin{bmatrix} k_1 & c_1 & & & & & & \\ & & k_2 & c_2 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & & & k_m & c_m \end{bmatrix}.$$

For a hysteretically damped system, the c_n are zero and the k_n are augmented with a loss factor η_n so that they take the form $k_n(1 + \mathrm{i}\eta_n \operatorname{sgn} \omega)$, where i is the imaginary unit.

If we know the disturbance inputs and critical outputs, we can use standard control-design techniques (such as decentralized H2 or H_∞ optimization) to design the TMDs. However, the disturbances and performance measures are often difficult to define, and it becomes more practical to maximize the minimum damping in a prescribed frequency range $[\omega_l, \omega_h]$. Thus, we obtain a minimax problem:

$$\max_{F \in \Omega} \left(\min_{i \in I} (\zeta_i(F)) \right)$$
$$I = \{ i | \omega_l \leqslant \omega_i \leqslant \omega_h, \omega_i = |eig(A + BFC)| \}, \tag{6}$$

where Ω is the set of F satisfying the block-diagonal constraint. To solve this minimax problem, we introduce the concept of a subgradient and its application to non-smooth optimization.

3. The subgradient and non-smooth optimization

It is well known that gradient-based algorithms are much more efficient than non-gradient-based algorithms (such as the simplex algorithm) in constrained or unconstrained optimization [27]. However $\min_{i \in I}(\zeta_i(F))$ is a non-smooth function, so we cannot use the well-known conjugate gradient or Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm. Non-smooth optimization problems have been examined extensively after the early studies of Polyak [27–29] and Dem'yanov [30,31].

The subgradient or ε-subgradient plays an important role in non-smooth optimization, similar to that of the gradient in smooth optimization. Given a convex function $f : R^n \to R$, a vector d is called a *subgradient* of f at x if

$$f(z) \geqslant f(x) + d'(z - x), \quad \forall z \in R^n. \tag{7}$$

The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the *subdifferential* of f at x. It is a nonempty, convex and compact set. The concepts of a subgradient and subdifferential are illustrated in Fig. 2 for the case where both f and x are scalars: from any point on a convex function, one can march along a line of any slope contained in the subdifferential without passing above the function f.

We can similarly define the subgradient for a concave function. From the definition, we can see that any subgradient is an ascent direction for a convex function $f : R^n \to R$. So *arbitrarily selected* subgradients—combined with a properly chosen step size [29]—will yield a descent sequence and converge to a stationary point or ε-stationary point. For the non-smooth function $\max_{i \in I} f_i(x)$ or $\min_{i \in I} f_i(x)$, where I is a finite index set, it is easy to find the entire subdifferential (set of all subgradients), and the steepest subgradient according to the following two theorems.

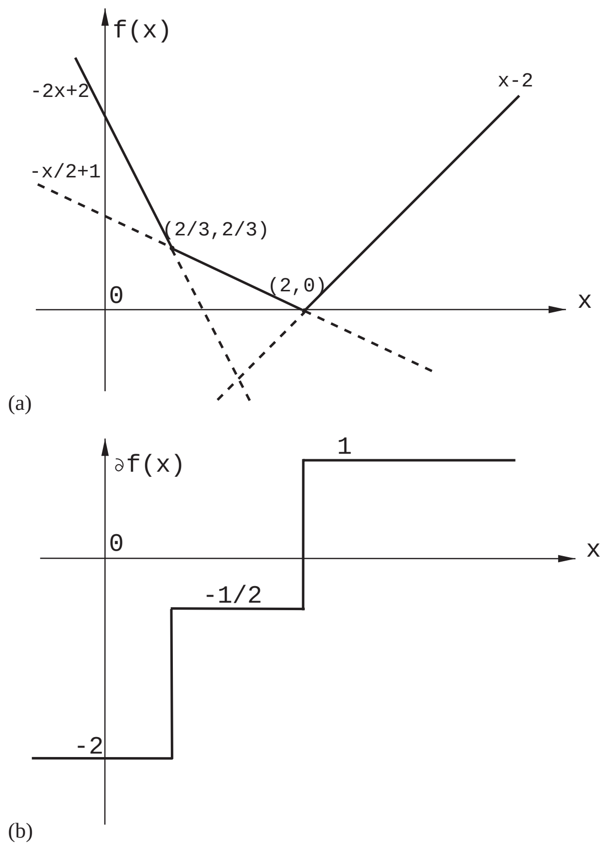

Fig. 2. Illustration of the subdifferential of a scalar non-smooth function of a scalar argument: (a) a function $f(x)$ and (b) its subdifferential $\partial f(x)$.

Theorem 1 (Danskin's [28]). *If $f_i(x): R^n \to R$ is smooth for all $i \in I$, then the subdifferential of $f(x) = \max_{i \in I} f_i(x)$ is*

$$\partial f(x) = conv\{\nabla_x f_{\bar{\imath}}(x) | \bar{\imath} \in \bar{I}(x)\}, \tag{8}$$

where $\bar{I}(x) = \{\bar{\imath} | f_{\bar{\imath}}(x) = \max_{i \in I} f_i(x)\}$, $\nabla_x f_{\bar{\imath}}(x)$ is the gradient of $f_{\bar{\imath}}(x)$ at x, and $conv(\cdot)$ denotes a convex hull.

Theorem 2 (Dem'yanov and Vasil'ev [31]). *A necessary condition for a continuous non-smooth (not necessarily convex) function $f(x): R^n \to R$ to attain a minimum at x^* is that $0 \in \partial f(x^*)$. If $0 \notin \partial f(x)$, then the direction*

$$-\arg \min_{d \in \partial f(x)} \|d(x)\|$$

is the steepest descent direction. (For a convex function, the condition above is sufficient.)

Theorems 1 and 2 suggest an approach for standard minimax problems (or max–min problems): Start with an arbitrary initial point x_0, evaluate the subdifferential $\partial f(x_0)$, use a one-dimensional optimization method to update x_0 in the direction of steepest descent given by Theorem 2, and then repeat the procedure. One might expect the limit point of this descent sequence to be a minimal point of the non-smooth function. However, due to the lack of smoothness, the limit point of the sequence generated by the above algorithm based on Theorem 2 may not even be a stationary point of $f(x)$. (An example is given by Dem'yanov [30].) Moreover, since any numerical algorithm based on the subgradient must be carried out in discrete steps, it is practical to introduce a scalar ε and define the ε-subgradient as follows:

Given a convex function $f: R^n \to R$, for a scalar $\varepsilon > 0$, we say that a vector d is an *ε-subgradient* of f at x if

$$f(z) + \varepsilon \geqslant f(x) + d'(z - x), \quad \forall z \in R^n. \tag{9}$$

The set of all ε-subgradients of f at x, denoted by $\partial_\varepsilon f(x)$, is called the ε-subdifferential of f at x. The following theorem is the basis of an ε-subgradient algorithm for minimax problems:

Theorem 3 (Dem'yanov [30]). *If $\varepsilon > 0$, $I_\varepsilon = \{j | [\max_{i \in I} f_i(x)] - f_j(x) < \varepsilon\}$, and $\partial_\varepsilon f(x) = conv\{\nabla_x f_j(x) | j \in I_\varepsilon(x)\}$ is not empty, then the search direction*

$$-\arg \min_{d \in \partial_\varepsilon f(x)} \|d(x)\|$$

and one-dimensional minimizing step size will yield a sequence whose limit point is an ε-stationary point of $f(x) = \max_{i \in I} f_i(x)$, which is an approximation to a stationary point with absolute error of at most ε.

4. Minimax algorithm for TMD design

With the above background of non-smooth optimization, we return to the TMD problem (6): determine the structure-constrained feedback gain F that maximizes the minimal damping in a certain frequency range. Our algorithm is based on Theorems 1, 3 and the following eigenvalue sensitivity formula:

Given a real-coefficient dynamic system $\dot{x} = (A + BFC)x$, the sensitivity of the jth eigenvalue λ_j to changes in the klth element of F is

$$\frac{\partial \lambda_j}{\partial F_{kl}} = \frac{w_j' b_k c_l v_j}{w_j' v_j}, \tag{10}$$

where v_j and w_j are the jth right and left eigenvectors of $A + BFC$, respectively, b_k is the kth column of B, and c_l is the lth row of C.

For systems with viscous damping, the matrices A, B, C, and F are real and the eigenvalues of $A + BFC$ are symmetric with respect to the real axis in the complex plane. We need only consider the poles and associated damping in the second quadrant. The damping ratio is

$$\zeta_j(F) = \frac{-\mathrm{Re}(\lambda_j)}{|\lambda_j|}. \tag{11}$$

Using the gradient chain rule, we write the sensitivities of the damping ratios with respect to the design parameters as

$$\frac{\partial \zeta_j}{\partial F_{kl}} = \frac{\partial \zeta_j}{\partial \lambda_j} \frac{\partial \lambda_j}{\partial F_{kl}}. \tag{12}$$

We now propose the following minimax algorithm for TMD design:

1. Choose the initial parameters—a block diagonal matrix F.
2. Solve for the steepest-descent subgradient $drt(F)$: Evaluate the eigenvalues and eigenvectors of $A + BFC$, find the set of indexes corresponding to modal frequencies whose damping ratio is close to the minimal damping inside the specified frequency band:

   $$I_\varepsilon(F) = \{j \mid \zeta_j(F) - \min_{i \in I} \zeta_i(F) \leqslant \varepsilon\}. \tag{13}$$

   Compute the gradient of the damping $\nabla_F \zeta_j(F)$ with respect to the free design variables in F for all $j \in I_\varepsilon(F)$ using (10) and (12). We obtain a convex hull $\partial_\varepsilon \zeta(F)$. Then solve a minimization problem to obtain the steepest-descent subgradient

   $$drt(F) = -\arg \min_{d \in \partial_\varepsilon \zeta(F)} \|d(F)\|. \tag{14}$$

   If $drt(F) = 0$, stop; otherwise go to step 3.
3. One-dimensional minimization: Search in the direction $drt(F)$ to determine the step size α which maximizes the function

   $$\min_{i \in I} \zeta_i(F + \alpha\, drt(F)).$$

   Update F with $F + \alpha\, drt(F)$. Then go to step 2.

Remark 1. To find the direction of steepest descent $drt(F)$ as defined by Eq. (14), we must solve a convex constrained linear least-squares problem:

$$\min_{d \in \partial_\varepsilon f(F)} \|d(F)\| = \min_{\beta_j \geqslant 0,\ \sum_j \beta_j = 1} \left\| \sum_{j \in I_\varepsilon} \beta_j \nabla_F f_j(F) \right\|. \tag{15}$$

Such problems can be solved efficiently using a standard code, such as the lsqlin function in the *Matlab Optimization Toolbox*.

Remark 2. Generally, gradient-based methods are not finitely convergent. So typically we will stop computation when $||drt(F)||$ or $\alpha||drt(F)||$ becomes sufficient small.

Remark 3. To make the approach more practical, we can also maximize the weighted minimal damping in a selected frequency range.

Remark 4. For a practical design, negative stiffness or damping are hard to construct. So in the case where there is a kinematic redundancy in the connection of the TMD, we should replace F_{kl} with F_{kl}^2. More generally, if we would like to constrain some parameter F_{kl} to be in some physically achievable internal $[r_1, r_2]$, we can specify F_{kl} with one parameter r:

$$F_{kl} = \tfrac{1}{2}(r_1 + r_2) + \tfrac{1}{2}(r_2 - r_1)\sin r$$

and evaluate the damping sensitivity with respect to the design parameter r.

Remark 5. For hysteretically damped systems, the matrices A and F are complex. There is little published work on optimization of such systems, though they are important in practice. To treat systems with hysteretic damping, we need only extend the eigenvalue sensitivity formula to the case of complex coefficients:

$$\frac{\partial \lambda_j}{\partial \mathrm{Re}(F_{kl})} = \frac{w_j' b_k c_l v_j}{w_j' v_j}, \quad \frac{\partial \lambda_j}{\partial \mathrm{Im}(F_{kl})} = \mathrm{i}\,\frac{w_j' b_k c_l v_j}{w_j' v_j}, \tag{16}$$

where i is the imaginary unit. Using these equations we can compute the sensitivity of the modal damping to changes in the real and imaginary parts of the design parameters in F.

5. Design examples

5.1. Two-DOF TMD: minimax, H2, and H_∞ *optima*

Consider a 2-DOF primary system that can translate in the x direction and rotate about the z-axis as shown in Fig. 3.

Our task is to choose k_1, k_2, c_1, and c_2 to damp the two modes of the main system. With initial guesses of $k_1 = k_2 = 500$ N/m and $c_1 = c_2 = 50$ N s/m, the minimax algorithm converges to $k_1 = 6038.93$ and $k_2 = 2679.96$ N/m along with $c_1 = 11.74$ and $c_2 = 5.94$ N s/m, producing a system with minimal modal damping of 8.77%.

To perform H2 and H_∞ optimization of the system, we take vertical displacement of the ground as the disturbance, and the displacement and rotation of the main mass gravity center as the cost output. The closed loop has one input and two outputs. We use gradient-based optimization to minimize the system H2 norm and linear matrix inequality (LMI) based iteration to minimize the system H_∞ norm as detailed by Zuo and Nayfeh [26]. The closed-loop

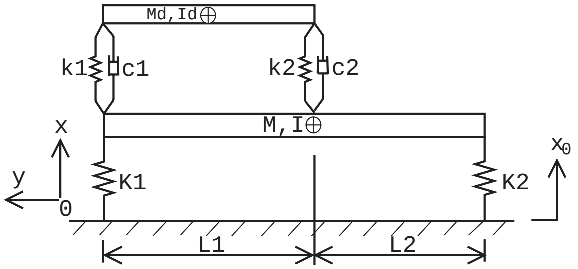

Fig. 3. Sketch of the 2-DOF system with a 2-DOF tuned-mass damper: $L_1 = 0.25$ m, $L_2 = 0.2$ m, $M = 5$ kg, $I = 0.1$ kg m^2, $K_1 = 50$ kN/m, $K_2 = 80$ kN/m, $M_d = 0.05$ M, and $I_d = 0.035$ I.

Table 1
Comparison of the performance obtained with the 2-DOF TMD under various designs: minimax, H2, and H_∞ optimization

	H_2 norm	H_∞ norm	Modal damping (%)
H_2 optimization	55.6620	11.9748	4.28, 5.13, 11.2, 3.77
H_∞ optimization	56.4402	11.0721	5.41, 6.53, 16.5, 3.81
Minimax	62.1083	16.4614	8.77, 8.77, 13.5, 8.77

performances produced by the minimax, H2, and H_∞ optimizations are compared in Table 1 and the frequency responses are shown in Fig. 4.

From Table 1 we see that the minimax design yields a system with highest damping but takes no account of the system zeros, the H2 design will produce a system with the smallest output variance under white noise input, and the H_∞ optimization achieves smallest peak magnitude for the worst-case sinusoidal input. So whether we should use minimax, H2, or H_∞ optimization for the design will depend on the performance requirements and our knowledge of the disturbances. In many cases it is difficult to define the disturbance inputs and the critical outputs and the minimax approach becomes somewhat more practical than input–output-based design. Moreover, unlike the H2 and H_∞ approaches, the minimax method is applicable to marginally stable systems or hysteretically damped systems.

5.2. Single MDOF vs. multiple SDOF TMDs for a free–free beam

Consider flexural, planar vibration of the free–free beam shown in Figs. 5 and 6. Its length is 1.829 m, its bending stiffness EI is 1.636×10^5 N m^2, its mass per unit length is 23.245 kg/m. Our goal is to damp the first three flexural modes. In order to describe the system in finite-dimensional state space, we discretize the beam into 12 segments with 13 nodes, each of which has three degrees of freedom in the plane. Note that this system is marginally stable due to the presence of rigid-body modes, and hence we can't use decentralized H2 or H_∞ optimization for the design.

Fig. 5 shows the set up of a 3-DOF TMD. Attached to the beam is a small rigid-body absorber, whose mass is 4% of that of the beam and whose length is 10% of that of the beam. It is mounted to the beam via three damped flexures at 55.4 mm from the neural axis of the beam. Fig. 6 shows

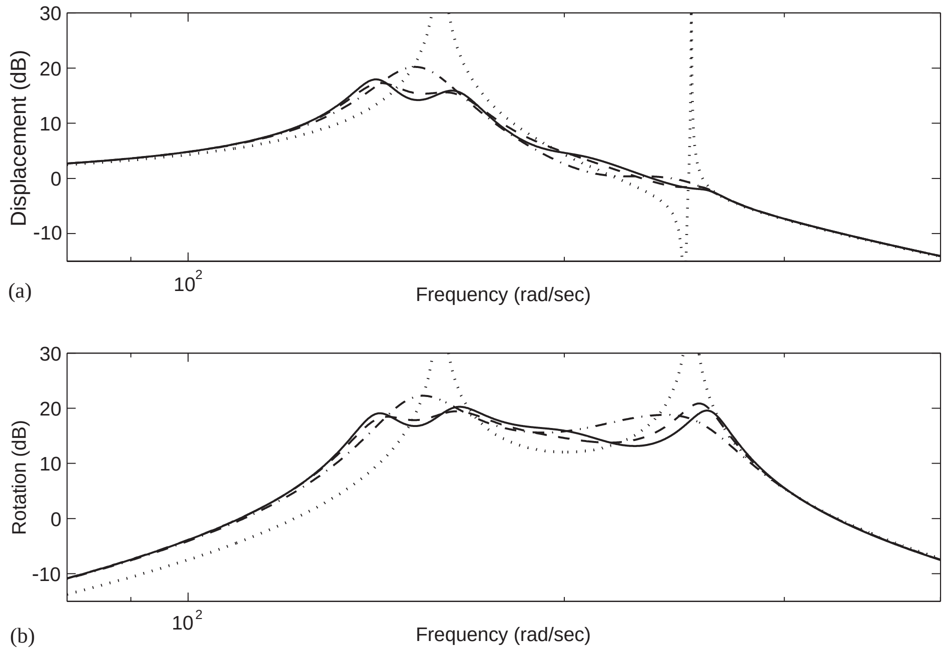

Fig. 4. Bode plots of transmission from ground vertical input x_0 to (a) the displacement x and (b) the rotation θ of the main mass: undamped (dotted), H2 optimal (solid), H_∞ optimal (dashed), minimax design (dash-dot).

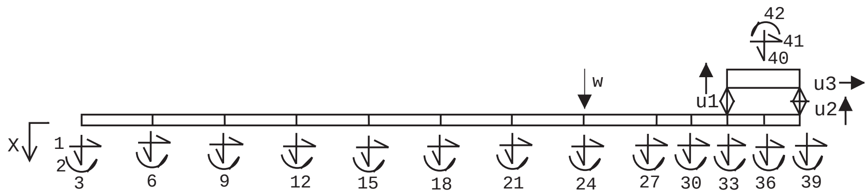

Fig. 5. One 3-DOF TMD for a 39-DOF discretized free–free beam.

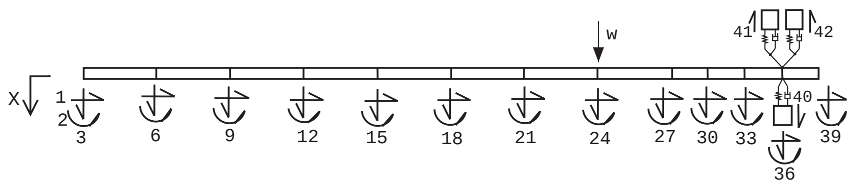

Fig. 6. Three SDOF TMDs for a 39-DOF discretized free–free beam.

three SDOF TMDs attached at the same location as the center of mass of the three DOF TMD. Each of the three dampers has the a mass equal to 1.333% of that of the beam.

For multiple SDOF TMDs, we begin by carrying out the traditional design by computing the equivalent main mass at the connection point for each mode based on the modal energy relations

Table 2
Comparison of the damping ratios (%) obtained with a 3 DOF TMD and three SDOF TMDs for the free–free beams of Figs. 5 and 6

Mode	3 SDOF TMDs traditional viscous	3 SDOF TMDs minimax viscous	3 DOF TMD minimax viscous	3 DOF TMD minimax hysteretic
1	3.38 7.22	6.78 8.74	10.81 10.81	10.67 10.67
2	2.44 5.92	6.17 6.17	9.83 9.92	9.70 9.71
3	1.52 4.62	4.38 6.22	6.79 7.12	6.67 6.67

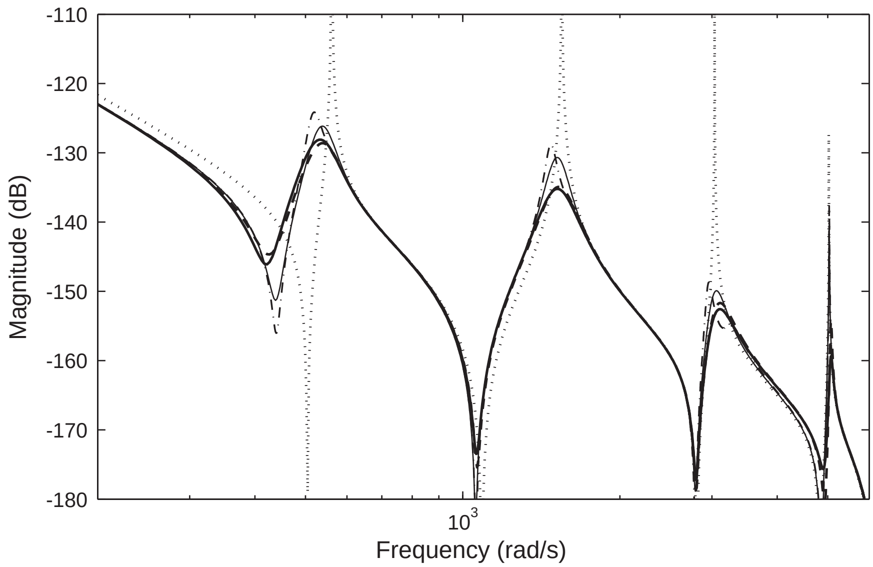

Fig. 7. Frequency response of the free–free beam: undamped (dashed), three SDOF TMDs traditional design (dash–dot), three SDOF TMDs minimax design (thinner solid), three-DOF viscous TMD minimax design (thicker solid), three-DOF hysteretic TMD minimax design (dots).

[32], and then choose the parameters for each TMD using Den Hartog's method by ignoring the coupling of other modes. The achieved damping ratios for the first three flexural modes of beam are shown in the first column of Table 2. Next, we cast the system as an 84 order plant with decentralized feedback and optimize the stiffness and damping of each connection using the minimax algorithm with weightings of 1.0:1.1:1.6 for the first three bending modes. The results for the SDOF TMDs and MDOF TMD are shown in the second and third columns of Table 2. For the MDOF TMD we also use the minimax algorithm to determine the optimal parameters for the case of hysteretic dampers and provide the results in the fourth column of Table 2.

The frequency responses from the exogenous force w to the collocated displacement are compared in Fig. 7. The minimax algorithm yields TMDs with a performance much better than the traditional design and the MDOF TMD performs better than multiple SDOF TMDs located at the same position. The performances of the optimal hysteretic and viscous TMDs are almost identical.

The location of the TMDs as well as the distribution of the mass among the multiple SDOF TMDs can have a significant effect on the performance. In the results shown in Table 2, the damping attained by the SDOF absorbers in the second bending mode of the beam has attained its (weighted) maximal value; if the mass of the corresponding absorber were increased at the expense of the other absorbers, the overall performance could be increased. If we place the three SDOF TMDs at the very tip of the beam, the minimax algorithm produces a maximized minimal damping of 9.78%, slightly smaller than the value of 10.81%, achieved by the MDOF TMD setup in Fig. 5.

5.3. Experiment: 2-DOF TMD for a free–free beam

Consider flexural, planar vibration of another free–free steel beam shown in Fig. 8.

Its cross section is $1.5 \times 6\ \text{in}^2$ and its length is 72.7 in. At one end of beam, we attach a 2 DOF steel block whose mass is 4% of that of beam, and whose dimensions are $7.8 \times 3.3 \times 1.0\ \text{in}^3$. The distance between the two damped mounts is 7.2 in. In contrast to the case presented in the preceding section, here the absorber is constrained from moving in the axial direction and hence has only two DOF. Our goal is to maximally damp the first two flexural vibration modes by optimal selection of the stiffness and damping in each mount.

Constraint of the movement of the beam in the axial direction will not influence the linearized bending model, and further the connections of TMD and beam are in x direction. Thus we take each node of the discretized beam to have only two degrees of freedom and obtain a 28-DOF system, which is cast as a 56 order plant with decentralized feedback. The minimax algorithm arrives at the optimal parameters

	Spring (N/m)	Damper (N s/m)
1	1.33×10^5	165.9
2	8.60×10^5	326.7

for a broad range of initial guesses. These parameters yield a system whose first four non-zero resonance frequencies and damping ratios are given in Table 3.

Based on the optimal parameters, we design spring–dashpot pairs in the form of flexures whose stiffness and damping are independently adjustable, as shown in Fig. 9. We size the blades to have a bending stiffness initially lower than that required by our nominal design and adjust the stiffness

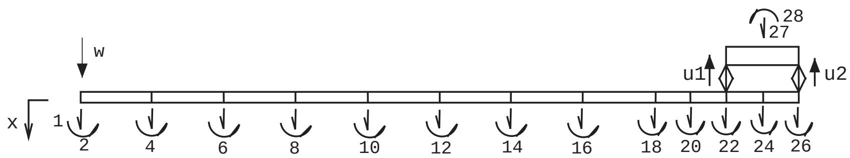

Fig. 8. Two-DOF TMD for a 26-DOF discretized free–free beam.

Table 3
Resonant frequencies and damping ratios achieved by minimax optimization of the two-DOF damper on a free–free beam

Mode	Damping ratio (%)	Frequency (Hz)
1	11.9	56.2
2	9.94	57.2
3	9.94	162
4	9.94	162

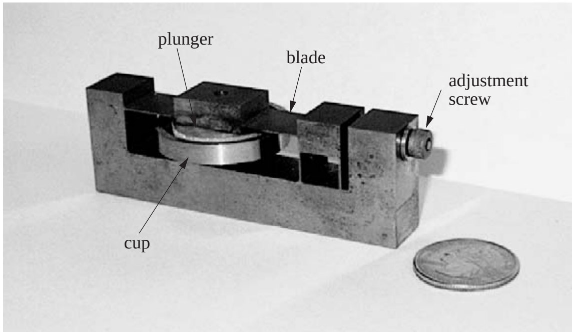

Fig. 9. Photograph of a flexure with adjustable stiffness and damping.

to the desired value by increasing the tension on the blade via the adjusting screw. The damping is produced by a viscous fluid that is sandwiched between the plunger and cup, and the damping coefficient is adjusted by moving the cup in the vertical direction.

In the experiment we hang the beam using latex tubing so as to approximate a free–free beam, and use an impact hammer and accelerometer to measure transfer functions. One typical transfer function (from force to acceleration at node 1 in Fig. 8) is shown in Fig. 10, where we plot the predicted and measured responses with and without the damper. As one would expect, the system initially exhibits almost no damping, with $\zeta \approx 10^{-4}$ for each of the first three modes. With the damper installed and properly tuned, each of the first two modes of the beam exhibit damping very close to that predicted.

6. Conclusions

Multi-degree-of-freedom (MDOF) tuned-mass-dampers (TMD) can be tuned to damp more than one mode of a primary system efficiently. In this paper, the problem of designing a MDOF TMD attached to a MDOF primary system is formulated as a decentralized static-output feedback problem. Then an ε-subgradient algorithm is presented that maximizes the minimum damping over a prescribed frequency range in order to obtain the optimal parameters of the MDOF TMD. In this approach, we can impose constraints on the ranges of the parameters and design for marginally stable and hysteretically damped systems directly.

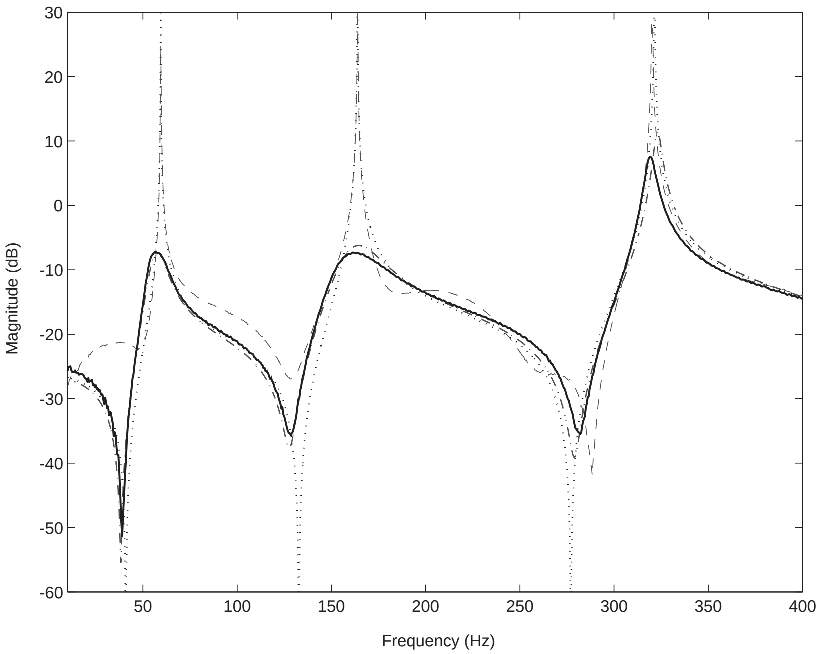

Fig. 10. Force-to-acceleration frequency response at Node 1 of the free–free beam: comparison of the predicted undamped response (dotted), measured undamped response (dashed), predicted damped response (dash–dot), and measured damped response (solid).

The minimax algorithm efficiently arrives at an optimum design for MDOF TMDs as well as multiple SDOF TMDs. It yields generally more damping than H2 or H_∞ optimizations, which produce designs with smaller responses for a specific set of disturbance inputs. The H2 and H_∞ optima feature some cancellation of the response of the various modes, whereas the minimax optimum simply maximizes the minimal damping. For the particular example of a free–free beam, we show that a MDOF TMD can be designed to provide higher damping in the first three flexural modes than can be attained by multiple SDOF TMDs at the same location and of the same total mass as the MDOF TMD. Finally, the experiment presented in this paper shows that it is practical to build a multi-DOF TMD that closely matches the ideal model.

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