Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 273 (2004) 175–184
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Multi-crack detection for beam by the natural frequencies

N.T. Khiem[a,*], T.V. Lien[b]

[a] Institute of Mechanics, National Centre for Natural Science and Technology of Vietnam, Vien Co Hoc, 264 Doi Cana, Hanoi, Viet Nam
[b] Hanoi University of Civil Engineering, 55, Giai Phong Road,Hanoi 264, Viet Nam

Received 20 January 2003; accepted 16 April 2003

Abstract

Multi-crack detection for beam by natural frequencies has been formulated in the form of a non-linear optimization problem, then solved by using the MATLAB functions. The spring model of crack is applied to establish the frequency equation based on the dynamic stiffness of multiple cracked beam. The equation is the basic instrument in solving the multi-crack detection of beam. The set of crack parameters to be detected includes not only the crack position and depth, but also the quantity of possible cracks. Numerical result obtained for a cantilever beam with single-, two- and three-cracks scenarios shows an efficiency and acceptability of the hereby proposed procedure.

1. Introduction

Crack is a damage that often occurs on members of structures and may cause serious failure of the structures. A crack must be detected in the early state. However, it is difficult to recognize a crack by using visual inspection techniques, and it may be detected usually by non-destructive techniques. In the last two decades, a lot of research efforts has been devoted to develop an effective approach for detecting crack in structures. The case of single crack detection in beam was studied in most publications by using the analytical model of a one-dimensional structure. The multi-crack detection problem in beams and frame structures is usually solved by using the finite element model (FEM) [1]. The FEM used in the crack detection problem proposes a model of crack as a uniform change of the element parameters, for instance, the modulus E or the geometry (A, I) or both as stiffness (EI). This approach is preferred in the application for large structures, but in fact it eliminates the significant effect of crack location (geometry of damage) in the

*Corresponding author. Tel.: +84-4-832-9705; fax: +84-4-8333-039.
E-mail address: ntkhiem@im01.ac.vn (N.T. Khiem).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00424-3

element. This fact has been notified by Banks et al. [2]. Obviously, the situation may be improved by refining the FE mesh. However, this, in consequence, makes the size of the FE model much larger in the end the FEM becomes ineffective in predicting the crack position and depth. The FEM can be utilized in the first stage of damage diagnosis, to determine zones where cracks may appear. For specification of crack location in elements the analytical model of the element is more useful. Thus, the idea that emerged involves the combining of the finite element approach with the analytical method for crack detection in large structures. The original aim was to develop the dynamic stiffness matrix (DSM) approach to crack detection in structures.

The DSM method was developed recently in studies [3–6], which among others, find the mention of Moon and Choi [6]. In the above-mentioned paper, the authors have presented an interesting comparison of natural frequencies computed by using the DSM method with frequencies obtained from the experiment. The error of computation compared with the experiment, as shown in Ref. [6], is about 1%. The DSM approach has been applied to study cracked beam [7,8] and, as shown in referred papers, the DSM method is effective not only in analysis but also in diagnosis of structures with cracks. In [9], the authors have used the DSM in multi-crack detection in beams by measured static displacement. The structural damage identification problem based on the DSM has been considered in general formulation by the authors of Ref. [10].

The focus of this paper is mainly on the problem of multi-crack detection for one-dimensional structures by natural frequencies. The procedure developed, hereby, is based on the DSM model of structures considered in Ref. [8] and *MatLab* functions in the optimization toolbox, so this can be extended to solve the problem of crack detection for more complete structures such as frames.

2. Formulation of the problem

A beam of length L, cross-section area $A = b \times h$, second moment of area I and Young's modulus E is considered. Suppose the beam has been cracked at a number of positions $x_1, \ldots, x_n$, where

$$x_0 = 0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec L = x_{n+1}.$$

The crack at x_j are modelled, as shown in Fig. 1, by rotational spring of stiffness $k_j = 1/\alpha_j$, where α_j is calculated by the formulas [11]

$$\alpha_j = \frac{6\pi(1 - v^2)h}{EI} I_c\left(\frac{a_j}{h}\right),$$

$$I_c(z) = 0.6272z^2 - 1.04533z^3 + 4.5948z^4 - 9.973z^5 + 20.2948z^6 - 33.0351z^7 + 47.1063z^8 - 40.7556z^9 + 19.6z^{10} \quad (1)$$

with the Poisson coefficient v, beam height h and crack depth a_j. Free vibration of the beam is described by the equation

$$\frac{d^4\Phi(x,\omega)}{dx^4} - \lambda^4\Phi(x,\omega) = 0, \quad (2)$$

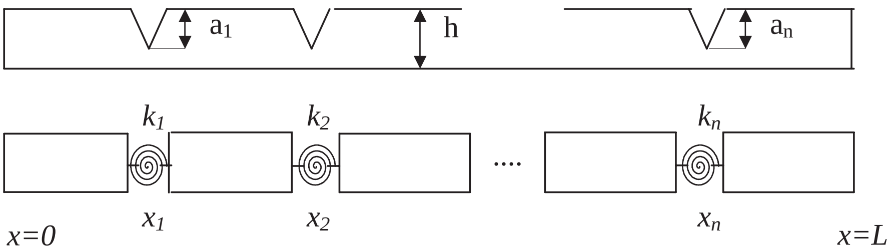

Fig. 1. Modelling of multiple cracked beam.

where $\lambda^4 = \omega^2 \rho A/(EI)$ or $\lambda = c_0\sqrt{\omega}$; $c_0 = [\rho A/(EI)]^{1/4}$. Eq. (2) must be investigated with the condition at the crack position x_j

$$\Phi(x_j - 0) = \Phi(x_j + 0), \quad \Phi''(x_j - 0) = \Phi''(x_j + 0), \quad \Phi'''(x_j - 0) = \Phi'''(x_j + 0),$$
$$\Phi'(x_j - 0) + \beta_j \Phi''(x_j - 0) = \Phi'(x_j + 0), \quad \beta_j = EI\alpha_j = \frac{EI}{k_j} \tag{3}$$

and the boundary conditions, which can be expressed in the form

$$\underline{B}_1^0 \begin{pmatrix} \Phi(+0) \\ \Phi'(+0) \end{pmatrix} + \underline{B}_2^0 \begin{pmatrix} EI\Phi'''(+0) \\ -EI\Phi''(+0) \end{pmatrix} = 0, \underline{B}_1^L \begin{pmatrix} \Phi(L-0) \\ \Phi'(L-0) \end{pmatrix} + \underline{B}_2^L \begin{pmatrix} -EI\Phi'''(L-0) \\ EI\Phi''(L-0) \end{pmatrix} = 0 \tag{4}$$

with the 2×2 dimension matrices $[B_j^{0,L}]$, $j = 1, 2$, of boundary parameters

$$\underline{B}_1^0 = \begin{bmatrix} B_{11}^0 & B_{12}^0 \\ B_{21}^0 & B_{22}^0 \end{bmatrix}, \quad \underline{B}_2^0 = \begin{bmatrix} B_{13}^0 & B_{14}^0 \\ B_{23}^0 & B_{24}^0 \end{bmatrix}, \quad \underline{B}_1^L = \begin{bmatrix} B_{11}^L & B_{12}^L \\ B_{21}^L & B_{22}^L \end{bmatrix}, \quad \underline{B}_2^L = \begin{bmatrix} B_{13}^L & B_{14}^L \\ B_{23}^L & B_{24}^L \end{bmatrix}.$$

Using the transfer matrix method, as described in Ref. [7], and with known boundary conditions, the frequency equation of the multiple cracked beams has the form

$$D(\omega, \bar{z}, \bar{\alpha}) = \det[A] = 0, \tag{5}$$

where $\bar{z} = \{x_1, ..., x_n\}^T$, $\bar{\alpha} = \{\alpha_1, ..., \alpha_n\}^T$ are the vectors of cracks position and magnitude, respectively, and A is a matrix of 4×4 dimension

$$A = \begin{bmatrix} B_0 \\ B_L Q \end{bmatrix} = \begin{bmatrix} B_{11}^0 & B_{12}^0 & B_{13}^0 & B_{14}^0 \\ B_{21}^0 & B_{22}^0 & B_{23}^0 & B_{24}^0 \\ \sum_i^4 B_{1j}^L Q_{j1} & \sum_i^4 B_{1j}^L Q_{j2} & \sum_i^4 B_{1j}^L Q_{j3} & \sum_i^4 B_{1j}^L Q_{j4} \\ \sum_i^4 B_{2j}^L Q_{j1} & \sum_i^4 B_{2j}^L Q_{j2} & \sum_i^4 B_{2j}^L Q_{j3} & \sum_i^4 B_{2j}^L Q_{j4} \end{bmatrix}.$$

In the last equation, Q_{jk}, j, $k = 1, 2, 3, 4$, are elements of the matrix

$$Q = T_{n+1} Q_n Q_{n-1} \dots Q_1 = T_{n+1} J_n T_n J_{n-1} \dots J_2 T_2 J_1 T_1,$$

where T_j, $j = 1, 2, \ldots, n+1$, have the form

$$T_j(\lambda, \ell_j) = \begin{bmatrix} K_1(\lambda\ell_j) & \lambda^{-1}K_2(\lambda\ell_j) & K_4(\lambda\ell_j)/EI\lambda^3 & -K_3(\lambda\ell_j)/EI\lambda^2 \\ \lambda K_4(\lambda\ell_j) & K_1(\lambda\ell_j) & K_3(\lambda\ell_j)/EI\lambda^2 & -K_2(\lambda\ell_j)/EI\lambda \\ -\lambda^3 EIK_2(\lambda\ell_j) & -\lambda^2 EIK_3(\lambda\ell_j) & -K_1(\lambda\ell_j) & \lambda K_4(\lambda\ell_j) \\ \lambda^2 EIK_3(\lambda\ell_j) & \lambda EIK_4(\lambda\ell_j) & \lambda^{-1}K_2(\lambda\ell_j) & -K_1(\lambda\ell_j) \end{bmatrix}$$

and

$$J_j = J(\alpha_j) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_j \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The notations $K_j(x)$, $j = 1, 2, 3, 4$, denote the functions

$$K_1(x) = \tfrac{1}{2}(\cosh x + \cos x), \quad K_3(x) = \tfrac{1}{2}(\cosh x - \cos x),$$
$$K_2(x) = \tfrac{1}{2}(\sinh x + \sin x), \quad K_4(x) = \tfrac{1}{2}(\sinh x - \sin x)$$

and $\ell_j = x_j - x_{j-1}$, $j = 1, \ldots, n+1$. With given crack and boundary parameters, solution of Eq. (5) yields the natural frequencies of multiple cracked beams as $\omega_j = \omega_j(\bar{z}, \bar{\alpha})$, $j = 1, 2, 3, \ldots$. The natural frequencies depending on the crack and boundary parameters were studied in Ref. [7]. In the present paper, Eq. (5) will be used basically to determine the crack parameters with given boundary conditions and natural frequencies. Let the known form of measurement of natural frequencies be denoted by a vector $\bar{\omega}^* = \{\omega_1^*, \ldots, \omega_m^*\}^T$. The problem now is to find (1) the number n of cracks that have possibly occurred in the beam, (2) the crack positions $\bar{z} = \{x_1, \ldots, x_n\}^T$, which are constrained by condition $0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec L$ and (3) the corresponding crack depths $\bar{a} = \{a_1, \ldots, a_n\}^T$ related to the crack magnitude α_j by Eq. (1). The last crack parameters in combination make up a vector of unknowns $\bar{y} = \{y_1, \ldots, y_{2n}\}^T = \{x_1, \ldots, x_n, a_1, \ldots, a_n\}^T$ for the crack detection problem. The number of cracks is also an unknown, but it is not included in the unknown vector because this parameter will be determined separately. Input for the problem is the measured natural frequencies $\bar{\omega}^* = \{\omega_1^*, \ldots, \omega_m^*\}^T$. Because of the measurements and modelling error, the multi-crack detection problem must be formulated in the form of a constrained non-linear optimization problem. First, due to relationship (1), one will have

$$\alpha_j = \frac{6\pi(1 - \nu^2)h}{EI} I_c(a_j/h) \equiv g(a_j) \equiv g_j(\bar{a}), \quad j = 1, \ldots, n.$$

Therefore, Eq. (5) can be rewritten as

$$D(\omega, \bar{z}, \bar{\alpha}) = D(\omega, \bar{z}, \bar{g}(\bar{a})) \equiv \hat{D}(\omega, \bar{z}, \bar{a}) = 0.$$

Suppose that m first solutions of the last equation with respect to ω are $\omega_1(\bar{z}, \bar{a}), \ldots, \omega_m(\bar{z}, \bar{a})$, i.e.,

$$\hat{D}\left[\omega_j(\bar{z}, \bar{a}), \bar{z}, \bar{a}\right] = 0, \quad j = 1, \ldots, m. \tag{6}$$

Now, the objective function may be defined by

$$f(\bar{y}) = \sum_{j=1}^{m} [\omega_j(\bar{z}, \bar{a}) - \omega_j^*]^2. \tag{7}$$

The inequality constraints can be expressed in the form

$$G_j(y_1, ..., y_n) = y_j - y_{j+1} \leqslant 0, \quad j = 1, ..., n-1. \tag{8}$$

The bounded domain for arguments is defined by $0 \leqslant y_j \leqslant L;\ 0 \leqslant y_{n+j} \leqslant h,\ j = 1, ..., n$. Thus, the non-linear optimization problem can be established in the form

$$\begin{aligned} &f(y_1, ..., y_{2n}) \Rightarrow \min, \\ &G_j(y_1, ..., y_n) \leqslant 0, \quad j = 1, 2, ..., n, \\ &0 \leqslant y_j \leqslant L, \quad 0 \leqslant y_{n+j} \leqslant h, \quad j = 1, 2, ..., n, \end{aligned} \tag{9}$$

where function f and G_j are defined in Eqs. (6)–(8). This is really standard constrained non-linear optimization problem of dimension $2n$. Here, the objective function is defined in the numerical form obtained each time by solving Eq. (6) regarding ω.

3. Method for solution

Solution of the multi-crack detection problem, thus, leads to determining a number of cracks n and then solving problem (9) with respect to the crack positions and depths. The procedure for solution of the problem can be formulated as follows:

Step 1: Suppose that there are n cracks that have possibly appeared in the beam. So, problem (9) would be available with m measured frequencies $\bar{\omega}^* = \{\omega_1^*, ..., \omega_m^*\}^T$.

Step 2: Using the MATLAB function *fmincon* to solve problem (9). In the process of solving problem (9), m solutions of Eq. (6) are required for each iteration. Let the solution of problem (9) be $x_1^{(n)}, ..., x_n^{(n)},\ a_1^{(n)}, ..., a_n^{(n)}$.

Step 3: The number of cracks is determined as follows: If among the $a_j^{(n)}, j = 1, ..., n$, there is no value close to zero and all $x_j^{(n)}, j = 1, ..., n$, are clearly separated, then seeking number of cracks N must be not less than n. In this case one has to increase the n and go to step 1. This procedure is continued until recognition of a close to zero crack depth or some crack positions close to eachother or to the free end. In that case, the number of cracks N would be chosen as the minimum among the number of non-zero crack depths and the number of separate crack positions.

Step 4: The different crack positions and corresponding depths recognized in Step 3 will be accepted as a solution of the multi-crack detection problem.

Of course, the number of cracks N suggested depends mainly on the number of frequencies measured, m. The assumed number of cracks should not exceed the number of measured frequencies because proposed number N leads to $2N$ unknowns to be determined. The great dimension compared to the quantity of input data may result in a poor solution of the problem. Evaluation of the number of cracks, as described above, requires to detail the conception "close to zero" of the estimated crack depths. To the author's knowledge, crack depth less than 10% of beam height is usually difficult to detect by natural frequencies. Therefore if detected, crack depth less than 10% of beam height is proposed to be "close to zero". The proposed procedure will be illustrated and validated by a case study presented in the next section.

4. Numerical results

Numerical results presented in this section have been obtained in consideration of a cantilever beam with the following parameter: beam length $L = 0.8\,\text{m}$; cross-section area $A = b \times h = 0.01 \times 0.02$ (m^2); Young's modulus $E = 2.1 \times 10^{11}$ (N/m^2), the Poisson coefficient $\nu = 0.25$ and material density $\rho = 7800\,(\text{kg/m}^3)$. The so-called measured frequencies are obtained not experimentally but from a simulation using the random number ε uniformly distributed in the interval $[-0.5, +0.5]$. Thus, pseudo-measured frequencies are taken as

$$\omega_j^* = \omega_j^c + err.\varepsilon_j, \quad j = 1, 2, ..., m, \tag{10}$$

Table 1
Solution of the optimization problem in the case of single crack

Proposed		Measurement error							
n	m	$err = 0\%$		$err = 2\%$		$err = 5\%$		$err = 7\%$	
		x	a	x	a	x	a	x	a
1	1	0.2700 (10%)	0.0062 (3.3%)	0.2650 (12%)	0.0061 (2%)	0.2494 (17%)	0.0059 (2%)	0.2760 (8%)	0.0068 (13%)
	2	0.3001 (0%)	0.0060 (0%)	0.3095 (3%)	0.0064 (7%)	0.3187 (6%)	0.0083 (38%)	0.2895 (4%)	0.0090 (50%)
2	1	0.2585 (14%)	0.0053 (12%)	0.2750 (8%)	0.0053 (12%)	0.2487 (17%)	0.0054 (10%)	0.2563 (15%)	0.0053 (12%)
		0.4069 (U)	0.0000 (U)	0.4415 (U)	0.0000 (U)	0.4998 (U)	0.0003 (U)	0.5074 (U)	0.0004 (U)
	2	0.3000 (0%)	0.0060 (0%)	0.3498 (17%)	0.0067 (12%)	0.2673 (11%)	0.0070 (17%)	0.2717 (9%)	0.0090 (50%)
		0.7512 (U)	0.0003 (U)			0.7902 (U)	0.0010 (U)	0.7970 (B)	0.0032 (B)
	3	0.3000 (0%)	0.0060 (0%)	0.2351 (22%)	0.0077 (28%)	0.2342 (22%)	0.0086 (43%)	0.2107 (30%)	0.0112 (87%)
		0.5068 (U)	0.0003 (U)	0.4102 (U)	0.0001 (U)	0.4201 (U)	0.0001 (U)	0.4003 (U)	0.0000 (U)
	4	0.2663 (U)	0.00 (U)	0.2624 (13%)	0.0059 (2%)	0.3237 (8%)	0.0081 (35%)	0.3531 (18%)	0.0075 (25%)
		0.3000 (0%)	0.0060 (0%)	0.5418 (U)	0.0002 (U)	0.4336 (U)	0.0003 (U)	0.4336 (U)	0.0003 (U)
3	1	0.2482 (17%)	0.0047 (22%)	0.2220 (26%)	0.0037 (38%)	0.2438 (19%)	0.0104 (73%)	0.2199 (27%)	0.0110 (83%)
		0.4500 (U)	0.0 (U)	0.4422 (U)	0.0 (U)	0.4523 (U)	0.0 (U)	0.4522 (U)	0.0 (U)
		0.7000 (U)	0.0000 (U)	0.7018 (U)	0.0000 (U)	0.7015 (U)	0.0000 (U)	0.7025 (U)	0.0000 (U)
	2	0.2970 (1%)	0.0059 (2%)	0.2982 (1%)	0.0090 (50%)	0.1732 (42%)	0.0061 (2%)	0.3219 (7%)	0.0117 (95%)
		0.6694 (U)	0.0006 (U)	0.5077 (U)	0.0 (U)	0.4535 (U)	0.0 (U)	0.4938 (U)	0.0 (U)
		0.7383 (U)	0.0004 (U)	0.5494 (U)	0.0002 (U)	0.7000 (U)	0.0001 (U)	0.7125 (U)	0.0000 (U)
	3	0.3000 (0%)	0.0060 (0%)	0.2515 (16%)	0.0054 (10%)	0.4790 (60%)	0.0076 (27%)	0.2407 (20%)	0.0096 (60%)
		0.3907 (U)	0.0 (U)	0.4500 (U)	0.0 (U)	0.5380 (U)	0.0 (U)	0.4500 (U)	0.0 (U)
		0.6083 (U)	0.0003 (U)	0.6000 (U)	0.0000 (U)	0.7689 (U)	0.0005 (U)	0.6000 (U)	0.0000 (U)
	4	0.2999 (0%)	0.0059 (2%)	0.1850 (U)	0.0 (U)	0.3490 (16%)	0.0056 (7%)	0.2042 (32%)	0.0093 (55%)
		0.3034 (R)	0.0059 (R)	0.2453 (18%)	0.0052 (13%)	0.3949 (U)	0.0 (U)	0.3915 (U)	0.0 (U)
		0.5220 (U)	0.0002 (U)	0.5752 (U)	0.0001(U)	0.5161 (U)	0.0000 (U)	0.6774 (U)	0.0000 (U)

x—crack position, a—crack depth.
Notation in brackets: R—crack detected repeatedly; U—uncracked (zero depth); B—detected crack closed to boundary; the numeral with percent-error of detection.

where *err* denotes a measurement error rate in percent, ε_j are the random numbers generated by the computer and ω_j^c are calculated frequencies with a simulated scenario of cracks. In this case study, the measurement error rate *err* is running through 0%, 2%, 5%, 7% and three scenarios have been thought of a cantilever with one, two and three cracks. The crack positions and depths resulting from solving the optimization problem for each sample of the random numbers $\{\varepsilon_j, j = 1, ..., m\}$ are averaged to be their mean values. The effect of the quantity of measured frequencies on the accuracy of crack detection is also considered in this example. So, the numerical solutions of the optimization problem obtained in the scenarios are presented in Tables 1–3. Numerals in brackets are percentages of error of detection and the letters in brackets imply

Table 2
Solution of the optimization problem in the case of two cracks

Proposed		Measurement error							
n	*m*	*err*=0%		*err*=2%		*err*=5%		*err*=7%	
		x	*a*	*x*	*a*	*x*	*a*	*x*	*a*
2	4	0.2958 (1%)	0.0059 (2%)	0.2860 (5%)	0.0075 (25%)	0.2821 (6%)	0.0085 (42%)	0.2250 (25%)	0.0111 (85%)
		0.4966 (1%)	0.0061 (2%)	0.5121 (2%)	0.0036 (40%)	0.5164 (3%)	0.0080 (33%)	0.5854 (17%)	0.0054 (10%)
	3	0.3153 (5%)	0.0063 (5%)	0.3032 (1%)	0.0064 (7%)	0.3335 (11%)	0.0063 (5%)	0.1368 (54%)	0.0045 (25%)
		0.5056 (1%)	0.0065 (8%)	0.5199 (4%)	0.0043 (28%)	0.4530 (9%)	0.0072 (20%)	0.4050 (19%)	0.0054 (10%)
	2	0.2807 (6%)	0.0057 (5%)	0.3964 (32%)	0.0045 (25%)	0.2160 (28%)	0.0078 (30%)	0.1732 (42%)	0.0067 (12%)
		0.5500 (10%)	0.0075 (25%)	0.5497 (10%)	0.0072 (20%)	0.5413 (8%)	0.0080 (33%)	0.5573 (11%)	0.0080 (33%)
	1	0.2518 (16%)	0.0053 (12%)	0.2543 (15%)	0.0048 (20%)	0.2504 (17%)	0.0073 (22%)	0.2855 (5%)	0.0084 (40%)
		0.5498 (10%)	0.0072 (20%)	0.5537 (11%)	0.0079 (32%)	0.5520 (10%)	0.0043 (28%)	0.4883 (2%)	0.0109 (82%)
3	5	0.2998 (0%)	0.0060 (0%)	0.3239 (8%)	0.0081 (35%)	0.2771 (8%)	0.0071 (18%)	0.3951 (32%)	0.0092 (53%)
		0.4998 (0%)	0.0060 (0%)	0.5235 (5%)	0.0035 (42%)	0.5220 (4%)	0.0085 (42%)	0.6935 (39%)	0.0080 (33%)
		0.6742 (U)	0.0001 (U)	0.7995 (B)	0.0123 (B)	0.7980 (B)	0.0041 (B)	0.7763 (U)	0.0000 (U)
	4	0.2976 (1%)	0.0059 (2%)	0.3279 (9%)	0.0039 (35%)	0.2067 (31%)	0.0071 (18%)	0.2733 (9%)	0.0101 (68%)
		0.4980 (0%)	0.0061 (2%)	0.4860 (3%)	0.0066 (10%)	0.6329 (27%)	0.0028 (53%)	0.5235 (5%)	0.0065 (8%)
		0.7929 (B)	0.0057 (B)	0.7906 (B)	0.0001 (B%)	0.7817 (B)	0.0160 (B)	0.7030 (U)	0.0000 (U)
	3	0.2152 (28%)	0.0045 (25%)	0.3108 (4%)	0.0085 (42%)	0.2429 (19%)	0.0101 (68%)	0.4183 (39%)	0.0042 (30%)
		0.4903 (0%)	0.0072 (20%)	0.5582 (12%)	0.0058 (3%)	0.4304 (U)	0.0001 (U)	0.5189 (R)	0.0092 (R)
		0.7877 (B)	0.0188 (B)	0.7999 (B)	0.0097 (B)	0.6585 (32%)	0.0013 (78%)	0.5190 (4%)	0.0069 (15%)
4	5	0.2998 (R)	0.0060 (R)	0.3083 (3%)	0.0085 (42%)	0.1490 (50%)	0.0062 (3%)	0.2942 (2%)	0.0107 (78%)
		0.2999 (0%)	0.0060 (0%)	0.4835 (3%)	0.0040 (33%)	0.5761 (15%)	0.0066 (10%)	0.5758 (15%)	0.0183 (205%)
		0.3003 (R)	0.0060 (R)	0.7967 (B)	0.0078 (B)	0.7915 (U)	0.0009 (U)	0.7941 (B)	0.0000 (B)
		0.5000 (0%)	0.0060 (0%)	0.7967(B)	0.0074 (B)	0.7993 (B)	0.0128 (B)	0.7941 (B)	0.0200 (B)
	4	0.2981 (1%)	0.0059 (2%)	0.2602 (13%)	0.0042 (30%)	0.2812 (6%)	0.0069 (15%)	0.2795 (7%)	0.0037 (38%)
		0.4983 (0%)	0.0060 (0%)	0.5015 (0%)	0.0078 (30%)	0.5180 (4%)	0.0076 (27%)	0.5173 (3%)	0.0085 (42%)
		0.6475 (U)	0.0002 (U)	0.7802 (B)	0.0063 (B)	0.7988 (B)	0.0000 (B)	0.7546 (U)	0.0000 (U)
		0.7941 (B)	0.0100 (B)	0.8000 (B)	0.0200 (B)	0.7990 (B)	0.0140 (B)	0.7912 (B)	0.0000 (U)
	3	0.3172 (6%)	0.0050 (17%)	0.2629 (12%)	0.0045 (25%)	0.4404 (47%)	0.0022 (63%)	0.5096 (70%)	0.0098 (63%)
		0.5164 (3%)	0.0066 (10%)	0.4476 (10%)	0.0069 (15%)	0.4413 (R)	0.0093 (R)	0.6844 (37%)	0.0060 (0%)
		0.8000 (B)	0.0178 (B)	0.7915 (B)	0.0067 (B)	0.7893 (R)	0.0139 (R)	0.6844 (U)	0.0002 (U)
		0.8000 (B)	0.0193 (B)	0.7915 (B)	0.0096 (B)	0.7893 (58%)	0.0066 (10%)	0.6844 (R)	0.0059 (R)

x—crack position, *a*—crack depth.
Actual cracks at positions $x^* = 0.3, 0.5$ m, with depths $a^* = 0.006, 0.006$ m; *n* is the number of cracks assumed and *m* is the number of frequencies measured (notation in brackets: R—crack detected repeatedly; U—uncracked (zero depth); B—detected crack closed to boundary; the numerals—error of detection in percent).

Table 3
Solution of the optimization problem in the case of three crack

Proposed		Measurement error (*err*)							
n	*m*	*err* = 0%		*err* = 2%		*err* = 5%		*err* = 7%	
		x	*a*	*x*	*a*	*x*	*a*	*x*	*a*
3	6	0.2000 (0%)	0.0070 (0%)	0.1926 (4%)	0.0063 (10%)	0.1701 (15%)	0.0055 (21%)	0.3705 (85%)	0.0066 (6%)
		0.4001 (0%)	0.0060 (0%)	0.3640 (9%)	0.0078 (30%)	0.4192 (5%)	0.0053 (12%)	0.5324 (6%)	0.0100 (67%)
		0.5000 (0%)	0.0050 (0%)	0.5983 (20%)	0.0044 (12%)	0.5283 (6%)	0.0092 (84%)	0.7481 (50%)	0.0076 (52%)
	5	0.2006 (0%)	0.0070 (0%)	0.2153 (8%)	0.0057 (19%)	0.1388 (31%)	0.0072 (3%)	0.4367 (9%)	0.0101 (44%)
		0.3957 (1%)	0.0060 (0%)	0.2571 (F)	0.0067 (F)	0.4774 (19%)	0.0055 (8%)	0.7293 (46%)	0.0108 (80%)
		0.5003 (0%)	0.0050 (0%)	0.3576 (11%)	0.0059 (11%)	0.5892 (18%)	0.0074 (48%)	0.7954 (B)	0.0059 (B)
	4	0.1792 (10%)	0.0066 (6%)	0.1854 (7%)	0.0062 (11%)	0.1385 (31%)	0.0035 (50%)	0.0838 (58%)	0.0043 (39%)
		0.3866 (3%)	0.0059 (2%)	0.3826 (4%)	0.0054 (10%)	0.3935 (2%)	0.0069 (15%)	0.4252 (6%)	0.0041 (32%)
		0.5125 (3%)	0.0057 (14%)	0.4563 (9%)	0.0073 (46%)	0.5773 (15%)	0.0083 (66%)	0.4253 (R)	0.0075 (R)
	3	0.1497 (25%)	0.0057 (19%)	0.1498 (25%)	0.0034 (51%)	0.1541 (23%)	0.0033 (53%)	0.1375 (31%)	0.0061 (13%)
		0.3391 (15%)	0.0067 (12%)	0.3817 (5%)	0.0068 (13%)	0.4179 (4%)	0.0082 (37%)	0.2976 (26%)	0.0073 (22%)
		0.5362 (7%)	0.0056 (12%)	0.4733 (5%)	0.0060 (20%)	0.5989 (20%)	0.0070 (40%)	0.5796 (16%)	0.0068 (36%)
4	6	0.1985 (1%)	0.0070 (0%)	0.1720 (14%)	0.0056 (20%)	0.1876 (6%)	0.0058 (17%)	0.2290 (15%)	0.0124 (77%)
		0.4134 (3%)	0.0061 (2%)	0.3975 (1%)	0.0047 (22%)	0.3586 (10%)	0.0069 (15%)	0.3943 (1%)	0.0055 (8%)
		0.5044 (1%)	0.0048 (4%)	0.5145 (3%)	0.0025 (50%)	0.6061 (21%)	0.0020 (60%)	0.5384 (U)	0.0000 (U)
		0.7637 (U)	0.0003 (U)	0.7468 (U)	0.0007 (U)	0.7999 (B)	0.0036 (B)	0.7970 (B)	0.0184 (B)
	5	0.2109 (5%)	0.0073 (4%)	0.2382 (19%)	0.0062 (11%)	0.2468 (23%)	0.0029 (59%)	0.0926 (54%)	0.0069 (1%)
		0.4407 (10%)	0.0072 (20%)	0.4034 (1%)	0.0080 (33%)	0.2468 (R)	0.0095 (R)	0.3962 (1%)	0.0098 (63%)
		0.6925 (39%)	0.0082 (64%)	0.5538 (11%)	0.0015 (70%)	0.5528 (11%)	0.0052 (13%)	0.6067 (21%)	0.0034 (32%)
		0.7582 (U)	0.0000 (U)	0.6820 (U)	0.0000 (U)	0.7245 (45%)	0.0071 (42%)	0.7969 (B)	0.0038 (B)
	4	0.1914 (2%)	0.0065 (7%)	0.1799 (10%)	0.0047 (33%)	0.2632 (32%)	0.0091 (30%)	0.1738 (13%)	0.0086 (23%)
		0.3475 (13%)	0.0061 (2%)	0.3522 (12%)	0.0053 (12%)	0.3497 (13%)	0.0059 (2%)	0.4349 (9%)	0.0032 (47%)
		0.4805 (4%)	0.0054 (8%)	0.4667 (7%)	0.0050 (0%)	0.3986 (20%)	0.0062 (24%)	0.4899 (U)	0.0007 (U)
		0.7169 (U)	0.0006 (U)	0.7293 (U)	0.0008 (U)	0.7017 (U)	0.0009 (U)	0.7308 (46%)	0.0076 (52%)
	3	0.1599 (20%)	0.0058 (17%)	0.1651 (17%)	0.0080 (14%)	0.1144 (43%)	0.0076 (9%)	0.0036 (U)	0.0000 (U)
		0.3114 (22%)	0.0057 (5%)	0.3263 (18%)	0.0031 (48%)	0.3323 (17%)	0.0074 (23%)	0.5565 (11%)	0.0077 (10%)
		0.4818 (4%)	0.0063 (26%)	0.5533 (11%)	0.0055 (10%)	0.4366 (13%)	0.0059 (18%)	0.7788 (F)	0.0108 (F)
		0.7291 (U)	0.0006 (U)	0.7182 (U)	0.0007 (U)	0.7146 (U)	0.0006 (U)	0.7846 (57%)	0.0067 (34%)
5	6	0.1888 (6%)	0.0066 (6%)	0.2033 (2%)	0.0044 (37%)	0.0936 (U)	0.0000 (U)	0.0829 (U)	0.0008 (U)
		0.4299 (7%)	0.0064 (7%)	0.4021 (1%)	0.0032 (47%)	0.4036 (102%)	0.0057 (18%)	0.3791 (90%)	0.0033 (53%)
		0.5396 (8%)	0.0047 (6%)	0.5970 (19%)	0.0075 (50%)	0.5749 (15%)	0.0098 (63%)	0.5298 (6%)	0.0089 (78%)
		0.6518 (U)	0.0000 (U)	0.6664 (U)	0.0000 (U)	0.6890 (U)	0.0000 (U)	0.6138 (23%)	0.0059 (18%)
		0.7994 (B)	0.0079 (B)	0.7996 (B)	0.0081 (B)	0.7859 (57%)	0.0200 (300%)	0.7962 (B)	0.0055 (B)
	5	0.2055 (3%)	0.0067 (4%)	0.1804 (10%)	0.0070 (0%)	0.1988 (1%)	0.0034 (51%)	0.2167 (8%)	0.0065 (7%)
		0.3684 (8%)	0.0067 (12%)	0.3308 (17%)	0.0022 (63%)	0.4149 (4%)	0.0064 (7%)	0.3993 (0%)	0.0071 (18%)
		0.5115 (U)	0.0007 (U)	0.4269 (15%)	0.0054 (8%)	0.5757 (15%)	0.0069 (38%)	0.3993 (R)	0.0083 (R)
		0.5195 (4%)	0.0043 (14%)	0.5512 (U)	0.0000 (U)	0.6857 (U)	0.0008 (U)	0.5828 (5%)	0.0037 (26%)
		0.7996 (B)	0.0049 (B)	0.7998 (B)	0.0136 (B)	0.7935 (B)	0.0098 (B)	0.6813 (U)	0.0000 (U)
	4	0.1359 (U)	0.0005 (U)	0.1871 (U)	0.0003 (U)	0.3238 (U)	0.0000 (U)	0.1521 (U)	0.0000 (U)
		0.3402 (70%)	0.0066 (6%)	0.3425 (71%)	0.0073 (4%)	0.5194 (4%)	0.0030 (50%)	0.2421 (21%)	0.0033 (53%)
		0.5120 (2%)	0.0044 (27%)	0.4475 (12%)	0.0052 (13%)	0.7119 (R)	0.0001 (R)	0.6052 (21%)	0.0096 (92%)
		0.6422 (28%)	0.0052 (4%)	0.6307 (26%)	0.0055 (10%)	0.7119 (42%)	0.0128 (113%)	0.7966 (B)	0.0119 (B)
		0.7983 (B)	0.0064 (B)	0.6915 (U)	0.0003 (U)	0.7734 (55%)	0.0169 (238%)	0.7969 (B)	0.0149 (B)

x—crack position, *a*—crack depth. *n*—number of cracks detected.
Actual cracks at positions x^*=0.2, 0.4, 0.5 m with depths $a^* = 0.007, 0.006, 0.005$ m, *n* is number of cracks assumed and *m* is the number of frequencies measured. Percentage-error of detection given in brackets, except the letters implying the following: R—crack detected repeatedly; U—uncracked (zero depth); B—detected crack closed to boundary; F—false detection.

false detections (F), repeated result of detection (R), cracks of zero depth detected (U) and the position of the detected crack close to the free end of the beam (B). Looking at Tables 1–3, it is interesting to note that the number of cracks is more correctly detected in comparison with the results of the crack position and depth detection, except in one case, where the measurement error reaches 7%. The incorrect detection of the number of cracks caused from the lack of measured frequencies compared with the number of unknowns is to be determined. The crack depth has been detected with less accuracy than the crack position. The tables show also the well-known fact that the more frequencies are measured the more exactly can crack positions be detected. In more detail, in order to correctly detect a single crack, it must be measured at least at two frequencies and the number of measured frequencies must be twice more than the number of cracks assumed. Of course, the measurement error decreases the accuracy of crack detection.

5. Conclusion

In the present paper, the DSM model has been successfully applied to detect numerous cracks in beams by natural frequencies. Especially, the problem of detecting the quantity of cracks is set-up and solved initially. The most stable result of the detection presented above is the number of cracks. The crack position diagnostics gives more accurate results in comparison with that of crack depth detection. The more natural frequencies are measured the more accurate is the crack position detected. But this fact does not come to the crack depth detection. The obtained results show that the procedure developed here works effectively only for measurement errors not exceeding 7%, but it is possible to extend it entirely for solving the multi-crack detection in more complete structures.

Acknowledgements

The authors are sincerely thankful to the Vietnam National Council in Natural Science for final support in completing this study.

References

[1] R. Ruotolo, C. Surace, Damage assessment of multiple cracked beams: numerical results and experimental validation, *Journal of Sound and Vibration* 206 (4) (1997) 567–588.

[2] H.T. Banks, D.J. Inman, D.J. Leo, Y. Wang, An experimentally validated damage detection theory in smart structures, *Journal of Sound and Vibration* 191 (5) (1996) 859–880.

[3] A.Y.T. Leung, Dynamic stiffness and response analysis, *Dynamics and Stability of System* 2 (1987) 125–137.

[4] A.Y.T. Leung, *Dynamic Stiffness and Substructures*, Springer, London, 1993, p. 242.

[5] A.Y.T. Leung, Dynamic stiffness for structures with distributed deterministic or random loads, *Journal of Sound and Vibration* 242 (3) (2001) 377–395.

[6] D.H. Moon, M.S. Choi, Vibration analysis for frame structures using transfer of dynamic stiffness coefficient, *Journal of Sound and Vibration* 234 (5) (2000) 725–736.

[7] N.T. Khiem, T.V. Lien, A simplified method for natural frequency analysis of multiple cracked beam, *Journal of Sound and Vibration* 245 (4) (2001) 737–751.

[8] N.T. Khiem, T.V. Lien, The dynamic stiffness matrix method in forced vibration analysis of multiple cracked beam, *Journal of Sound and Vibration* 254 (3) (2002) 541–555.
[9] T.V. Lien, N.T. Khiem, Static diagnosis of multiple cracked beam, *Vietnam Journal of Mechanics, NCNS&T of Vietnam* 23 (4) (2002) 224–233.
[10] U. Lee, J. Shin, A frequency–domain method of structural damage identification formulated from the dynamic stiffness equation of motion, *Journal of Sound and Vibration* 257 (4) (2002) 615–634.
[11] T.G. Chondros, A.D. Dimarogonas, J. Yao, A continuous cracked beam theory, *Journal of Sound and Vibration* 215 (1) (1998) 17–34.