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On an iterative general-order perturbation method for multiple structural damage detection

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Abstract

A general order perturbation method involving multiple perturbation parameters is developed for eigenvalue problems with changes in the stiffness parameters. The perturbation solutions and eigenparameter sensitivities of all orders are derived explicitly. The perturbation method is used iteratively in conjunction with an optimization method to identify the stiffness parameters of structures. The generalized inverse method is used efficiently with the first order perturbations, and the gradient and quasi-Newton methods are used with the higher order perturbations. Numerical simulations on discrete and continuous structural systems demonstrated the robustness of the algorithm in detecting the locations and extent of small to large levels of damage. The effects of measurement noise and reduced measurements on the performance of the algorithm are evaluated.

1. Introduction

Structural damage detection using changes in vibration characteristics has received much attention in recent years and a summary overview of the subject is provided in Ref. [1]. The problem is closely related to that of updating a mathematical model using test data, and can be classified as an inverse minimization problem. The methods available in the literature can be broadly divided into three categories: direct methods [2–5], iterative methods [6], and control-based eigenstructure assignment methods [7,8]. The direct methods, such as the optimal matrix updating algorithms [2–5], identify the damage locations and extent in a single iteration. Based on first order sensitivity analysis, Lin et al. [6] determined iteratively the modified structural parameters by minimizing the differences between the model and test data. In addition to inverse

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modelling, damage detection studies need to address practical issues including measurement noise and reduced measurements. The latter results from experimental measurement of a lesser number of degrees of freedom than that of the analytical model [9]. Model reduction [10,11] and eigenvector expansion [9,12] techniques have been used to handle the incomplete measurement problem.

Various sensitivity analyses were developed over the past few decades. Fox and Kapoor [13] derived the rates of change of eigenparameters with respect to structural design parameters. Rogers [14] extended their work to non-symmetric eigenvalue problems. Nelson [15] presented a simplified procedure for calculating the eigenvector derivatives for both symmetric and non-symmetric eigensystems. Lin et al. [6] applied sensitivity analysis to frequency response functions. Wanxie and Gengdong [16] applied the stationarity of Rayleigh's quotient to the second order sensitivity analysis of multimodal eigenvalues. Wicher and Nalecy [17] determined the second order sensitivity matrix of structural systems in the frequency domain.

Wilkinson [18] first developed the perturbation theory for eigenvalue problems. Brandon [19] calculated the second order sensitivities of eigenvalues and eigenvectors using perturbation analysis. Ryland and Meirovitch [20] developed a second order perturbation method to calculate the changes of eigenparameters with small changes in the mass and stiffness matrices. Kan and Chopra [21] and Tsicnias and Hutchinson [22] derived the second order perturbation solutions for a torsionally coupled building. While most analyses involve perturbation of a single parameter, Stahara [23] presented a first order, multiple-parameter perturbation procedure for designing optimized profiles of turbomachinery blades. A higher order, multiple-parameter perturbation procedure involving cross-product terms has not been available in the literature.

In this work a multiple-parameter, general order perturbation method is developed. The changes in the stiffness parameters are used as the perturbation parameters. By equating the coefficients of like order terms involving the same perturbation parameters in the normalization relations of eigenvectors and the eigenvalue problem, the perturbation solutions of all orders are derived. The sensitivities of eigenparameters of all orders are obtained. The perturbation method is used in an iterative manner with an optimization method to identify the stiffness parameters of structures. Extensive results on a serial mass–spring system and a fixed–fixed beam illustrated the robustness of the algorithm. Simulated noise and incomplete eigenvector measurements are included in the beam examples.

2. Methodology

The method presented below can simultaneously identify the unknown stiffness parameters and is formulated as a damage detection problem. Since the effects of the changes in the inertial properties of a damaged structure are usually relatively small, only the changes in the stiffness properties due to structural damage are considered.

Consider an N-degree-of-freedom, linear, time-invariant, self-adjoint system with distinct eigenvalues. The stiffness parameters of the undamaged structure are denoted by G_{hi} ($i = 1, 2, \ldots, m$), where m is the number of the stiffness parameters. Structural damage is characterized by reductions in the stiffness parameters. The estimated stiffness parameters of the damaged structure before each iteration are denoted by G_i ($i = 1, 2, \ldots, m$), and its stiffness matrix, which depends linearly on G_i, is denoted by $\mathbf{K} = \mathbf{K}(\mathbf{G})$, where $\mathbf{G} = [G_1, G_2, \ldots, G_m]^{\mathrm{T}}$. Here the superscript T denotes matrix transpose. The eigenvalue problem of the structure with stiffness parameters G_i is

$$\mathbf{K}\phi^k = \lambda^k \mathbf{M}\phi^k, \tag{1}$$

where $\mathbf{M}$ is the constant mass matrix, and $\lambda^k = \lambda^k(\mathbf{G})$ and $\phi^k = \phi^k(\mathbf{G})$ $(k = 1, 2, \ldots, N)$ are the kth eigenvalue and mass-normalized eigenvector, respectively. It is noted that $\lambda_k = \omega_k^2$, where ω_k is the kth natural frequency of the structure. The normalized eigenvectors of Eq. (1) satisfy the orthonormality relations

$$(\phi^k)^{\mathrm{T}}\mathbf{M}\phi^u = \delta_{ku}, \quad (\phi^k)^{\mathrm{T}}\mathbf{K}\phi^u = \lambda^k\delta_{ku}, \tag{2}$$

where $1 \leqslant u \leqslant N$ and δ_{ku} is the Kronecker delta. Before the first iteration, the initial stiffness parameters of the damaged structure are assumed to be $G_i^{(0)} = \sigma_i G_{hi}$ $(i = 1, 2, \ldots, m)$, where $0 < \sigma_i \leqslant 1$, and the eigenvalue problem (1) corresponds to that of the structure with stiffness parameters $G_i^{(0)}$. If there is no prior knowledge of the integrity of the structure, one can start the iteration from the stiffness parameters of the undamaged structure and set $\sigma_i = 1$. Let G_{di} $(i = 1, 2, \ldots, m)$ denote the stiffness parameters of the damaged structure. The eigenvalue problem of the damaged structure is

$$\mathbf{K}_d\phi_d^k = \lambda_d^k \mathbf{M}\phi_d^k, \tag{3}$$

where $\mathbf{K}_d = \mathbf{K}(\mathbf{G}_d)$ is the stiffness matrix with $\mathbf{G}_d = [G_{d1}, G_{d2}, \ldots, G_{dm}]^{\mathrm{T}}$, and $\lambda_d^k = \lambda^k(\mathbf{G}_d)$ and $\phi_d^k = \phi^k(\mathbf{G}_d)$ are the kth eigenvalue and mass-normalized eigenvector, respectively. The stiffness matrix $\mathbf{K}_d$ is related to $\mathbf{K}$ through the Taylor expansion

$$\mathbf{K}_d = \mathbf{K}(\mathbf{G}_d) = \mathbf{K} + \sum_{i=1}^{m} \frac{\partial \mathbf{K}}{\partial G_i}\delta G_i, \tag{4}$$

where $\delta G_i = G_{di} - G_i$ $(i = 1, 2, \ldots, m)$ are the changes in the stiffness parameters, and the higher order derivatives of $\mathbf{K}$ with respect to G_i vanish because $\mathbf{K}$ is assumed to be a linear function of G_i. Based on the finite element model, the global stiffness matrix of a continuous structure satisfies Eq. (4) as its higher order derivatives with respect to each element stiffness parameter vanish.

Let the kth eigenvalue and mass-normalized eigenvector of the damaged structure be related to λ^k and ϕ^k through

$$\begin{aligned}\lambda_d^k = \lambda^k &+ \sum_{i=1}^{m} \lambda_{(1)i}^k \delta G_i + \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_{(2)ij}^k \delta G_i \delta G_j + \cdots \\ &+ \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{m}\cdots\sum_{t=1}^{m}}_{p\ \ summations} \lambda_{(p)ij\ldots t}^k \delta G_i \delta G_j \cdots \delta G_t + e_\lambda^k,\end{aligned} \tag{5}$$

$$\phi_d^k = \phi^k + \sum_{i=1}^{m} \mathbf{z}_{(1)i}^k \delta G_i + \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{z}_{(2)ij}^k \delta G_i \delta G_j + \cdots + \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{m} \cdots \sum_{t=1}^{m}}_{p \text{ summations}} \mathbf{z}_{(p)ij\cdots t}^k \delta G_i \delta G_j \cdots \delta G_t + \mathbf{e}_\phi^k, \tag{6}$$

where $\lambda_{(1)i}^k, \lambda_{(2)ij}^k, \ldots$, and $\lambda_{(p)ij\cdots t}^k$ are the coefficients of the first, second, …, and pth order perturbations for the eigenvalue, $\mathbf{z}_{(1)i}^k, \mathbf{z}_{(2)ij}^k, \ldots$, and $\mathbf{z}_{(p)ij\cdots t}^k$ are the coefficient vectors of the first, second, …, and pth order perturbations for the eigenvector, and e_λ^k and $\mathbf{e}_\phi^k$ are the residuals of order $p+1$. Note that the numbers in the parentheses in the subscripts of the coefficients and coefficient vectors indicate the orders of the terms. By the Taylor expansion, one has for any $p \geqslant 1$,

$$p!\lambda_{(p)ij\cdots t}^k = \frac{\partial^p \lambda^k}{\partial G_i \partial G_j \cdots \partial G_t}, \quad p!\mathbf{z}_{(p)ij\cdots t}^k = \frac{\partial^p \phi^k}{\partial G_i \partial G_j \cdots \partial G_t}. \tag{7}$$

By Eqs. (7), $\lambda_{(p)ij\cdots t}^k$ and $\mathbf{z}_{(p)ij\cdots t}^k$ are symmetric in the p indices, $i, j, \ldots$, and t. The right-hand sides of Eqs. (7) are the pth order sensitivities of the eigenvalues and eigenvectors with respect to the stiffness parameters.

Using the normalization relations of the eigenvectors, ϕ^k and ϕ_d^k, and symmetry of the coefficient vectors in Eq. (6), as indicated earlier, one obtains

$$\begin{aligned} 1 &= (\phi_d^k)^{\mathrm{T}} \mathbf{M} \phi_d^k \\ &= \left(\phi^k + \sum_{i=1}^{m} \mathbf{z}_{(1)i}^k \delta G_i + \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{z}_{(2)ij}^k \delta G_i \delta G_j + \cdots + \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{m} \cdots \sum_{t=1}^{m}}_{p \text{ summations}} \mathbf{z}_{(p)ij\cdots st}^k \delta G_i \delta G_j \cdots \delta G_s \delta G_t + \ldots \right)^{\mathrm{T}} \\ &\quad \times \mathbf{M} \left(\phi^k + \sum_{i=1}^{m} \mathbf{z}_{(1)i}^k \delta G_i + \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{z}_{(2)ij}^k \delta G_i \delta G_j + \cdots + \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{m} \cdots \sum_{t=1}^{m}}_{p \text{ summations}} \mathbf{z}_{(p)ij\cdots st}^k \delta G_i \delta G_j \cdots \delta G_s \delta G_t + \cdots \right) \end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{i=1}^{m} [(\phi^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)i}^k + (\mathbf{z}_{(1)i}^k)^{\mathrm{T}} \mathbf{M} \phi^k] \delta G_i \\
&+ \sum_{i=1}^{m} \sum_{j=i}^{m} \frac{1}{R_{ij}} \{2!(\phi^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)ij}^k + [(\mathbf{z}_{(1)i}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)j}^k + (\mathbf{z}_{(1)j}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)i}^k] \\
&+ 2!(\mathbf{z}_{(2)ij}^k)^{\mathrm{T}} \mathbf{M} \phi^k\} \delta G_i \delta G_j + \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{l=j}^{m} \frac{1}{R_{ijl}} \{3!(\phi^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(3)ijl}^k \\
&+ 2![(\mathbf{z}_{(1)i}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)jl}^k + (\mathbf{z}_{(1)j}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)il}^k + (\mathbf{z}_{(1)l}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)ij}^k] \\
&+ 2![(\mathbf{z}_{(2)jl}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)i}^k + (\mathbf{z}_{(2)il}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)j}^k + (\mathbf{z}_{(2)ij}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)l}^k] \\
&+ 3!(\mathbf{z}_{(3)ijl}^k)^{\mathrm{T}} \mathbf{M} \phi^k\} \delta G_i \delta G_j \delta G_l \\
&+ \cdots + \underbrace{\sum_{i=1}^{m} \sum_{j=i}^{m} \cdots \sum_{t=s}^{m}}_{p \ \text{summations}} \frac{1}{R_{ij\cdots st}} \{p!(\phi^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p)ij\cdots st}^k \\
&+ (p-1)![(\mathbf{z}_{(1)i}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-1)jl\cdots st}^k + (\mathbf{z}_{(1)j}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-1)il\cdots st}^k + \cdots + (\mathbf{z}_{(1)t}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-1)ij\cdots s}^k] \\
&+ 2!(p-2)![(\mathbf{z}_{(2)ij}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-2)l\cdots st}^k + (\mathbf{z}_{(2)il}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-2)j\cdots st}^k + \cdots + (\mathbf{z}_{(2)st}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(p-2)ij\cdots r}^k] + \\
&\cdots + (p-2)!2![(\mathbf{z}_{(p-2)l\cdots st}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)ij}^k + (\mathbf{z}_{(p-2)j\cdots st}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)il}^k + \cdots + (\mathbf{z}_{(p-2)ij\cdots r}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(2)st}^k] \\
&+ (p-1)![(\mathbf{z}_{(p-1)jl\cdots st}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)i}^k + (\mathbf{z}_{(p-1)il\cdots st}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)j}^k + \cdots + (\mathbf{z}_{(p-1)ij\cdots s}^k)^{\mathrm{T}} \mathbf{M} \mathbf{z}_{(1)t}^k] \\
&+ p!(\mathbf{z}_{(p)ij\cdots st}^k)^{\mathrm{T}} \mathbf{M} \phi^k\} \delta G_i \delta G_j \cdots \delta G_s \delta G_t + \cdots . \qquad (8)
\end{aligned}
$$

For any pth ($p \geqslant 1$) order term in the last expression of Eq. (8), the coefficient $R_{ij\cdots t}$ is defined by $R_{ij\cdots t} = X_1!X_2!\cdots X_a!$, where $X_1, X_2, \ldots,$ and X_a $(1 \leqslant a \leqslant p)$ are the numbers of the first, second, …, and last distinct indices within indices, $i, j, \ldots,$ and t, with $X_1 + X_2 + \cdots + X_a = p$. For instance, $R_{1234} = 1!1!1!1! = 1$ with $a = 4$, and $R_{112223} = 2!3!1! = 2!3!$ with $a = 3$. Some explanations of the general pth order term in the last expansion in Eqs. (8) are in order. It consists of $p + 1$ types of terms ordered from the beginning to the end of the expression within the braces, with each type of terms except the first and last ones enclosed in square brackets. The cth $(1 \leqslant c \leqslant p + 1)$ type of terms is obtained by multiplying a $(c - 1)$th order term in the expansion of $(\phi_d^k)^{\mathrm{T}}$ by a $(p - c + 1)$th order term in the expansion of $\mathbf{M}\phi_d^k$. For the cth type of term, where $2 \leqslant c \leqslant p$, the p indices, $i, j, \ldots,$ and t are distributed in the subscripts of the two vectors pre- and post-multiplying $\mathbf{M}$, whose numbers of indices in the subscripts are $c - 1$ and $p - c + 1$, respectively. For the first and last ($(p + 1)$th) types of terms, all the p indices lie in the subscripts of the vectors post- and pre-multiplying $\mathbf{M}$, respectively. The number of terms within each set of square brackets equals the number of different combinations of indices in the subscripts of the vectors pre- and post-multiplying $\mathbf{M}$. When all the p indices, $i, j, \ldots,$ and t, have distinct values, due to symmetry of these vectors in their indices, terms of the cth $(1 \leqslant c \leqslant p + 1)$ type, involving different permutations of the same indices in the subscripts of the vectors, are equal and combined, resulting in the scaling factor $(c - 1)!(p - c + 1)!$ in front of the square brackets. Consequently, the indices in the second through pth summations range from the values of their preceding summation indices to m. When any of the p indices, $i,j,\ldots,$ and t, have equal values, the corresponding terms in each type are given by those in the previous case divided by $R_{ij\cdots t}$. For instance, the fourth order term of the form $\delta G_1 \delta G_2^3$ in the expansion of $(\phi_d^k)^{\mathrm{T}}\mathbf{M}\phi_d^k$ can be obtained from the expression for the pth order term in Eq. (8):

$$\begin{aligned}
&\frac{1}{1!3!}\{4!(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(4)1222} + 3![(\mathbf{z}^k_{(1)1})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)222} + (\mathbf{z}^k_{(1)2})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)122} + (\mathbf{z}^k_{(1)2})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)122}\\
&\quad + (\mathbf{z}^k_{(1)2})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)122}] + 2!2![(\mathbf{z}^k_{(2)12})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)22} + (\mathbf{z}^k_{(2)12})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)22}\\
&\quad + (\mathbf{z}^k_{(2)12})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)22} + (\mathbf{z}^k_{(2)22})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)12} + (\mathbf{z}^k_{(2)22})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)12} + (\mathbf{z}^k_{(2)22})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)12}]\\
&\quad + 3![(\mathbf{z}^k_{(3)222})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)1} + (\mathbf{z}^k_{(3)122})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)2} + (\mathbf{z}^k_{(3)122})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)2} + (\mathbf{z}^k_{(3)122})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)2}]\\
&\quad + 4!(\mathbf{z}^k_{(4)1222})^{\mathrm{T}}\mathbf{M}\phi^k\}\delta G_1\delta G_2^3 = \{4(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(4)1222} + [(\mathbf{z}^k_{(1)1})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)222} + 3(\mathbf{z}^k_{(1)2})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)122}]\\
&\quad + 4(\mathbf{z}^k_{(2)12})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)22} + [(\mathbf{z}^k_{(3)222})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)1} + 3(\mathbf{z}^k_{(3)122})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)2}]\\
&\quad + 4(\mathbf{z}^k_{(4)1222})^{\mathrm{T}}\mathbf{M}\phi^k\}\delta G_1\delta G_2^3,
\end{aligned}\tag{9}$$

where $p=4$ and the four indices involved, $i,j,l,$ and o, are $i=1$ and $j=l=o=2$.

Equating the coefficients of the δG_i $(i=1,2,\ldots,m)$ terms in Eq. (8) and using symmetry of the mass matrix yields

$$(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)i} = 0. \tag{10}$$

Equating the coefficients of the $\delta G_i\delta G_j$ terms and using symmetry of $\mathbf{M}$ and $\mathbf{z}^k_{(2)ij}$ yields

$$(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)ij} = -\frac{1}{2!}(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)j} \tag{11}$$

for all $i,j=1,2,\ldots,m$. Following a similar procedure, one obtains

$$(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(3)ijl} = -\frac{2!}{3!}[(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)jl} + (\mathbf{z}^k_{(1)j})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)li} + (\mathbf{z}^k_{(1)l})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)ij}] \tag{12}$$

for $i,j,l=1,2,\ldots,m$. Equating the coefficients of the $\delta G_i\delta G_j\ldots\delta G_s\delta G_t$ terms of pth order in Eqs. (8) yield

$$\begin{aligned}
(\phi^k)^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p)ij\cdots t} = {}& -\frac{1}{2(p!)}\{(p-1)![(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)jl\cdots st} + (\mathbf{z}^k_{(1)j})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)il\cdots st}\\
& + \cdots + (\mathbf{z}^k_{(1)t})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots s}] + 2!(p-2)![(\mathbf{z}^k_{(2)ij})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)l\cdots st}\\
& + (\mathbf{z}^k_{(2)il})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)j\cdots st} + \cdots + (\mathbf{z}^k_{(2)st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)ij\cdots q}] + \cdots\\
& + (p-2)!2![(\mathbf{z}^k_{(p-2)l\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)ij} + (\mathbf{z}^k_{(p-2)j\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)il}\\
& + \cdots + (\mathbf{z}^k_{(p-2)ij\cdots r})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)st}] + (p-1)![(\mathbf{z}^k_{(p-1)jl\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)i}\\
& + (\mathbf{z}^k_{(p-1)il\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)j} + \cdots + (\mathbf{z}^k_{(p-1)ij\cdots s})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)t}]\}
\end{aligned}\tag{13}$$

for $i,j,\ldots,t=1,2,\ldots,m$. The $p-1$ types of terms, enclosed in the $p-1$ sets of square brackets on the right-hand side of Eq. (13), are ordered from the beginning to the end of the expression within the braces, and their structures are readily observed. When p is odd, by symmetry of $\mathbf{M}$ and $\mathbf{z}^k_{(p)ij\cdots t}$, the yth $(1\leqslant y\leqslant (p-1)/2)$ type of terms is identical to the $(p-y)$th type, and the two types of terms can be combined. Similarly, when p is even, the yth ($1 \leqslant y \leqslant p/2-1$) type of terms equals and can be combined with the $(p-y)$th type of terms. In this case, however, there is a separate type, the $(p/2)$th type, of terms in the middle of the expression.

Substituting Eqs. (4)–(6) into Eq. (3) yields

$$\begin{aligned}&\left\{\mathbf{K}+\sum_{i=1}^{m}\frac{\partial \mathbf{K}}{\partial G_i}\delta G_i\right\}\left\{\phi^k+\sum_{i=1}^{m}\mathbf{z}^k_{(1)i}\delta G_i+\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{z}^k_{(2)ij}\delta G_i\delta G_j\right.\\&\quad\left.+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\mathbf{z}^k_{(3)ijl}\delta G_i\delta G_j\delta G_l+\cdots\right\}\\&=\left\{\lambda^k+\sum_{i=1}^{m}\lambda^k_{(1)i}\delta G_i+\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda^k_{(2)ij}\delta G_i\delta G_j\right.\\&\quad\left.+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\lambda^k_{(3)ijl}\delta G_i\delta G_j\delta G_l+\cdots\right\}\mathbf{M}\\&\quad\left\{\phi^k+\sum_{i=1}^{m}\mathbf{z}^k_{(1)i}\delta G_i+\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{z}^k_{(2)ij}\delta G_i\delta G_j\right.\\&\quad\left.+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\mathbf{z}^k_{(3)ijl}\delta G_i\delta G_j\delta G_l+\cdots\right\}.\end{aligned}\tag{14}$$

Equating the coefficients of the δG_i ($i=1,2,\ldots,m$) terms in Eq. (14) yields

$$\mathbf{K}\mathbf{z}^k_{(1)i}+\frac{\partial \mathbf{K}}{\partial G_i}\phi^k=\lambda^k\mathbf{M}\mathbf{z}^k_{(1)i}+\lambda^k_{(1)i}\mathbf{M}\phi^k.\tag{15}$$

Expanding $\mathbf{z}^k_{(1)i}$ using normalized eigenvectors of Eq. (1) as basis vectors, one has

$$\mathbf{z}^k_{(1)i}=\sum_{u=1}^{N}P^k_{(1)iu}\phi^u,\tag{16}$$

where $P^k_{(1)iu}$ is the coefficient of the uth term and the number in the parentheses in its subscript corresponds to that of $\mathbf{z}^k_{(1)i}$. Pre-multiplying Eq. (15) by $(\phi^k)^{\mathrm{T}}$ and using Eqs. (1), (2), and (16) yields

$$\lambda^k_{(1)i}=(\phi^k)^{\mathrm{T}}\frac{\partial \mathbf{K}}{\partial G_i}\phi^k.\tag{17}$$

Substituting Eq. (16) into Eq. (10) and using Eq. (2) yields

$$P^k_{(1)ik}=0.\tag{18}$$

Pre-multiplying Eq. (15) by $(\phi^v)^{\mathrm{T}}$, where $1\leqslant v\leqslant N$ and $v\neq k$, and using Eqs. (1), (2), and (16) yields

$$P^k_{(1)iv}=\frac{1}{\lambda^k-\lambda^v}(\phi^v)^{\mathrm{T}}\frac{\partial \mathbf{K}}{\partial G_i}\phi^k.\tag{19}$$

By Eqs. (16), (18), and (19) we have determined $\mathbf{z}^k_{(1)i}$.

Equating the coefficients of the $\delta G_i \delta G_j$ terms in Eq. (14) yields

$$\begin{aligned} &2!\mathbf{K}\mathbf{z}^k_{(2)ij} + \frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(1)j} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(1)i} \\ &\quad = 2!\lambda^k \mathbf{M}\mathbf{z}^k_{(2)ij} + \lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(1)j} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(1)i} + 2!\lambda^k_{(2)ij}\mathbf{M}\phi^k. \end{aligned} \tag{20}$$

Expanding $\mathbf{z}^k_{(2)ij}$ using normalized eigenvectors of Eq. (1) as basis vectors, one has

$$\mathbf{z}^k_{(2)ij} = \sum_{u=1}^{N} P^k_{(2)iju}\phi^u, \tag{21}$$

where $P^k_{(2)iju}$ is the coefficient of the uth term and the number in the parentheses in its subscript corresponds to that of $\mathbf{z}^k_{(2)ij}$. Pre-multiplying Eq. (20) by $(\phi^k)^{\mathrm{T}}$ and using Eqs. (1), (2), (10), and (21) yields

$$\lambda^k_{(2)ij} = \frac{1}{2!}(\phi^k)^{\mathrm{T}}\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(1)j} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(1)i}\right]. \tag{22}$$

Substituting Eq. (21) into Eq. (11) and using Eq. (2) yields

$$P^k_{(2)ijk} = -\frac{1}{2!}(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)j}. \tag{23}$$

Pre-multiplying Eq. (20) by $(\phi^v)^{\mathrm{T}}$, where $1 \leqslant v \leqslant N$ and $v \neq k$, and using Eqs. (1), (2), and (21) yields

$$\begin{aligned} P^k_{(2)ijv} = &\frac{1}{2!(\lambda^k - \lambda^v)}(\phi^v)^{\mathrm{T}}\left\{\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(1)j} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(1)i}\right]\right. \\ &\left. - [\lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(1)j} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(1)i}]\right\}. \end{aligned} \tag{24}$$

By Eqs. (21), (23), and (24) we have determined $\mathbf{z}^k_{(2)ij}$.

Equating the coefficients of the $\delta G_i \delta G_j \delta G_l$ terms in Eq. (14) yields

$$\begin{aligned} &3!\mathbf{K}\mathbf{z}^k_{(3)ijl} + 2!\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(2)jl} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(2)li} + \frac{\partial \mathbf{K}}{\partial G_l}\mathbf{z}^k_{(2)ij}\right] \\ &\quad = 3!\lambda^k\mathbf{M}\mathbf{z}^k_{(3)ijl} + 2![\lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(2)jl} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(2)li} + \lambda^k_{(1)l}\mathbf{M}\mathbf{z}^k_{(2)ij}] \\ &\qquad + 2![\lambda^k_{(2)jl}\mathbf{M}\mathbf{z}^k_{(1)i} + \lambda^k_{(2)li}\mathbf{M}\mathbf{z}^k_{(1)j} + \lambda^k_{(2)ij}\mathbf{M}\mathbf{z}^k_{(1)l}] + 3!\lambda^k_{(3)ijl}\mathbf{M}\phi^k. \end{aligned} \tag{25}$$

Expanding $\mathbf{z}^k_{(3)ijl}$ using normalized eigenvectors of Eq. (1) as basis vectors, one has

$$\mathbf{z}^k_{(3)ijl} = \sum_{u=1}^{n} P^k_{(3)ijlu}\phi^u, \tag{26}$$

where $P^k_{(3)ijlu}$ is the coefficient of the uth term and the number in the parentheses in its subscript corresponds to that of $\mathbf{z}^k_{(3)ijl}$. Pre-multiplying Eq. (25) by $(\phi^k)^{\mathrm{T}}$ and using Eqs. (1), (2), (10), and (26) yields

$$\lambda^k_{(3)ijl} = \frac{2!}{3!}(\phi^k)^{\mathrm{T}}\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(2)jl} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(2)li} + \frac{\partial \mathbf{K}}{\partial G_l}\mathbf{z}^k_{(2)ij} - \lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(2)jl} - \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(2)li} - \lambda^k_{(1)l}\mathbf{M}\mathbf{z}^k_{(2)ij}\right]. \tag{27}$$

Substituting Eq. (26) into Eq. (12) and using Eq. (2) yields

$$P^k_{(3)ijlk} = -\frac{2!}{3!}[(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)jl} + (\mathbf{z}^k_{(1)j})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)li} + (\mathbf{z}^k_{(1)l})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)ij}]. \tag{28}$$

Pre-multiplying Eq. (25) by $(\phi^v)^{\mathrm{T}}$, where $1 \leqslant v \leqslant N$ and $v \neq k$, and using Eqs. (1), (2), and (26) yields

$$P^k_{(3)ijlv} = \frac{2!}{3!(\lambda^k - \lambda^v)}(\phi^v)^{\mathrm{T}}\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(2)jl} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(2)li} + \frac{\partial \mathbf{K}}{\partial G_l}\mathbf{z}^k_{(2)ij} - \lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(2)jl} - \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(2)li} - \lambda^k_{(1)l}\mathbf{M}\mathbf{z}^k_{(2)ij} - \lambda^k_{(2)jl}\mathbf{M}\mathbf{z}^k_{(1)i} - \lambda^k_{(2)li}\mathbf{M}\mathbf{z}^k_{(1)j} - \lambda^k_{(2)ij}\mathbf{M}\mathbf{z}^k_{(1)l}\right]. \tag{29}$$

By Eqs. (26), (28), and (29) we have determined $\mathbf{z}^k_{(3)ijl}$.

We proceed now to derive the perturbation solutions for the general pth order terms in Eqs. (5) and (6). Equating the coefficients of the $\delta G_i \delta G_j \cdots \delta G_s \delta G_t$ terms of order p in Eq. (14) yields

$$\begin{aligned} &p!\mathbf{K}\mathbf{z}^k_{(p)ij\cdots st} + (p-1)!\left[\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(p-1)j\cdots st} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots + \frac{\partial \mathbf{K}}{\partial G_t}\mathbf{z}^k_{(p-1)ij\cdots s}\right] \\ &= p!\lambda^k\mathbf{M}\mathbf{z}^k_{(p)ij\cdots st} + (p-1)![\lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(p-1)j\cdots st} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots \\ &\quad + \lambda^k_{(1)t}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots s}] + 2!(p-2)![\lambda^k_{(2)ij}\mathbf{M}\mathbf{z}^k_{(p-2)l\cdots st} + \lambda^k_{(2)il}\mathbf{M}\mathbf{z}^k_{(p-2)j\cdots st} \\ &\quad + \cdots + \lambda^k_{(2)st}\mathbf{M}\mathbf{z}^k_{(p-2)ij\cdots q}] + \cdots + p!\lambda^k_{(p)ij\cdots st}\mathbf{M}\phi^k. \end{aligned} \tag{30}$$

Expanding $\mathbf{z}^k_{(p)ij\cdots st}$ using normalized eigenvectors of Eq. (1) as basis vectors, one has

$$\mathbf{z}^k_{(p)ij\cdots st} = \sum_{u=1}^{n} P^k_{(p)ij\cdots stu}\phi^u, \tag{31}$$

where $P^k_{(p)ij\cdots stu}$ is the coefficient of the uth term and the number in the parentheses in its subscript corresponds to that of $\mathbf{z}^k_{(p)ij\cdots st}$. Pre-multiplying Eq. (30) by $(\phi^k)^{\mathrm{T}}$ and using Eqs. (1), (10), (31) and orthonormality relations of eigenvectors yields

$$\begin{aligned} \lambda^k_{(p)ij\cdots st} = \frac{1}{p!}(\phi^k)^{\mathrm{T}}\Bigg[&(p-1)!\left(\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(p-1)j\cdots st} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots + \frac{\partial \mathbf{K}}{\partial G_t}\mathbf{z}^k_{(p-1)ij\cdots s}\right) \\ &-(p-1)!(\lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(p-1)j\cdots st} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots + \lambda^k_{(1)t}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots s}) \\ &-2!(p-2)!(\lambda^k_{(2)ij}\mathbf{M}\mathbf{z}^k_{(p-2)l\cdots st} + \lambda^k_{(2)il}\mathbf{M}\mathbf{z}^k_{(p-2)j\cdots st} + \cdots + \lambda^k_{(2)st}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots r}) \\ &-\cdots-(p-2)!2!(\lambda^k_{(p-2)l\cdots st}\mathbf{M}\mathbf{z}^k_{(2)ij} \\ &+ \lambda^k_{(p-2)j\cdots st}\mathbf{M}\mathbf{z}^k_{(2)il} + \cdots + \lambda^k_{(p-2)ij\cdots r}\mathbf{M}\mathbf{z}^k_{(2)st})\Bigg]. \end{aligned} \tag{32}$$

The pth order sensitivities of eigenvalues are obtained from Eqs. (7) and (32). They depend on the eigenvalue and eigenvector sensitivities of orders up to $p-2$ and $p-1$, respectively. Substituting

Eq. (31) into Eq. (13) and using Eq. (2) yields

$$\begin{aligned} P^k_{(p)ij\cdots stk} = & -\frac{1}{2(p!)}\{(p-1)![(\mathbf{z}^k_{(1)i})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)jl\cdots st} + (\mathbf{z}^k_{(1)j})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)il\cdots st} + \cdots \\ & + (\mathbf{z}^k_{(1)t})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots s}] + 2!(p-2)![(\mathbf{z}^k_{(2)ij})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)l\cdots st} \\ & + (\mathbf{z}^k_{(2)il})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)j\cdots st} + \cdots + (\mathbf{z}^k_{(2)st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(p-2)ij\cdots q}] + \cdots \\ & + (p-2)!2![(\mathbf{z}^k_{(p-2)l\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)ij} + (\mathbf{z}^k_{(p-2)j\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)il} + \cdots \\ & + (\mathbf{z}^k_{(p-2)ij\cdots r})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(2)st}] + (p-1)![(\mathbf{z}^k_{(p-1)jl\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)i} \\ & + (\mathbf{z}^k_{(p-1)il\cdots st})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)j} + \cdots + (\mathbf{z}^k_{(p-1)ij\cdots s})^{\mathrm{T}}\mathbf{M}\mathbf{z}^k_{(1)t}]\}. \end{aligned} \tag{33}$$

Pre-multiplying Eq. (30) by $(\phi^v)^{\mathrm{T}}$, where $1 \leqslant v \leqslant N$ and $v \neq k$, and using Eqs. (1), (2), and (31) yields

$$\begin{aligned} P^k_{(p)ij\cdots stv} = & \frac{1}{p!(\lambda^k - \lambda^v)}(\phi^v)^{\mathrm{T}}\Bigg[(p-1)!\left(\frac{\partial \mathbf{K}}{\partial G_i}\mathbf{z}^k_{(p-1)j\cdots st} + \frac{\partial \mathbf{K}}{\partial G_j}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots + \frac{\partial \mathbf{K}}{\partial G_t}\mathbf{z}^k_{(p-1)ij\cdots s}\right) \\ & -(p-1)!(\lambda^k_{(1)i}\mathbf{M}\mathbf{z}^k_{(p-1)j\cdots st} + \lambda^k_{(1)j}\mathbf{M}\mathbf{z}^k_{(p-1)i\cdots st} + \cdots + \lambda^k_{(1)t}\mathbf{M}\mathbf{z}^k_{(p-1)ij\cdots s}) \\ & -2!(p-2)!(\lambda^k_{(2)ij}\mathbf{M}\mathbf{z}^k_{(p-2)l\cdots st} + \lambda^k_{(2)il}\mathbf{M}\mathbf{z}^k_{(p-2)j\cdots st} + \cdots + \lambda^k_{(2)st}\mathbf{M}\mathbf{z}^k_{(p-2)ij\cdots r}) - \cdots \\ & -(p-1)!(\lambda^k_{(p-1)j\cdots st}\mathbf{M}\mathbf{z}^k_{(1)i} + \lambda^k_{(p-1)i\cdots st}\mathbf{M}\mathbf{z}^k_{(1)j} + \cdots + \lambda^k_{(p-1)ij\cdots s}\mathbf{M}\mathbf{z}^k_{(1)t})\Bigg]. \end{aligned} \tag{34}$$

By Eqs. (31), (33), and (34) we have determined $\mathbf{z}^k_{(p)ij\cdots st}$. The pth order sensitivities of eigenvectors can be subsequently obtained from Eq. (7). They depend on the eigenvalue and eigenvector sensitivities of orders up to $p-1$.

Eqs. (5) and (6) serve both the forward and inverse problems. In the former one determines the changes in the eigenparameters with changes in the stiffness parameters. Damage detection is treated as an inverse problem, in which one identifies iteratively the changes in the stiffness parameters from a selected set of measured eigenparameters of the damaged structure: λ^k_d ($k = 1, 2, \ldots, n_\lambda$) and ϕ^k_d ($k = 1, 2, \ldots, n_\phi$), where $1 \leqslant n_\lambda, n_\phi \leqslant N$. Here, λ^k_d and ϕ^k_d are assumed to be the perfect eigenparameters; simulated noise is included in the measured eigenparameters in Section 4.2. Often we choose a set of n measured eigenparameter pairs to detect damage, i.e., $n_\lambda = n_\phi = n$. Let the number of the measured degrees of freedom of ϕ^k_d be N_m; $N_m = N$ and $N_m < N$ when we have complete and incomplete eigenvector measurements, respectively. With reduced measurements the unmeasured degrees of freedom of ϕ^k_d are estimated first using a modified eigenvector expansion method, as shown in Section 4.2, and ϕ^k_d is normalized subsequently. Only the component equations corresponding to the measured degrees of freedom of ϕ^k_d are used in Eq. (6). The system equations (5) and (6) involve $n_\lambda + n_\phi N_m$ equations with m unknowns, which are in general determinate if $n_\lambda + n_\phi N_m = m$, under-determined if $n_\lambda + n_\phi N_m < m$, and over-determined if $n_\lambda + n_\phi N_m > m$. In the first iteration, λ^k and ϕ^k in Eqs. (5) and (6) correspond to the eigenparameters of the structure with the initial stiffness parameters $G_i^{(0)}$. With the perturbation terms determined as shown above, the changes in the stiffness parameters $\delta G_i^{(1)}$, where the number in the superscript denotes the iteration number, can be solved from Eqs. (5) and (6) using an optimization method discussed in Section 3. The estimated stiffness parameters of the damaged structure are updated by $G_i^{(1)} = G_i^{(0)} + \delta G_i^{(1)}$. With the updated stiffness parameters, the eigenparameters, λ^k and ϕ^k, in Eqs. (5) and (6) are re-calculated from the

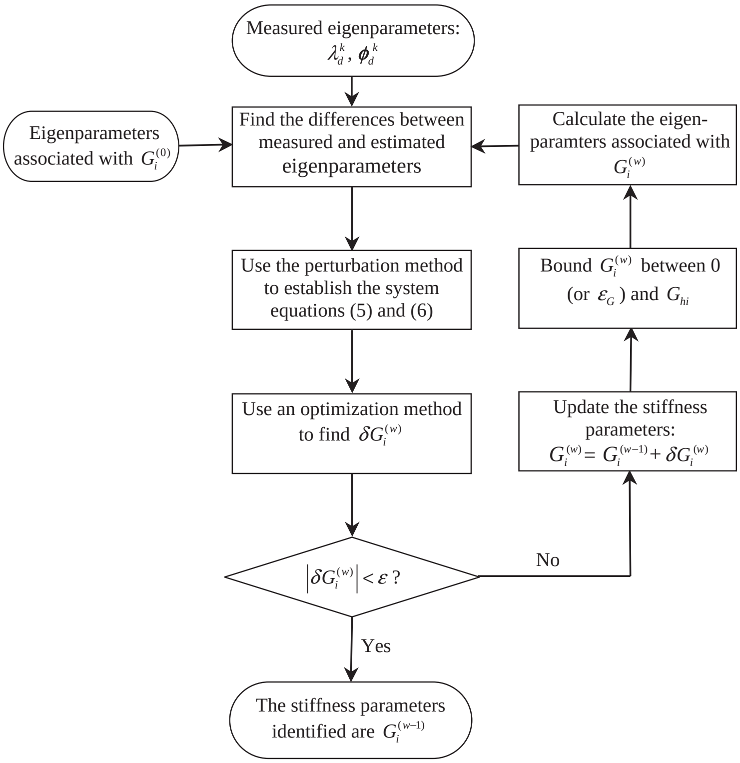

Fig. 1. Inverse algorithm for identifying the stiffness parameters of the damaged structure from a select set of measured eigenparameters. The flowchart for the quasi-Newton methods to find the optimal solutions to the system equations is shown in Fig. 2.

eigenvalue problem (1) and the perturbation terms are re-evaluated. One subsequently finds $\delta G_i^{(2)}$ using the same method as that in the first iteration. This process is continued until the termination criterion, $|\delta G_i^{(L)}| < \varepsilon$, where L is the last iteration number and ε is some small constant, is satisfied for all $i = 1, 2, \ldots, m$. Note that after the wth ($1 \leqslant w < L$) iteration, we set $G_i^{(w)}$ to G_{hi} if $G_i^{(w)} > G_{hi}$, and to zero or some small stiffness value ε_G if $G_i^{(w)} < 0$. The flowchart for the iterative algorithm is shown in Fig. 1. When a single iteration is used, the iterative method becomes a direct method.

3. Optimization methods

Neglecting the residuals of order $p+1$ in Eqs. (5) and (6) yields a system of polynomial equations of order p. When $n_\lambda + n_\phi N_m \leqslant m$, $\delta G_i^{(w)}$ ($i = 1, 2, \ldots, m$) at the wth iteration can be solved from the resulting equations. There are generally an infinite number of solutions when $n_\lambda + n_\phi N_m < m$, a unique solution when $n_\lambda + n_\phi N_m = m$ and $p = 1$, and a finite number of solutions when $n_\lambda + n_\phi N_m = m$ and $p > 1$. When $n_\lambda + n_\phi N_m > m$, one generally cannot find $\delta G_i^{(w)}$ to satisfy all the equations, and an optimization method can be used to find $\delta G_i^{(w)}$ which minimize an objective function related to the errors in satisfying these equations. We use here the same notations, e_λ^k and $\mathbf{e}_\phi^k$, to denote the errors in satisfying the system equations (5) and (6), respectively. Consider the objective function

$$J = \sum_{k=1}^{n_\lambda} W_\lambda^k (e_\lambda^k)^2 + \sum_{k=1}^{n_\phi} W_\phi^k (\mathbf{e}_\phi^k)^{\mathrm{T}} (\mathbf{e}_\phi^k), \tag{35}$$

where W_λ^k $(k = 1, 2, \ldots, n_\lambda)$ and W_ϕ^k $(k = 1, 2, \ldots, n_\phi)$ are the weighting factors, and J is a function of $\delta G_i^{(w)}$ when one substitutes the expressions for e_λ^k and $\mathbf{e}_\phi^k$ in Eqs. (5) and (6) into Eq. (35). When the first order perturbations are retained in Eqs. (5) and (6), the least-squares method [24] can be used to determine $\delta G_i^{(w)}$ which minimize Eq. (35) with $W_\lambda^k = W_\phi^k = 1$. Other weighting factors can be handled by dividing Eqs. (5) and (6) by $1/\sqrt{W_\lambda^k}$ and $1/\sqrt{W_\phi^k}$, respectively. The method determines essentially the generalized inverse of the resulting system matrix, and is also known as the generalized inverse method. When the perturbations up to the pth $(p \geqslant 1)$ order are included in Eqs. (5) and (6), the gradient and quasi-Newton methods [25] can be used to determine $\delta G_i^{(w)}$ iteratively. Unlike the generalized inverse method, the methods are applicable when other objective functions are defined. While the optimization methods are introduced for over-determined systems, they can be used when $n_\lambda + n_\phi N_m \leqslant m$, in which case $J = 0$ (i.e., $e_\lambda = \mathbf{e}_\phi = 0$) when the optimal solutions are reached.

3.1. Gradient method

To minimize the objective function in Eq. (35) at the wth iteration, one can use the algorithm

$$\delta \mathbf{G}_{(b)}^{(w)} = \delta \mathbf{G}_{(b-1)}^{(w)} - \alpha_b \mathbf{f}_{b-1} \tag{36}$$

to update the changes in the stiffness parameters, where $\delta \mathbf{G}_{(b)}^{(w)} = (\delta G_{1(b)}^{(w)}, \delta G_{2(b)}^{(w)}, \ldots, \delta G_{m(b)}^{(w)})^{\mathrm{T}}$, $\alpha_b \geqslant 0$ is the step size, and $\mathbf{f}_{b-1}$ equals the gradient vector $\mathbf{g}_{b-1} = (\partial J/\partial G_1^{(w)}, \partial J/\partial G_2^{(w)}, \ldots, \partial J/\partial G_m^{(w)})^{\mathrm{T}}$ associated with $\delta G_{(b-1)}^{(w)}$. Note that the subscript b $(b \geqslant 1)$ in all the variables in Eq. (36) denotes the number of nested iterations. The initial values used are $\delta G_{i(0)}^{(w)} = 0$. The nested iteration is terminated when $\alpha_b \|\mathbf{g}_{b-1}\|_\infty < \gamma$, where $\|\cdot\|_\infty$ is the infinity norm and γ is some small constant, or the number of nested iterations exceeds an acceptable number, D.

3.2. Quasi-Newton methods

Due to its successive linear approximations to the objective function, the gradient method can progress slowly when approaching a stationary point. The quasi-Newton methods can provide a remedy to the problem by using essentially quadratic approximations to the objective function near the stationary point. The iteration scheme of these methods is given by Eq. (36) with $\mathbf{f}_{b-1} = \mathbf{B}_{b-1}\mathbf{g}_{b-1}$, where $\mathbf{B}_{b-1}$ is an approximation to the inverse of the Hessian matrix used at the bth nested iteration, and the other variables the same as those defined in Section 3.1. Initially, we set $\delta G_{i(0)}^{(w)} = 0$ and $\mathbf{B}_0 = \mathbf{I}$, the identity matrix. The matrix $\mathbf{B}_b$ is updated using either the DFP formula [25]

$$\mathbf{B}_b = \mathbf{B}_{b-1} + \frac{(\delta \mathbf{G}_{(b)}^{(w)} - \delta \mathbf{G}_{(b-1)}^{(w)})(\delta \mathbf{G}_{(b)}^{(w)} - \delta \mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}}}{(\delta \mathbf{G}_{(b)}^{(w)} - \delta \mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}} (\mathbf{g}_b - \mathbf{g}_{b-1})}$$

$$
-\frac{[\mathbf{B}_{b-1}(\mathbf{g}_b-\mathbf{g}_{b-1})][\mathbf{B}_{b-1}(\mathbf{g}_b-\mathbf{g}_{b-1})]^{\mathrm{T}}}{(\mathbf{g}_b-\mathbf{g}_{b-1})^{\mathrm{T}}\mathbf{B}_{b-1}(\mathbf{g}_b-\mathbf{g}_{b-1})} \tag{37}
$$

or the BFGS formula [25]

$$
\begin{aligned}
\mathbf{B}_b = \mathbf{B}_{b-1} &+ \left[1+\frac{(\mathbf{g}_b-\mathbf{g}_{b-1})^{\mathrm{T}}\mathbf{B}_{b-1}(\mathbf{g}_b-\mathbf{g}_{b-1})}{(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}}(\mathbf{g}_b-\mathbf{g}_{b-1})}\right] \\
&\times\frac{(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}}}{(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}}(\mathbf{g}_b-\mathbf{g}_{b-1})} \\
&-\frac{(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})(\mathbf{g}_b-\mathbf{g}_{b-1})^{\mathrm{T}}\mathbf{B}_{b-1}+\mathbf{B}_{b-1}(\mathbf{g}_b-\mathbf{g}_{b-1})(\mathbf{g}_b-\mathbf{g}_{b-1})^{\mathrm{T}}}{(\delta\mathbf{G}_{(b)}^{(w)}-\delta\mathbf{G}_{(b-1)}^{(w)})^{\mathrm{T}}(\mathbf{g}_b-\mathbf{g}_{b-1})}.
\end{aligned} \tag{38}
$$

The nested iteration is terminated when $\alpha_b\|\mathbf{B}_{b-1}\mathbf{g}_{b-1}\|_\infty < \gamma$ or the number of iterations exceeds D. The flowchart for the quasi-Newton methods, including the step size search procedure as outlined below, is shown in Fig. 2.

3.3. Step size search procedure

The optimal step size for the gradient and quasi-Newton methods is determined in each nested iteration to minimize the function $J(\delta\mathbf{G}_{(b-1)}^{(w)}-\alpha_b\mathbf{f}_{b-1}) = F(\alpha_b)$ with respect to α_b. The search procedure is divided into two phases: an initial search to bracket the optimum α_b^* and a golden section search to locate α_b^* within the bracket (Fig. 2).

3.3.1. Initial bracketing

Choose the starting point $x_1 = 0$ and an initial increment $\Delta > 0$. Let $x_2 = x_1 + \Delta$, $F_1 = F(x_1)$, and $F_2 = F(x_2)$. Since for the gradient and quasi-Newton methods, $\mathbf{f}_0 = \mathbf{g}_0$ and it is along a descent direction of J when Δ is sufficiently small, one has $F_2 < F_1$. Rename 2Δ as Δ, and let $x_3 = x_2 + \Delta$ and $F_3 = F(x_3)$. If $F_3 > F_2$, stop and α_b^* is contained in the interval (x_1, x_3). Otherwise, rename x_2 as x_1 and x_3 as x_2, then F_2 becomes F_1 and F_3 becomes F_2. Rename 2Δ as Δ, and let $x_3 = x_2 + \Delta$ and $F_3 = F(x_3)$. Compare F_3 and F_2, and repeat the above procedure if $F_3 < F_2$ until $F_3 > F_2$, with the final interval (x_1, x_3) containing α_b^*.

3.3.2. Golden section search

If $|x_3 - x_2| > |x_2 - x_1|$, define a new point:

$$
x_4 = x_2 + 0.382(x_3 - x_2). \tag{39}
$$

Otherwise, rename x_1 as x_3 and x_3 as x_1, and then define x_4 using Eq. (39). Let $F_4 = F(x_4)$. If $F_2 < F_4$, rename x_4 as x_3, then F_4 becomes F_3. Otherwise, rename x_2 as x_1 and x_4 as x_2, then F_2 becomes F_1 and F_4 becomes F_2. Compare $|x_3 - x_2|$ and $|x_2 - x_1|$, and repeat the above procedure until $|x_3 - x_1|\,\|\mathbf{f}_{b-1}\|_\infty < \varepsilon_\alpha$, where ε_α is some small constant. Then choose $\alpha_b^* = (x_1 + x_3)/2$.

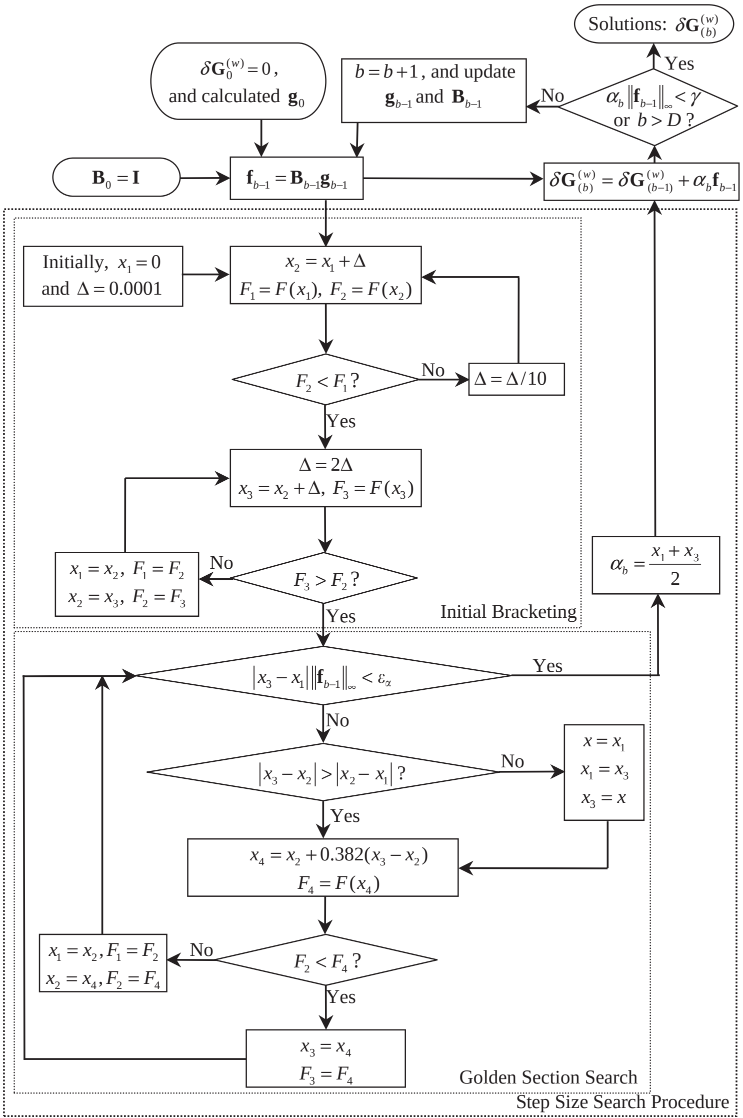

Fig. 2. Quasi-Newton methods along with the step size search procedure for finding the optimal solutions to the system equations.

4. Results and discussion

4.1. Mass–spring system

The algorithm developed in the previous sections is used to identify the stiffness parameters of a N-degree-of-freedom system consisting of a serial chain of masses and springs (Fig. 3). Let the masses of the system be $M_i = 1$ kg ($i = 1, 2, \ldots, N$), and the stiffnesses of the undamaged springs

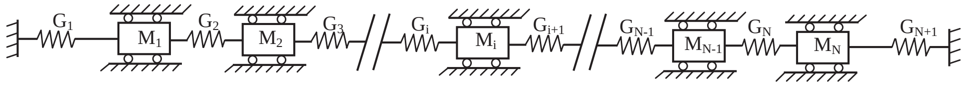

Fig. 3. Schematic of a serial mass–spring system.

be $G_{hi} = 1$ N/m $(i = 1, 2, \ldots, m)$, where $m = N + 1$. The system is said to have a small, medium and large level of damage if the maximum reduction in the stiffnesses is within 30%, between 30% and 70% and over 70%, respectively. The mass matrix $\mathbf{M}$ is an $N \times N$ identity matrix, and the stiffness matrix $\mathbf{K}$ is a banded matrix with entries $\mathbf{K}_{ii} = G_i + G_{i+1}$ $(i = 1, 2, \ldots, N)$, $\mathbf{K}_{i(i+1)} = \mathbf{K}_{(i+1)i} = -G_{i+1}$ $(i = 1, 2, \ldots, N-1)$, and all other entries equal to zero. The matrices $\partial\mathbf{K}/\partial G_1$ and $\partial\mathbf{K}/\partial G_N$ have a unit value in entries (1, 1) and (N, N), respectively, and zero entries elsewhere. The non-zero entries of the matrices $\partial\mathbf{K}/\partial G_i$ $(i = 2, 3, \ldots, N-1)$ are $(\partial\mathbf{K}/\partial G_i)_{(i-1)(i-1)} = (\partial\mathbf{K}/\partial G_i)_{ii} = 1$ and $(\partial\mathbf{K}/\partial G_i)_{i(i-1)} = (\partial\mathbf{K}/\partial G_i)_{(i-1)i} = -1$.

We look at a forward problem first with $N = 3$ and $m = 4$. The stiffnesses of the damaged system are $G_{d1} = G_{d3} = 1$ N/m, $G_{d2} = 0.3$ N/m and $G_{d4} = 0$ N/m. The undamaged system is considered as the unperturbed system and the damaged system as the perturbed system. Based on the eigenparameters of the undamaged system, the eigensolutions of the damaged system are obtained using the first, second, and third order perturbations, as shown in Table 1. The results show that even with the large changes in stiffness, the third order perturbation solutions compare favorably with the exact solutions for the damaged system. The higher order perturbation solutions can be used for large order systems when their direct eigensolutions become costly.

Consider the damage detection problem now with $N = 9$, $m = 10$, $G_{d5} = 0.5$ N/m, $G_{d8} = 0.7$ N/m, $G_{d10} = 0.8$ N/m, and $G_{di} = 1$ N/m $(i = 1, 2, 3, 4, 6, 7, 9)$. We set $W_\lambda^k = W_\phi^k = 1$, $\varepsilon = 0.001$, $\gamma = 10^{-10}$, $\sigma_i = 1$ for all i, and $D = 500$; the actual numbers of nested iterations in all the cases are much smaller than D. All the degrees of freedom of an eigenvector are assumed to be measured; hence $N_m = N$. Since vanishing stiffness in any spring other than the two end ones in Fig. 3 can result in two decoupled subsystems, when $G_i^{(w)} < 0$ we set $G_i^{(w)}$ to $\varepsilon_G = 0.1$ N/m in the first two iterations and to 0.01 N/m in the remaining iterations. A relatively large value is assigned to ε_G in the initial iterations to avoid close eigenvalues in the mass–spring system and small denominators in Eq. (19). This improves convergence especially when a small number of eigenparameter pairs are used. A smaller value is used for ε_G in the later iterations to improve the accuracy of stiffness estimation when there is a large level of stiffness reduction. Using the first order perturbations and different numbers of eigenparameter pairs, the maximum errors in estimating the stiffnesses of the damaged system at the wth iteration, defined by

$$E = \max_{1 \leqslant i \leqslant N} \frac{|G_i^{(w)} - G_{di}|}{G_{di}}, \tag{40}$$

are shown in Fig. 4(a) and (b) for all the iterations. When $n = 1$, the error decreases slowly, though monotonically, and there is an estimation error of 1.5% at the end of iteration. While the errors can increase with the iteration number before approaching zero for $n = 3$, they decrease monotonically for $n = 2$ and $n \geqslant 4$. All the stiffnesses are exactly identified at the end of iteration when $n \geqslant 2$. Note that the number of the system equations equals and exceeds the number of unknowns when $n = 1$ and $n \geqslant 2$ respectively. Since the system equations are linear, they have a

Table 1
Eigensolutions of the damaged system from an eigenvalue problem solver (exact) and perturbation analysis

Eigenparameters	Exact	Unperturbed	Perturbed		
			First order	Second order	Third order
λ^1	0.10602	0.58579	0.30576	0.15670	0.10090
λ^2	1.27538	2.00000	1.15000	1.25500	1.29344
λ^3	2.21859	3.41421	2.14424	2.18830	2.20567
ϕ^1	$\begin{Bmatrix} 0.16516 \\ 0.65733 \\ 0.73528 \end{Bmatrix}$	$\begin{Bmatrix} 0.50000 \\ 0.70711 \\ 0.50000 \end{Bmatrix}$	$\begin{Bmatrix} 0.28523 \\ 0.68836 \\ 0.74129 \end{Bmatrix}$	$\begin{Bmatrix} 0.16111 \\ 0.66444 \\ 0.79452 \end{Bmatrix}$	$\begin{Bmatrix} 0.12907 \\ 0.65040 \\ 0.76653 \end{Bmatrix}$
ϕ^2	$\begin{Bmatrix} 0.95541 \\ 0.07840 \\ -0.28470 \end{Bmatrix}$	$\begin{Bmatrix} 0.70711 \\ -0.00000 \\ -0.70711 \end{Bmatrix}$	$\begin{Bmatrix} 0.95459 \\ 0.10607 \\ -0.45962 \end{Bmatrix}$	$\begin{Bmatrix} 1.00188 \\ 0.14319 \\ -0.31776 \end{Bmatrix}$	$\begin{Bmatrix} 0.97976 \\ 0.12310 \\ -0.26810 \end{Bmatrix}$
ϕ^3	$\begin{Bmatrix} 0.24478 \\ -0.74952 \\ 0.61507 \end{Bmatrix}$	$\begin{Bmatrix} 0.50000 \\ -0.70711 \\ 0.50000 \end{Bmatrix}$	$\begin{Bmatrix} 0.36477 \\ -0.72586 \\ 0.60871 \end{Bmatrix}$	$\begin{Bmatrix} 0.29635 \\ -0.74132 \\ 0.62482 \end{Bmatrix}$	$\begin{Bmatrix} 0.26445 \\ -0.74875 \\ 0.62362 \end{Bmatrix}$

unique solution when $n = 1$, and J has a unique minimum when $n \geqslant 2$. With the small γ the gradient method and the quasi-Newton methods using the DFP and BFGS formulas yield exactly the same results as the generalized inverse method (not shown here). Because the generalized inverse method does not involve any nested iteration, it is the most efficient one among the four methods and will be used with the first order perturbations in all the cases considered in this paper. While not shown here, the results indicated that the quasi-Newton methods converge faster than the gradient method and the BFGS method has the similar performance to the DFP method. In what follows the BFGS method will be used with the higher order perturbations. With the second order perturbations the errors shown in Fig. 4(c) decrease monotonically for all n. While the use of the second order perturbations improves the accuracy of stiffness estimation in each iteration and reduces the number of iterations, it takes a much longer time to compute the higher order perturbations and the associated optimal solutions.

When only the first few eigenvalues are used, for instance, $n_\lambda = 5$ and $n_\phi = 0$, the stiffnesses identified with the first order perturbations

$$\mathbf{G}^{(4)} = (0.875, 0.976, 0.926, 0.864, 0.699, 0.699, 0.864, 0.926, 0.976, 0.875)^{\mathrm{T}}\ \mathrm{N/m}, \tag{41}$$

where the number in the superscript denotes the last iteration number, correspond to those of a different system with the same eigenvalues for the first five modes as the damaged system. The same stiffnesses are identified with the second order perturbations. Similarly, when the first eigenvector is used, i.e., $n_\phi = 1$ and $n_\lambda = 0$, the stiffnesses identified with the first order perturbations are those of a different system with the same eigenvector for the first mode as the damaged system:

$$\mathbf{G}^{(9)} = (0.989, 0.991, 0.995, 1, 0.520, 0.829, 0.940, 0.668, 0.959, 0.768)^{\mathrm{T}}\ \mathrm{N/m}. \tag{42}$$

With the second order perturbations the stiffnesses of the damaged system are identified. The stiffnesses identified are not unique because the system equations in each iteration are

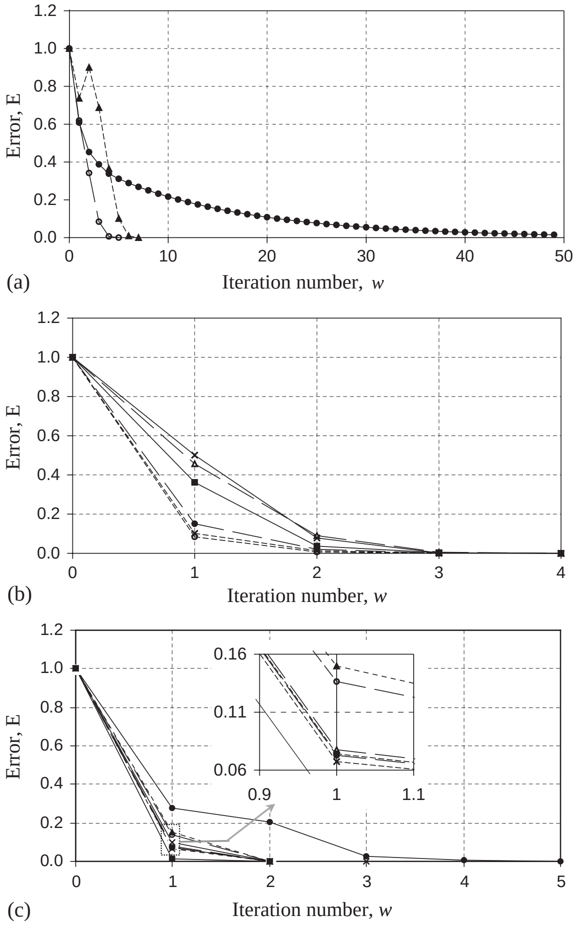

Fig. 4. Estimation errors in each iteration for the system with $N = 9$, $m = 10$, $G_{d5} = 0.5$ N/m, $G_{d8} = 0.7$ N/m, $G_{d10} = 0.8$ N/m, and $G_{di} = 1$ N/m ($i = 1, 2, 3, 4, 6, 7, 9$): (a) $p = 1$ and $n = 1, 2, 3$; (b) $p = 1$ and $n = 4, 5, \ldots, 9$; (c) $p = 2$ and $n = 1, 2, \ldots, 9$. The errors at $w = 1$ in the expanded view in (c) decrease in the order $n = 3, 2, 4, 9, 7, 8$, with the lines for $n = 5$ and 7 virtually indistinguishable. —●—, $n = 1$; —○—, $n = 2$; ···▲···, $n = 3$; – △ –, $n = 4$; – × –, $n = 5$; –■–, $n = 6$; –●–, $n = 7$; ··· × ···, $n = 8$; ···○···, $n = 9$.

under-determined. The solution given by the generalized inverse method here is the minimum norm solution [26]. Increasing the number of eigenparameters used can avoid this problem.

If the system has a large level of damage, i.e., $G_{d5} = 0.3$ N/m, $G_{d10} = 0.1$ N/m, with the other parameters unchanged, the stiffnesses of the damaged system are identified with the first order perturbations after 55 iterations when $n = 1$ and 6 iterations when $n = 2$ and 3 (Fig. 5(a)). For $n \geqslant 4$, the errors decrease monotonically and the number of iterations is reduced slightly, as shown in Fig. 5(b). With the second order perturbations the errors shown in Fig. 5(c) decrease monotonically for all n, and the number of iterations for $n = 1$ is reduced from 55 in Fig. 5(a) to 4.

Finally, consider a large order system with a large level of damage: $N = 39$, $m = 40$, $G_{d12} = 0.7$ N/m, $G_{d19} = G_{d37} = 0.1$ N/m, $G_{d28} = 0.8$ N/m, $G_{di} = 1$ N/m ($1 \leqslant i \leqslant 40$ and $i \neq 12, 19, 28, 37$), and the other parameters are the same as those in the previous example. With the first order

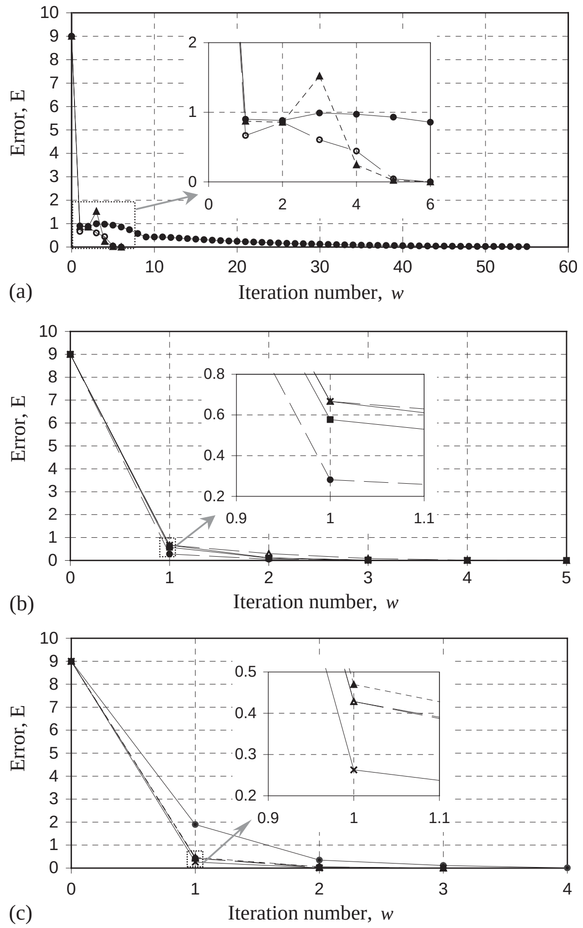

Fig. 5. Estimation errors in each iteration for the system with $G_{d5} = 0.3$ N/m, $G_{d10} = 0.1$ N/m, and all the other parameters the same as those in Fig. 2: (a) $p = 1$ and $n = 1, 2, 3$, with an expanded view for $1 \leqslant w \leqslant 6$; (b) $p = 1$ and $n = 4, 5, 6, 7$, with an expanded view near $w = 1$; (c) $p = 2$ and $n = 1, 2, \ldots, 5$. The errors at $w = 1$ in the expanded view in (c) decrease in the order $n = 3, 2, 4, 5$, with the lines for $n = 2$ and 4 virtually indistinguishable. —●—, $n = 1$; —○—, $n = 2$; ···▲···, $n = 3$; – △ –, $n = 4$; – × –, $n = 5$; – ▩ –, $n = 6$; – ● –, $n = 7$.

perturbations the exact stiffnesses are identified after 57 iterations when $n = 1$, as shown in Fig. 6(a). Using a larger number of eigenparameter pairs significantly reduces the number of iterations, as shown in Fig. 6(b). With the second order perturbations the exact stiffnesses are identified after 6 iterations when $n = 1$ and 3 iterations when $n = 2$ (Fig. 6(c)).

4.2. Fixed–fixed beam

The algorithm developed in Sections 2 and 3 is applied to detecting structural damage in an aluminum beam with fixed boundaries. The beam of length $L_t = 0.7$ m, width $W = 0.0254$ m, and

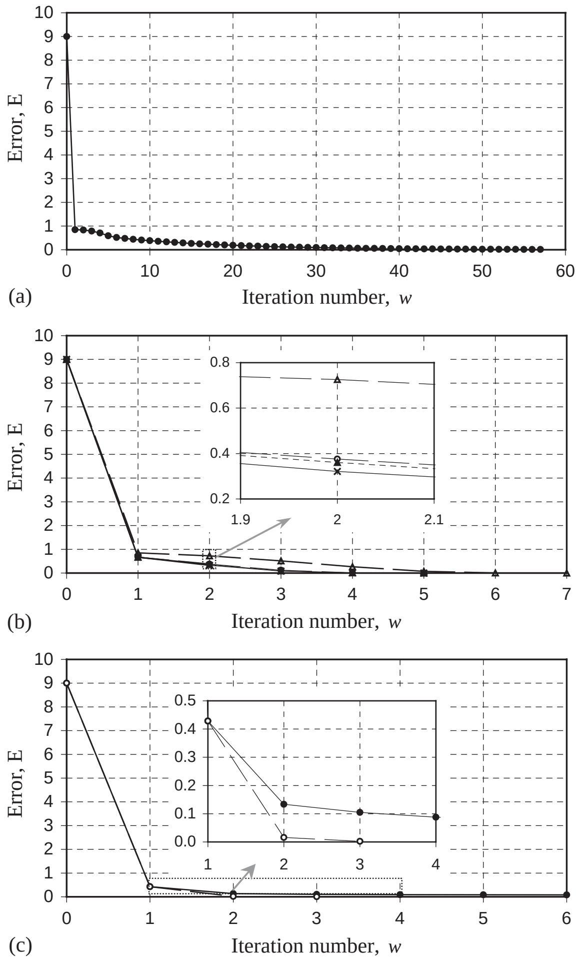

Fig. 6. Estimation errors in each iteration for the system with $N = 39$, $m = 40$, $G_{d12} = 0.7$ N/m, $G_{d19} = G_{d37} = 0.1$ N/m, $G_{d28} = 0.8$ N/m, and all the other stiffness parameters equal to 1 N/m: (a) $p = 1$ and $n = 1$; (b) $p = 1$ and $n = 2, 3, 4, 5$; (c) $p = 2$ and $n = 1, 2$, with an expanded view for $1 \leqslant w \leqslant 4$. The errors at $w = 2$ in the expanded view in (b) decrease in the order $n = 4, 2, 3, 5$. —●—, $n = 1$; −∘−, $n = 2$; ⋯▲⋯, $n = 3$; −△−, $n = 4$; −×−, $n = 5$.

thickness $H = 0.0031$ m has an area moment of inertia $I = \frac{1}{12} WH^3 = 6.3058 \times 10^{-11}$ m^4 and a mass density $\rho = 2715$ kg/m^3. The finite element method is used to model its transverse vibration. The beam is divided into N_e elements, as shown in Fig. 7, with the length of each element being $l_e = L_t/N_e$, and there are $N_e + 1$ nodes. With V_i and θ_i denoting the translational and rotational displacements at node i ($i = 1, 2, \ldots, N_e + 1$), the displacement vector of the ith ($i = 1, 2, \ldots, N_e$) element is $[V_i, \theta_i, V_{i+1}, \theta_{i+1}]^T$. The Young's modulus is assumed to be constant over each beam element and that of the ith element is denoted by G_i. The Young's modulus of the undamaged beam is $G_h = 69 \times 10^9$ N/m^2. Hence $G_{hi} = G_h$ for $i = 1, 2, \ldots, m$, where $m = N_e$. Small to large levels of damage correspond to reductions of the moduli defined in Section 4.1. The mass and

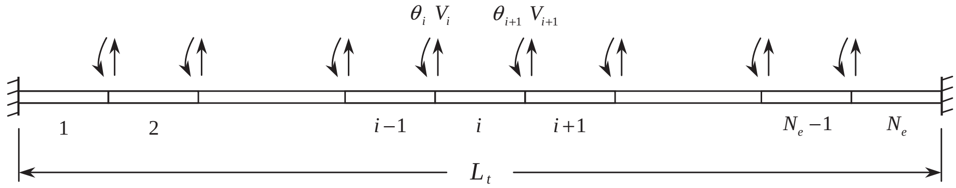

Fig. 7. Finite element model of a fixed–fixed beam. The element numbers and the displacements for the ith element are indicated.

stiffness matrices of the ith beam element are

$$\mathbf{M}_i^e = \frac{\rho W H l_e}{420}\begin{bmatrix} 156 & -22l_e & 54 & 13l_e \\ -22l_e & 4l_e^2 & -13l_e & -3l_e^2 \\ 54 & -13l_e & 156 & 22l_e \\ 13l_e & -3l_e^2 & 22l_e & 4l_e^2 \end{bmatrix}, \quad \mathbf{K}_i^e = \frac{G_i I}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}. \tag{43}$$

Using the standard assembly process yields the $2(N_e+1)\times 2(N_e+1)$ global mass and stiffness matrices. Constraining the translational and rotational displacements of the two nodes at the boundaries to zero yields the $N \times N$ $\mathbf{M}$ and $\mathbf{K}$ matrices, where $N = 2(N_e - 1)$ is the degrees of freedom of the system. The displacement vector of the system, involving the displacements of the second through N_eth node, is $[V_2, \theta_2, V_3, \theta_3, \ldots, V_{N_e}, \theta_{N_e}]^{\mathrm{T}}$. The matrix $\partial\mathbf{K}/\partial G_i$ $(i = 1, 2, \ldots, m)$ can be obtained from $\mathbf{K}$ by setting $G_i = 1$ and $G_1 = \cdots = G_{i-1} = G_{i+1} = \cdots = G_N = 0$. The parameters W_λ^k, W_ϕ^k, ε, γ, σ_i, and D are set to the same values as those in Section 4.1, and ε_G is set to $0.15G_h$ in the first two iterations and to $0.05G_h$ in the remaining iterations. The first order perturbations are used below unless indicated otherwise.

Consider first the cases with $N_e = m = 10$ and $N = 18$. When the system has a medium level of damage:

$$\mathbf{G}_d = (1, 1, 1, 1, 0.5, 1, 1, 0.7, 1, 0.8)^{\mathrm{T}} \times G_h, \tag{44}$$

the stiffness parameters of the damaged system are identified after 6 iterations with $n = 1$. When the system has a large level of damage:

$$\mathbf{G}_d = (1, 1, 1, 1, 0.3, 1, 1, 0.7, 1, 0.1)^{\mathrm{T}} \times G_h, \tag{45}$$

the stiffness parameters of the damaged system are identified after 7 iterations with $n = 1$. Consider next the cases with $N_e = 20$, 40, and 80. For the systems with medium and large levels of damage, the stiffness parameters of the first 10 elements are given by Eqs. (44) and (45), respectively, and those of the remaining elements are G_h. In all the cases the stiffness parameters of the damaged systems are identified within 10 iterations when $n = 1$. The numbers of iterations are reduced slightly when the second order perturbations are used. Note that all the degrees of freedom are measured here and the system equations are over-determined when $n = 1$.

When only the translational degrees of freedom of an eigenvector are measured, a modified eigenvector expansion method is used to estimate the unmeasured rotational degrees of freedom. To this end, ϕ_d^k is partitioned in the form $\phi_d^k = [(\phi_{dm}^k)^{\mathrm{T}}, (\phi_{du}^k)^{\mathrm{T}}]^{\mathrm{T}}$, where ϕ_{dm}^k and ϕ_{du}^k are the measured and unmeasured degrees of freedom of ϕ_d^k, respectively. Similarly, ϕ^k in Eq. (6) is partitioned in the form $\phi^k = [(\phi_m^k)^{\mathrm{T}}, (\phi_u^k)^{\mathrm{T}}]^{\mathrm{T}}$, where ϕ_m^k and ϕ_u^k correspond to the measured and unmeasured components of ϕ_d^k, respectively. Since ϕ_{dm}^k, ϕ_m^k and ϕ_u^k are known in each iteration, ϕ_{du}^k can be estimated from $\phi_{du}^k = [(\phi_m^k)^+ \phi_{dm}^k]\phi_u^k$, where the superscript + denotes generalized inverse. Once the rotational degrees of freedom of ϕ_{du}^k are determined, ϕ_d^k and ϕ^k are converted to their original forms and ϕ_d^k is mass-normalized. Since only the component equations corresponding to the measured degrees of freedom of ϕ_d^k are used, the system equations (5) and (6) are determinate when $n = 1$. Using this method the exact stiffness parameters of the damaged systems considered above can be identified. For the 10- and 20-element beams with the medium levels of damage, the stiffness parameters are identified after 5 and 23 iterations, respectively, with $n_\lambda = 2$ and $n_\phi = 1$. For the 10- and 20-element beams with the large levels of

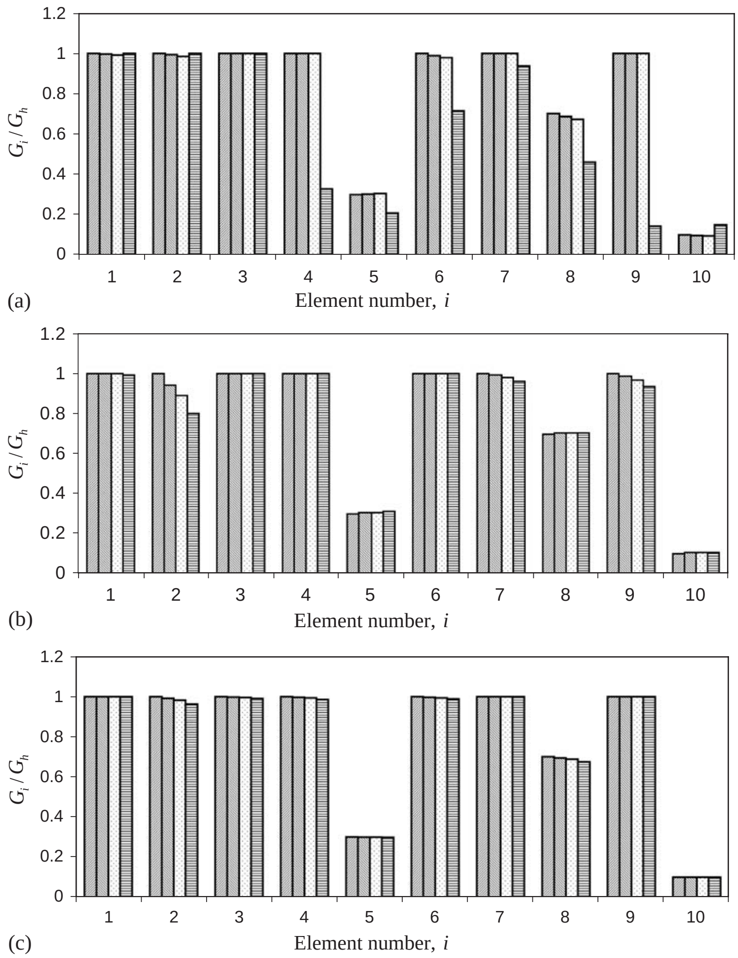

Fig. 8. Stiffness parameters identified with complete eigenvector measurements and different noise levels (▧, $\upsilon = 5\%$; □, $\upsilon = 10\%$; ▤, $\upsilon = 20\%$) for the 10-element beam with a large level of damage: (a) $n = 1$; (b) $n = 2$; and (c) $n = 3$. The stiffness parameters identified when $\upsilon = 0$ (▨) are the exact values.

damage, the stiffness parameters are identified after 9 iterations with $n_\lambda = 3$ and $n_\phi = 1$ and 10 iterations with $n = 2$ respectively.

The effects of measurement noise on the performance of the algorithm are evaluated for the 10-element beam with the large level of damage. Simulated noise, similar to that in Ref. [27], is included in the measured eigenparameters:

$$\lambda_d^k = \lambda_d^{*k} + \upsilon R_\lambda^k \lambda_d^{*k}, \quad \phi_d^k = \phi_d^{*k} + \upsilon \mathbf{R}_\phi^k \phi_d^{*k}, \tag{46}$$

where λ_d^{*k} and ϕ_d^{*k} are the kth perfect eigenvalue and eigenvector, respectively, R_λ^k is a uniformly distributed random variable in the interval $[-1, 1]$, $\mathbf{R}_\phi^k$ is a diagonal matrix whose diagonal entries are independently, uniformly distributed random variables in the interval $[-1, 1]$, and $\upsilon \in [0, 1]$ is the noise level. Note that R_λ^k and $\mathbf{R}_\phi^k$ are generated for each measured mode. Each random parameter is generated 10 times and the average is used. Three different noise levels are considered: $\upsilon = 5\%$, 10%, and 20%. When all the degrees of freedom are measured, the stiffness parameters identified with $n = 1$, 2, and 3 are shown in Fig. 8(a), (b), and (c), respectively. When only the translational degrees of freedom are measured, the eigenvector expansion method described above is used and the stiffness parameters identified with $n = 2$ and 3 are shown in Fig. 9(a) and (b), respectively. The stiffness parameters corresponding to $\upsilon = 0\%$ in Figs. 8 and 9 are the exact values. In the presence of noise the stiffness parameters can be accurately identified with an increased number of measured eigenparameters. The second order perturbations do not provide much advantage when there are errors in the measured eigenparameters.

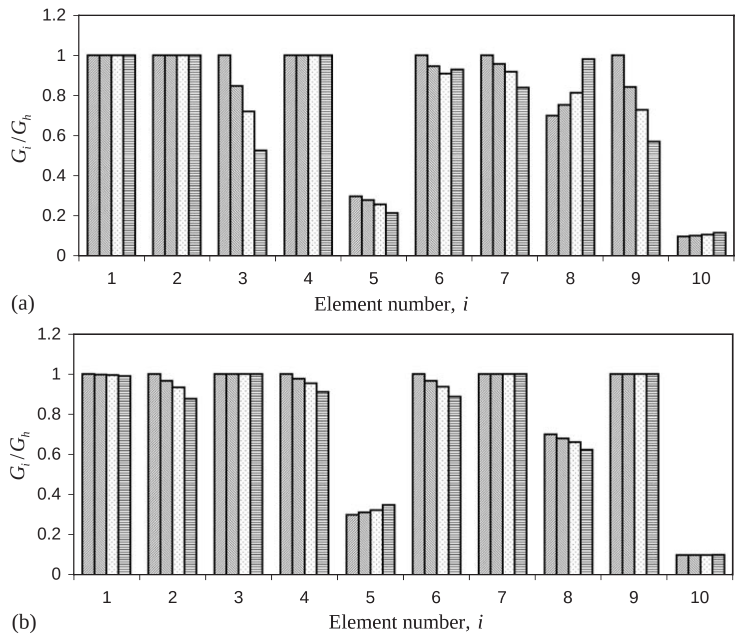

Fig. 9. Stiffness parameters identified with reduced measurements and different noise levels (▧, $\upsilon = 5\%$; □, $\upsilon = 10\%$; ▤, $\upsilon = 20\%$) for the 10-element beam with a large level of damage: (a) $n = 2$ and (b) $n = 3$. The stiffness parameters identified when $\upsilon = 0$ (▨) are the exact values.

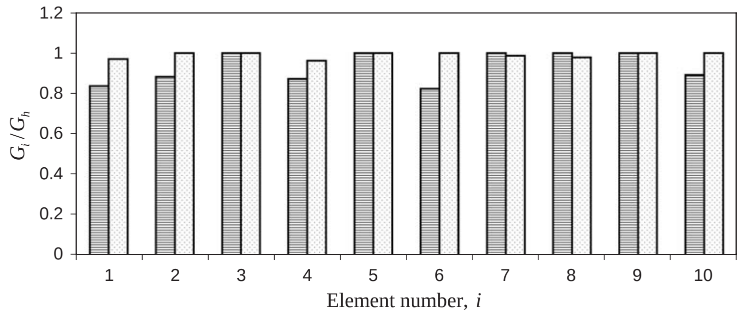

Fig. 10. Stiffness parameters identified with reduced eigenvector measurements and 20% noise for the 10-element undamaged beam: ▤, $n = 2$; □, $n = 3$.

Finally, a false-positive study is conducted. When 20% noise is included in the measured eigenparameters of the 10-element undamaged beam and only the translational degrees of freedom are measured, the stiffness parameters identified with $n = 2$ and 3 are shown in Fig. 10. The maximum error in the identified stiffness parameters is much smaller than the noise level when $n = 3$.

5. Concluding remarks

The sensitivities of eigenparameters of all orders are derived for the first time using a multiple-parameter, general order perturbation method. The higher order solutions are used to estimate the changes in the eigenparameters with large changes in the stiffness parameters. The perturbation method is combined with an optimization method to form an iterative damage detection algorithm. Numerical simulations show that with a small number of measured eigenparameter pairs, the stiffness parameters of the damaged systems can be accurately identified. By linking the damage detection algorithm to a standard finite element code, the method can be applied to large operational structures. It can detect a large level of damge with severe mismatch between the eigenparameters of the damaged and undamaged structures.

Acknowledgements

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