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Journal of Sound and Vibration 273 (2004) 429–440

JOURNAL OF SOUND AND VIBRATION

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Letter to the Editor

On the semi-discretization method for feedback control design of linear systems with time delay

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Received 13 January 2003; accepted 24 July 2003

1. Introduction

Time delay can be due to the delayed system dynamics, and can also be due to the signal transport delay in control systems. Effects of time delay on the stability and performance of control systems have been a subject of many studies. Yang and Wu [1] and Stepan [2] have studied structural systems with time delay. In Ref. [3], a time delay filter is developed to design a fuel/time optimal control. The time delay feedback control is designed in Ref. [4] to regulate the librational motion of gravity-gradient satellites in an elliptic orbit. Vibration suppression using delayed resonator has been studied by Olgac and Holm-Hansen [5] and Filipovic and Olgac [6]. A study on stability and performance of feedback controls with multiple time delays is reported in Ref. [7] by considering the roots of the closed-loop characteristic equation. For linear time-invariant (LTI) systems with time delay, methods such as root locus and Nyquist criterion are well-established. A survey of more advanced stability analysis for delayed linear systems is presented in Ref. [8].

Delayed control systems in the continuous time domain have been extensively studied by fully discretizing the system in the time domain. Cai and Huang [9] studied an optimal vibration controller with a delayed feedback for a building model where they used standard discretization techniques. Pinto and Goncalves [10] fully discretized a non-linear SDOF system to study control problems with time delay. Klein and Ramirez [11] studied MDOF delayed optimal regulator controllers with a hybrid discretization technique where they partitioned the state equation into discrete and continuous portions. Recently, Insperger and Stepan extended the method of semi-discretization, which is common in structural dynamics and fluid mechanics [12,13], to the delayed ordinary differential equations and demonstrated the powerful aspects of the method over the full discretization approach [14,15]. The merit of the semi-discretization method lies in that it makes use of the exact solution of linear systems over a short time interval to construct the mapping of a finite dimensional state vector for the system with time delay.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.06.023

The objective of this paper is to investigate ways to further improve the accuracy of the solutions obtained by the semi-discretization method due to Insperger and Stepan. We shall consider the stability and performance analysis of feedback controls of time-invariant and periodic linear systems with time delay. The computational efficiency and accuracy of the proposed improvement is demonstrated in the numerical examples.

The paper is organized as follows. Section 2 describes the semi-discretization method, and the proposed ways to improve its accuracy. In Section 3 we discuss measures for comparison to benchmark the proposed improvement. In Section 4, two examples are presented to demonstrate the effectiveness of the improvement.

2. Method of semi-discretization

Consider a second order periodic system with time delay under a PD control,

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + a_2(t)x(t) = -k_p x(t-\tau) - k_d \dot{x}(t-\tau), \tag{1}$$

where $x(t)$ is the system response, the coefficients $a_1(t)$ and $a_2(t)$ are periodic functions of time with period T, τ is a constant time delay, and k_p and k_d are the proportional and derivative gains. The following discussion is applicable to higher order linear systems as well as to PID controls. For brevity, we restrict ourselves only to the PD control of the second order system.

Because of the time delay, the state vector of this simple system is no longer just $(x(t), \dot{x}(t))$, but $(x(t), \dot{x}(t), x(t-\tau_1), \dot{x}(t-\tau_1))$ for all $0<\tau_1 \leqslant \tau$, which has an infinite dimension.

Let us discretize the period T into an integer k intervals of length Δt such that $T = k\Delta t$. For the sake of simplicity, we assume that the time delay $\tau = n\Delta t$, where n is an integer. When an integer n cannot be found, discretization of the time delay τ will be approximate. Details on how to treat this case can be found in Refs. [14,15]. We introduce the following notations:

$$\begin{aligned} &x(t_i - \tau) = x((i-n)\Delta t) = x_{i-n}, \quad \dot{x}(t_i - \tau) = \dot{x}((i-n)\Delta t) = \dot{x}_{i-n}, \\ &a_1(t_i) = a_{1i}, \quad a_2(t_i) = a_{2i}, \quad -k_d\dot{x}_{i-n} - k_p x_{i-n} \equiv f_{i-n} \end{aligned} \tag{2}$$

and distinguish the following three approximations.

2.1. Zeroth order approximation

Consider Eq. (1) in a time interval $t \in [t_i, t_{i+1}]$ where $t_i = i\Delta t,\ i = 0, 1, 2, \ldots, k$. According to Refs. [14,15], the system is approximated by the following time-invariant linear equation with a constant excitation due to the delayed response:

$$\ddot{x}(t) + a_{1i}\dot{x}(t) + a_{2i}x(t) = f_{i-n}, \quad t \in [t_i, t_{i+1}]. \tag{3}$$

We shall refer to the solution of this system as the zeroth order approximation. Note that a_{1i}, a_{2i} and f_{i-n} are the corresponding functions evaluated at the beginning of the time interval t_i and the effect of the terminal values over the discrete time interval is ignored.

2.2. Improved zeroth order approximation

To search for improvements to the above solution, we consider the following time-invariant approximation of Eq. (1) over the time interval $[t_i, t_{i+1}]$:

$$\ddot{x}(t) + \tilde{a}_{1i}\dot{x}(t) + \tilde{a}_{2i}x(t) = \tilde{f}_{i-n}, \quad t \in [t_i, t_{i+1}], \tag{4}$$

where

$$\begin{aligned} &\tilde{a}_{1i} = \delta_1 a_{1i} + (1-\delta_1)a_{1i+1}, \quad \tilde{a}_{2i} = \delta_2 a_{2i} + (1-\delta_2)a_{2i+1}, \\ &\tilde{f}_{i-n} = \delta_3 f_{i-n} + (1-\delta_3) f_{i-n+1}, \end{aligned} \tag{5}$$

and where δ_m $(m = 1, 2, 3)$ are constants in the interval between 0 and 1.

A simple improvement can be achieved if we choose $\delta_m = \frac{1}{2}$ $(m = 1, 2, 3)$, where we have the mid-point approximation and $\tilde{a}_{1i}$, $\tilde{a}_{2i}$, and $\tilde{f}_{i-n}$ are the averages of the initial and terminal values. The mid-point approximation is still zeroth order. However, it involves the terminal value of functions over the time interval. We will refer to the solutions with the mid-point approximation as the improved zeroth order solution. Note that, when $\delta_m = 1$ we obtain the zeroth order approximation.

2.3. First order approximation

When $\tilde{f}_{i-n}$ is taken as a linear function of time over the discretized interval as

$$\tilde{f}_{i-n} = f_{i-n} + \frac{(f_{i-n+1} - f_{i-n})}{\Delta t}(t - t_i), \quad t \in [t_i, t_{i+1}] \tag{6}$$

and $\tilde{a}_{1i}$ and $\tilde{a}_{2i}$ remain the average of the initial and terminal values with $\delta_m = \frac{1}{2}$ $(m = 1, 2)$, we refer to this as the first order approximation. Our discussions will be centered around these three approximation schemes.

The exact solutions of the time invariant linear system (4) with any one of the three approximation schemes are obtainable in closed form, and lead to the relationship

$$\begin{Bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{Bmatrix} = \begin{bmatrix} \alpha_{1i} & \alpha_{2i} \\ \beta_{1i} & \beta_{2i} \end{bmatrix} \begin{Bmatrix} x_i \\ \dot{x}_i \end{Bmatrix} + \begin{Bmatrix} \alpha_{3i} & \alpha_{4i} \\ \beta_{3i} & \beta_{4i} \end{Bmatrix} \begin{Bmatrix} f_{i-n} \\ f_{i-n+1} \end{Bmatrix}. \tag{7}$$

We provide the explicit expressions of the coefficients α's and β's when $\tilde{a}_1^2 \neq 4\tilde{a}_2$ in Table 1 where $\kappa = \sqrt{\tilde{a}_1^2 - 4\tilde{a}_2}$, $\Lambda_1 = \exp(-\frac{1}{2}\Delta t\,(\tilde{a}_1 + \kappa))$ and $\Lambda_2 = \exp(-\frac{1}{2}\Delta t\,(\tilde{a}_1 - \kappa))$.

Define an $(n+2)$ dimensional state vector as

$$\mathbf{y}_i = \{\dot{x}_i \;\; x_i \;\; f_{i-1} \;\; \cdots \;\; f_{i-n+1} \;\; f_{i-n}\}^{\mathrm{T}}. \tag{8}$$

A mapping of $\mathbf{y}_i$ over the interval $[t_i, t_{i+1}]$ can be obtained as

$$\mathbf{y}_{i+1} = \mathbf{A}_i \mathbf{y}_i, \tag{9}$$

Table 1
Zeroth and first order approximation coefficients. When $\delta_3 = \frac{1}{2}$, it corresponds to the improved zeroth order approximation

	Zeroth order	First order
α_1	$\frac{\Lambda_1(\kappa - \tilde{a}_1) + \Lambda_2(\tilde{a}_1 + \kappa)}{2\kappa}$	$\frac{\Lambda_1(\kappa - \tilde{a}_1) + \Lambda_2(\tilde{a}_1 + \kappa)}{2\kappa}$
α_2	$\frac{-\Lambda_1 + \Lambda_2}{\kappa}$	$\frac{-\Lambda_1 + \Lambda_2}{\kappa}$
α_3	$\frac{\delta_3(-2\kappa + (\kappa + \tilde{a}_1)\Lambda_2 + (\kappa - \tilde{a}_1)\Lambda_1)}{2\kappa\tilde{a}_2}$	$\frac{2\tilde{a}_1\kappa + (\tilde{a}_2((\tilde{a}_1 - \kappa)\Delta t - 2) + \tilde{a}_1^2 - \tilde{a}_1\kappa)\Lambda_1 - (\tilde{a}_2(\Delta t(\tilde{a}_1 + \kappa) - 2) + \tilde{a}_1^2 + \tilde{a}_1\kappa)\Lambda_2}{2\tilde{a}_2^2\Delta t\kappa}$
α_4	$\frac{(1 - \delta_3)(-2\kappa + (\kappa + \tilde{a}_1)\Lambda_2 + (\kappa - \tilde{a}_1)\Lambda_1)}{2\kappa\tilde{a}_2}$	$\frac{-2\tilde{a}_1\kappa + (-\tilde{a}_1^2 + \tilde{a}_1\kappa + 2\tilde{a}_2)\Lambda_1 + (\tilde{a}_1^2 - 2\tilde{a}_2 + \tilde{a}_1\kappa)\Lambda_2}{\tilde{a}_2^2\Delta t\kappa}$
β_1	$\frac{\tilde{a}_2(\Lambda_1 - \Lambda_2)}{\kappa}$	$\frac{\tilde{a}_2(\Lambda_1 - \Lambda_2)}{\kappa}$
β_2	$\frac{\Lambda_1(\kappa + \tilde{a}_1) + \Lambda_2(\kappa - \tilde{a}_1)}{2\kappa}$	$\frac{\Lambda_1(\kappa + \tilde{a}_1) + \Lambda_2(\kappa - \tilde{a}_1)}{2\kappa}$
β_3	$\frac{\delta_3(\Lambda_2 - \Lambda_1)}{\kappa}$	$\frac{\Lambda_1(\kappa - 2\tilde{a}_2\Delta t - \tilde{a}_1) + \Lambda_2(\tilde{a}_1 + 2\tilde{a}_2\Delta t + \kappa) - 2\kappa}{2\tilde{a}_2\Delta t\kappa}$
β_4	$\frac{(1 - \delta_3)(\Lambda_2 - \Lambda_1)}{\kappa}$	$\frac{2\kappa + \Lambda_1(\tilde{a}_1 - \kappa) - \Lambda_2(\tilde{a}_1 + \kappa)}{2\tilde{a}_2\Delta t\kappa}$

where the transition matrix is

$$\mathbf{A}_i = \begin{bmatrix} \beta_{2i} & \beta_{1i} & 0 & 0 & \cdots & \beta_{4i} & \beta_{3i} \\ \alpha_{2i} & \alpha_{1i} & 0 & 0 & \cdots & \alpha_{4i} & \alpha_{3i} \\ -k_d & -k_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{10}$$

The mapping of the state vector over one period $T = k\Delta t$ is therefore

$$\mathbf{y}_{j+1} = \mathbf{\Phi}\mathbf{y}_j, \tag{11}$$

where the mapping matrix $\mathbf{\Phi}$ is given by

$$\mathbf{\Phi} = \mathbf{A}_{k-1}\mathbf{A}_{k-2}\cdots\mathbf{A}_1\mathbf{A}_0. \tag{12}$$

The index j ($j = 0, 1, \ldots$) refers to the number of periods, i.e. $\mathbf{y}_j$ is the state vector at the beginning of the jth period. This is the essence of the method of semi-discretization, regardless of the approximation scheme adopted. The stability of the system is determined by the eigenvalues of $\mathbf{\Phi}$. Let $\lambda_{\max}$ denote the largest absolute value of eigenvalues of the matrix $\mathbf{\Phi}$. When $\lambda_{\max} < 1$, $\mathbf{\Phi}$ is a contraction, and the system is asymptotically stable. The stability boundary is given by $\lambda_{\max} = 1$.

Note that as Δt gets smaller the accuracy of $\mathbf{\Phi}$ improves. On the other hand, the shorter the time intervals, the larger the dimension of $\mathbf{\Phi}$, resulting in the need for more computational effort. In the following sections, we demonstrate how each approximation scheme affects the computational accuracy and efficiency of the semi-discretization method.

3. Comparison measures

3.1. Stability bounds of control gains

To compare the accuracy of the three approximation schemes, we need a measure. Since the exact solutions for periodic systems are not available, we shall consider the following LTI system, for which we can obtain exact stability bounds of the control gains:

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = -k_d\dot{x}(t-\tau) - k_p x(t-\tau), \tag{13}$$

where ζ is the damping ratio, and ω is the natural frequency. The characteristic equation of the closed-loop system is

$$s^2 + 2\zeta\omega s + \omega^2 + k_d s \exp(-\tau s) + k_p \exp(-\tau s) = 0, \tag{14}$$

where s is the Laplace variable. The roots of Eq. (14) are the closed-loop poles. By studying the stability of the closed-loop poles, we can find the exact ranges of the control gains k_d and k_p that stabilize the system. Let k_d^* and k_p^* be a pair of control gains within the stability boundary of the controlled system. Keeping either of the control gains constant, we can determine the upper and lower limits of the other control gain that renders the system marginally stable. We label these exact gains as k_d^u, k_d^l, k_p^u and k_p^l, where the superscript u and l, respectively, stands for upper and lower bounds.

Because the system in Eq. (13) is autonomous, we can arbitrarily select a period $T > \tau$ to construct the mapping $\mathbf{\Phi}$. By fixing one of k_d^* and k_p^* in turn and varying the other, we can obtain an approximate value for the upper and lower bound of the control gains crossing the stability boundary defined by $\lambda_{\max} = 1$. These approximate gains corresponding to the exact ones k_d^u, k_d^l, k_p^u and k_p^l are denoted as $\tilde{k}_d^u$, $\tilde{k}_d^l$, $\tilde{k}_p^u$ and $\tilde{k}_p^l$.

We use the following root mean square error as a measure of accuracy of the semi-discretization method:

$$k_{er} = \tfrac{1}{2}\sqrt{(k_d^u - \tilde{k}_d^u)^2 + (k_d^l - \tilde{k}_d^l)^2 + (k_p^u - \tilde{k}_p^u)^2 + (k_p^l - \tilde{k}_p^l)^2}. \qquad (15)$$

3.2. Optimal feedback gains

Eq. (11) indicates that the smaller $\lambda_{\max}$ of $\mathbf{\Phi}$ is, the faster the system converges to zero. $\lambda_{\max}$ can be considered as a measure of the control performance. If we restrict our interest in a finite region in the parametric space (k_p, k_d) where the system is stable, we can find an optimal pair of control gains (k_p, k_d) in the region to minimize $\lambda_{\max}$. This leads to the following optimization problem for control gains:

$$\min_{k_p,k_d} \,[\max(|\lambda(\mathbf{\Phi})|)] \quad \text{subject to } \lambda_{\max} < 1. \qquad (16)$$

This optimization offers a different approach to the design of feedback controls for linear systems with time delay. The control performance criterion is the decay rate of the mapping $\mathbf{\Phi}$ over one period.

We can also study the effects of the three approximation schemes on the optimal gains. The convergence of the control gains and $\lambda_{\max}$ as a function of discretization level offers a qualitative measure for comparison, and will be considered hereafter.

3.3. Response in time domain

Finally, we simulate the system response and compare the decay rate of the response to that predicted by the semi-discretization method with different approximation schemes. In the numerical examples, we examine the decay rate of the L_2 norm of the state vector $\mathbf{y}$. This comparison is amenable to both time-invariant and periodic systems.

4. Numerical examples

4.1. Time-invariant system

We first consider a second order autonomous system defined in Eq. (13) with $\zeta = 0.05$, $\omega = 2$ and $\tau = \pi/2$. We have selected a period $T = \pi > \tau$ to construct the mapping.

Table 2
Exact and approximate stability bounds of control gains with varying discretization levels

Discretization level	Solution method	$k_p = -0.1356$		$k_d = -0.3898$		Error
		Lower	Upper	Lower	Upper	k_{er}
	Exact	−1.75862692	0.19715358	−1.85247965	1.364948976	n/a
10	Zeroth	−1.556284	0.1918105	−1.5558121	1.4998466	1.9182036e − 1
	Improved	−1.792398	0.19921478	−1.84305351	1.41216126	2.9421873e − 2
	First order	−1.7935024	0.19887852	−1.8705159	1.37106345	1.9887050e − 2
20	Zeroth	−1.6557696	0.1938158	−1.69536752	1.43163415	9.9651774e − 2
	Improved	−1.7635742	0.19733034	−1.8401601	1.38495437	1.2005144e − 2
	First order	−1.7672229	0.19758328	−1.85698090	1.36647265	4.9157433e − 3
40	Zeroth	−1.7068267	0.19531915	−1.77153523	1.39809059	5.0835515e − 2
	Improved	−1.7581437	0.19703111	−1.84436498	1.37407107	6.1096020e − 3
	First order	−1.7607684	0.19726091	−1.85360447	1.36532958	1.2255089e − 3
60	Zeroth	−1.7240170	0.1958938	−1.79796493	1.3869977	3.4122699e − 2
	Improved	−1.7576467	0.1970252	−1.84662891	1.37083623	4.1793607e − 3
	First order	−1.7595781	0.19720128	−1.85297953	1.36511811	5.4439084e − 4
80	Zeroth	−1.732641	0.1961951	−1.81138360	1.38146835	2.5680546e − 2
	Improved	−1.7576447	0.19703972	−1.84792522	1.36929181	3.1851518e − 3
	First order	−1.75916182	0.19718041	−1.85276083	1.36504411	3.0616550e − 4
100	Zeroth	−1.7378257	0.196380	−1.81950120	1.37815617	2.0586993e − 2
	Improved	−1.7577224	0.1970541	−1.84875598	1.368388439	2.5750569e − 3
	First order	−1.7589692	0.19717075	−1.85265961	1.365009861	1.9593027e − 4

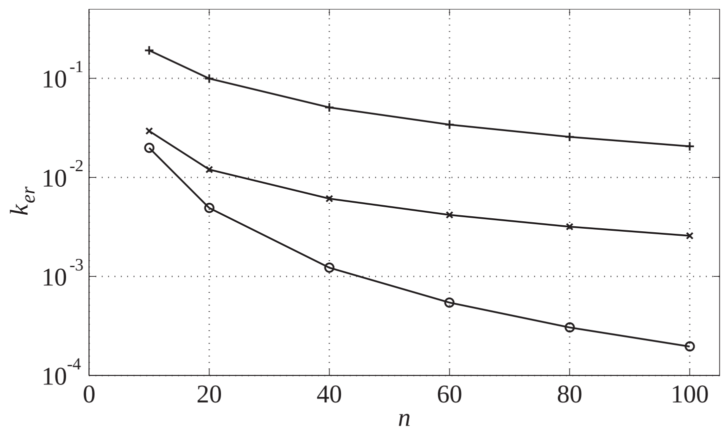

Fig. 1. Variation of the control gain error k_{er} with discretization level n. $-+-+-$, zeroth order; $-\times-\times-$, improved zeroth order; –◦–◦–, first order.

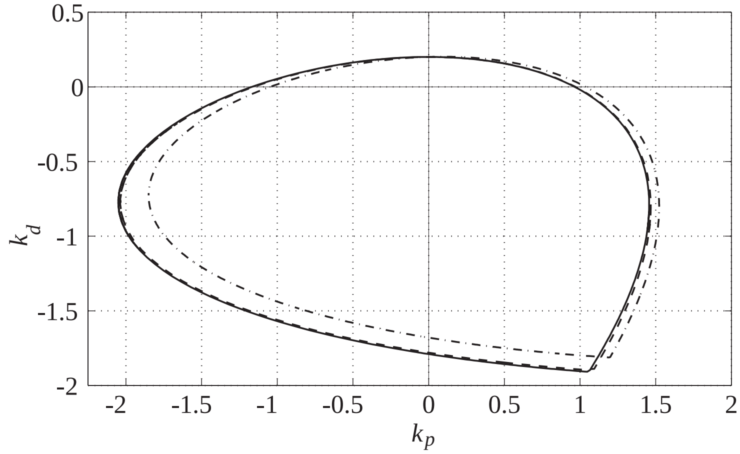

Fig. 2. Stability boundaries of the second order linear time-invariant system with time delay. $\tau = \pi/2$ and $n = 20$: $- \cdot - \cdot -$, zeroth order approximation: $- \; - \; -$, improved zeroth order approximation; -----, first order approximation.

Table 3
Optimal control gains and the largest absolute value of eigenvalues of $\mathbf{\Phi}$ with varying discretization levels

Discretization level	Approximation	Optimal gains k_p	k_d	$\lambda_{\max}$
20	Zeroth	−2.016893	−0.3090877	0.00155351
	Improved zeroth	−2.035019	−0.2839209	0.00344162
	First order	−2.034412	−0.2832894	0.00335362
40	Zeroth	−2.02756	−0.2973511	0.00222317
	Improved zeroth	−2.034192	−0.2830767	0.00336122
	First order	−2.034040	−0.2829187	0.00333904
60	Zeroth	−2.029776	−0.292568	0.00265072
	Improved zeroth	−2.034037	−0.2829196	0.0033463
	First order	−2.033967	−0.2828478	0.0033365
80	Zeroth	−2.030838	−0.2901451	0.0028373
	Improved zeroth	−2.033984	−0.2828655	0.0033410
	First order	−2.033947	−0.2828261	0.0033354
100	Zeroth	−2.031465	−0.2886839	0.0029433
	Improved zeroth	−2.033959	−0.2828401	0.0033386
	First order	−2.033935	−0.282815	0.0033350
120	Zeroth	−2.031879	−0.2877072	0.0030119
	Improved zeroth	−2.033946	−0.2828265	0.0033372
	First order	−2.033929	−0.2828085	0.00333479

In Table 2, we present the solutions for the upper and lower stability bounds of the control gains with different discretization levels. We used $k_d^* = -0.1356$ and $k_p^* = -0.3898$, the optimal control gains by the zeroth order approximation with $n = 20$. These solutions are compared with the exact values. The results in the table are also plotted in Fig. 1. The figure shows that the convergence of the first order approximation is far superior to that of the zeroth order approximations. At $n = 10$, its error is comparable to that of the zeroth order approximation at $n = 100$. Since the dimension of the matrix $\mathbf{A}_i$ is $(n+2) \times (n+2)$, and $\mathbf{\Phi}$ is a product of $k > n$

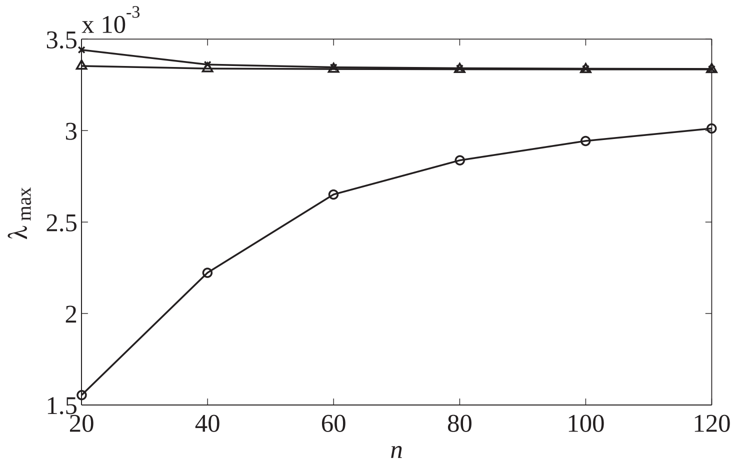

Fig. 3. Variation of the largest absolute value of eigenvalues of $\mathbf{\Phi}$ with discretization level n. –◦–◦–, zeroth order; $-\times-\times-$, improved zeroth order; –Δ–Δ–, first order approximation.

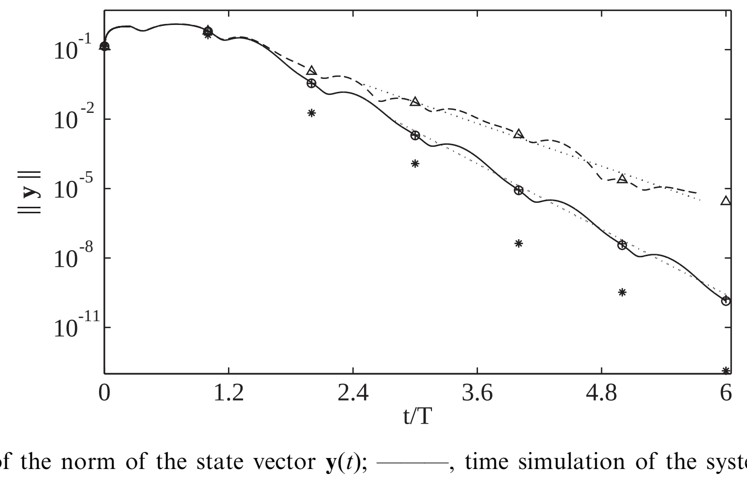

Fig. 4. Time history of the norm of the state vector $\mathbf{y}(t)$; ———, time simulation of the system with optimal gains computed by the first order approximation; $(k_p, k_d) = (-2.03441, -0.28329)$ and $\lambda_{\max} = 3.3536\text{e}-3$. Corresponding mapping by the improved zeroth order approximation (+) and by the first order approximation (◦). — — — —, Time simulation of the system with optimal gains computed by the zeroth order approximation; $(k_p, k_d) = (-2.01689, -0.30909)$ and $\lambda_{\max} = 1.5535\text{e}-3$. Corresponding mapping by the zeroth order approximation (∗), and mapping by the first order approximation (Δ). ⋯⋯, The logarithmic curve fit. In each of the mappings $n = 20$ is used.

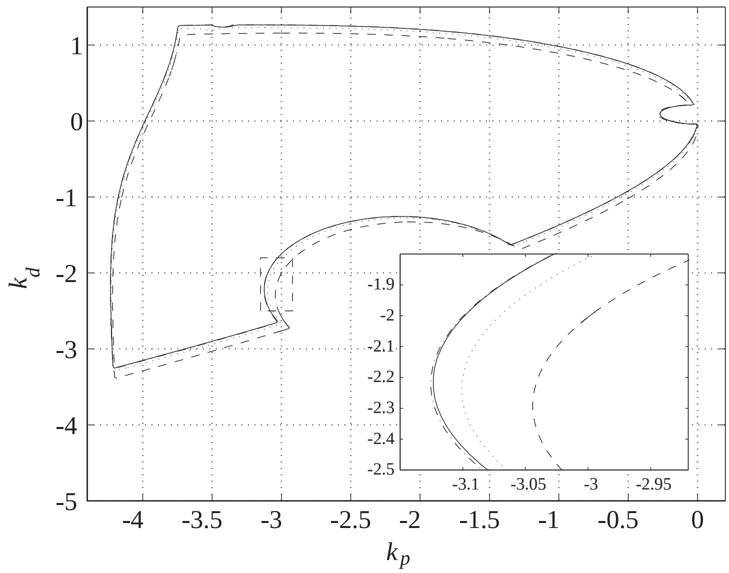

Fig. 5. Stability boundary in (k_p, k_d) plane, a section of the graph is enlarged to show the detail; ———, first order approximation with $n = 12$; $- \cdot - \cdot -$, improved zeroth order approximation with $n = 12$; - - - -, zeroth order approximation with $n = 12$; $\cdots\cdots$, zeroth order approximation with $n = 40$.

matrices $\mathbf{A}_i$, the computational effort to form $\mathbf{\Phi}$ is proportional to $(n+2) \times (n+2) \times k \sim O(n^3)$. Thus, the first order approximation provides about 1000 fold computational efficiency increase as compared to the zeroth order scheme. The increase in the computational efficiency significantly speeds up the optimization solution process, which involves repeated calculations of $\mathbf{\Phi}$ and its eigenvalues. Fig. 1 also shows that for a given discretization level the accuracy of the first order approximation is higher than those of the zeroth order approximations.

Fig. 2 shows the effect of the three approximation schemes on the stability boundary in the control gain space. The stable region is inside the closed curve. For $n = 20$, we note that there is a slight difference between the improved zeroth order solution and the first order solution. On the other hand, the difference between the stability boundary predicted by the first order approach and that by the zeroth order is considerably bigger. We can conclude that the formulations involving the terminal values essentially improve accuracy. The stability boundary obtained by the zeroth order approximation approaches that by the first order approximation as n increases beyond 40, and deviations increase when n gets smaller.

4.2. Periodic system

The strength of the semi-discretization method lies in its ability to handle periodic systems. To demonstrate this point, we consider the classical Mathieu equation with a delayed feedback control,

$$\ddot{x}(t) + (\delta + 2\varepsilon \cos 2t)x(t) = -k_d\dot{x}(t-\tau) - k_p x(t-\tau), \tag{17}$$

where $\varepsilon = 1$, $\delta = 4$, and the period of the system is $T = \pi$. We assume a time delay $\tau = \pi/4$. The uncontrolled system is parametrically unstable.

Table 3 shows the optimal feedback gains and associated largest absolute value $\lambda_{\max}$ of eigenvalues of $\mathbf{\Phi}$. The variation of $\lambda_{\max}$ with discretization level is depicted in Fig. 3. The solutions obtained by the first order and improved zeroth order approximation converge faster than that by the zeroth order approximation. Fig. 4 shows the time history of the norm of the state vector $\mathbf{y}(t)$. The figure validates that the decay rate, characterized by $\lambda_{\max}$, obtained by the first order and improved zeroth order approximation do converge to the exact ones.

Finally, we present the stability boundaries of the control gains with $\lambda_{\max} = 1$ using different approximations in Fig. 5. The shape of the stability region is more complex than that of the time-invariant system. The irregular geometry is a reflection of the complex behavior of the periodic system. This figure demonstrates again that the proposed first order and improved zeroth approximations improve the accuracy and efficiency of the semi-discretization method even for periodic systems.

5. Concluding remarks

The semi-discretization method is efficient and accurate for analysis and control design of time-invariant and periodic linear systems with time delay. Three approximation schemes in conjunction to the semi-discretization method have been studied in this paper. Different measures for comparison of the approximation schemes have been considered. Extensive numerical results show that the proposed first order and improved zeroth order approximations increase the efficiency and accuracy of the method as compared to the zeroth order approximation.

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