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Journal of Sound and Vibration 273 (2004) 451–455
JOURNAL OF SOUND AND VIBRATION
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Letter to the Editor

Improved approximate formulas for the natural frequencies of simply supported Bernoulli–Euler beams with rotational restrains at the ends

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Received 18 March 2003; accepted 2 August 2003

1. Introduction

An almost semi-centennial formula by Newmark and Veletsos [1] for the determination of natural frequencies of simply supported Bernoulli–Euler beams with rotational restrains at the ends has been discussed recently in Ref. [2] by Maurizi et al. Liu [3] has also established a simplified formula for restrained cantilever beams. The simplified formula is useful, for example, in assisting the structural engineers to get a quick estimation of the natural frequencies in the preliminary design stage. In Ref. [2], the maximum relative errors of the formula by Newmark and Veletsos were found to be around 2.5% for the fundamental frequency and lower for higher frequencies. In this letter, improved formulas with reduced relative errors are given.

Consider a simply supported beam with modulus of flexural rigidity EI, mass density per unit length ρA and span length between supports L. The two ends are simply supported and restrained by two rotational springs with linear stiffness K_{r_1} and K_{r_2}. An end is hinged if $K_r = 0$ and clamped if $K_r \to \infty$.

It can be shown that the exact nth circular natural frequency ω_n for the uniform Bernoulli–Euler beam is given by

$$\omega_n = \lambda_n^2 \sqrt{\frac{EI}{\rho A L^4}}, \tag{1}$$

where λ_n is the non-dimensional frequency parameter and is the nth non-zero root of the following transcendental equation (e.g. Refs. [2,4,5]):

$$2R_1 R_2 \varphi_1(\lambda)\lambda^2 + (R_1 + R_2)\varphi_6(\lambda)\lambda - \varphi_4(\lambda) = 0 \tag{2}$$

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.08.047

with

$$R_1 = EI/(K_{r_1}L), \quad R_2 = EI/(K_{r_2}L), \quad \varphi_1(\lambda) = \sin(\lambda)\sinh(\lambda), \tag{3a}$$

$$\varphi_4(\lambda) = \cos(\lambda)\cosh(\lambda) - 1 \quad \text{and} \quad \varphi_6(\lambda) = \sin(\lambda)\cosh(\lambda) - \sinh(\lambda)\cos(\lambda). \tag{3b}$$

The determination of the roots from Eq. (2) for various values of R_1 and R_2 could be clumsy. It is desirable to have a simple but yet accurate formula for the evaluation of the natural frequencies directly.

In Ref. [1], λ_n is effectively approximated by

$$\lambda_n = \pi\sqrt{\left(n + \frac{1}{2}\left(\frac{\beta_1}{5n+\beta_1}\right)\right)\left(n + \frac{1}{2}\left(\frac{\beta_2}{5n+\beta_2}\right)\right)}, \tag{4}$$

where $\beta_1 = 1/R_1$ and $\beta_2 = 1/R_2$.

Maurizi et al. [2] have verified that the above formula is accurate with around 2% in relative errors and slightly larger in absolute errors. In the following, a more accurate formula is proposed.

2. Simplified equation

Since the natural frequencies of the restrained beam must lie between the natural frequencies of a hinged–hinged beam (with $\lambda_n = n\pi$) and a clamped–clamped beam, it is permissible to express λ_n as

$$\lambda_n = n\pi + \varepsilon_n \tag{5}$$

with $\varepsilon_n \geqslant 0$ and $n = 1, 2, \ldots$.

In this case, the following simplifications can be made:

$$\sin(\lambda_n) = (-1)^n \sin(\varepsilon_n), \quad \cos(\lambda_n) = (-1)^n \cos(\varepsilon_n), \tag{6a}$$

$$\sinh(\lambda_n) \approx \exp(\lambda_n)/2, \quad \cosh(\lambda_n) \approx \exp(\lambda_n)/2. \tag{6b}$$

The approximations in Eq. (6b) are very reasonable as $\exp(\lambda_n)$ is much bigger than $\exp(-\lambda_n)$. For example, when $\lambda = \pi, \exp(\lambda)/\exp(-\lambda) \approx 535$.

As a result, substituting Eq. (6) into Eq. (2), one has

$$(-1)^n(2R_1R_2\sin(\varepsilon_n)\lambda_n^2 + (R_1+R_2)(\sin(\varepsilon_n) - \cos(\varepsilon_n))\lambda_n - \cos(\varepsilon_n))\frac{\exp(\lambda_n)}{2} + 1 = 0. \tag{7}$$

Again, as $\exp(\lambda_n)$ should be much bigger than unity especially for large n, Eq. (7) could be approximated as

$$2R_1R_2\sin(\varepsilon_n)\lambda_n^2 + (R_1+R_2)(\sin(\varepsilon_n) - \cos(\varepsilon_n))\lambda_n - \cos(\varepsilon_n) = 0. \tag{8}$$

Rearranging the terms, ε_n can be evaluated as

$$\varepsilon_n = \tan^{-1}\left(\frac{(R_1+R_2)\lambda_n + 1}{2R_1R_2\lambda_n^2 + (R_1+R_2)\lambda_n}\right). \tag{9}$$

As λ_n is not known initially, it can be approximated by $n\pi$ in Eq. (9). Hence, λ_n in Eq. (5) could be approximated by

$$\begin{aligned}\lambda_n &= n\pi + \tan^{-1}\left(\frac{(R_1+R_2)n\pi+1}{2R_1R_2n^2\pi^2+(R_1+R_2)n\pi}\right)\\ &= n\pi + \tan^{-1}\left(\frac{(\beta_1+\beta_2)n\pi+\beta_1\beta_2}{2n^2\pi^2+(\beta_1+\beta_2)n\pi}\right).\end{aligned} \tag{10}$$

The complexity of Eq. (10) is similar to Eq. (4). Both formulas give the exact values for simply supported beams (i.e., $\lambda_n = n\pi$ when $\beta_1 = \beta_2 = 0$). Table 1 shows the λ_n for the first five natural frequencies of various elastically restrained beam obtained by using Eq. (4) (denoted as N–V) and Eq. (10) (denoted as present). It can be seen that the present approximate formula is more accurate in general.

For the present formula, the relative errors are less than 0.1% for the third and higher natural frequencies in general. The maximum relative errors are around 0.3% when β_1 and β_2 are around 10. For the N–V formula, the maximum errors are also around 0.3% for the third natural frequencies when the support conditions are close to a clamped–hinged beam. In fact, it has been shown [6] that λ_n for the higher natural frequencies for the clamped–hinged and clamped–clamped beams can be given by $(n+1/4)\pi$ and $(n+1/2)\pi$, respectively. The present formula gives the correct values for these two limiting cases while N–V formula only gives the correct values for the clamped–clamped conditions.

3. Iteration formula

For the first two natural frequencies, the relative errors are slightly higher. It can be verified that for the N–V formula, the maximum relative error occurs when β_1 is large and β_2 is small (i.e., close to the clamped–hinged condition). The errors are around 2.5% and 0.7% for the first and second natural frequencies, respectively. For the present formula, it can be verified that the relative errors are less than 0.4% and 0.2%, respectively, for the first and second natural frequencies in general. The maximum error occurs when both β_1 and β_2 are around 10. The maximum errors are around 2.0% and 0.7% for the first and second natural frequencies, respectively.

The main source of errors in the lower frequencies is due to the approximation of λ_n by $n\pi$ in Eq. (9). The accuracy can be improved if the following iterations are carried out:

$$\lambda_n^{(i+1)} = n\pi + \tan^{-1}\left(\frac{(R_1+R_2)\lambda_n^{(i)}+1}{2R_1R_2(\lambda_n^{(i)})^2+(R_1+R_2)\lambda_n^{(i)}}\right), \quad i = 0, 1, 2, \ldots \tag{11}$$

with $\lambda_n^{(0)} = n\pi$. Table 2 shows the results obtained by carrying out one and two iterations. It can be seen that the results can be improved significantly by using just one iteration for $\beta_1 = \beta_2 = 10$.

It can be verified that by carrying out one iteration, the maximum error still occurs when both β_1 and β_2 are around 10. However, the errors are reduced to around 0.5% and 0.03% for the first and second natural frequencies, respectively. When two iterations are used, the relative error of the first and second natural frequencies can be further reduced. However, the maximum errors

Table 1
Comparison of the frequency parameters λ between the exact values, the N–V method and the present method

β_1	β_2	Mode number					
		1	2	3	4	5	
0	0	3.141593	6.283185	9.424778	12.566371	15.707963	Exact
		3.141593	6.283185	9.424778	12.566371	15.707963	Eq. (4) (N–V)
		0.00%	0.00%	0.00%	0.00%	0.00%	Error
		3.141593	6.283185	9.424778	12.566371	15.707963	Eq. (10) (Present)
		0.00%	0.00%	0.00%	0.00%	0.00%	Error
0.01	10	3.666010	6.688156	9.752074	12.840018	15.942637	Exact
		3.629408	6.665157	9.734409	12.825899	15.931101	Eq. (4) (N–V)
		−1.00%	−0.34%	−0.18%	−0.11%	−0.07%	Error
		3.693926	6.701113	9.758974	12.844070	15.945200	Eq. (10) (Present)
		0.76%	0.19%	0.07%	0.03%	0.02%	Error
10	10	4.155664	7.068249	10.065679	13.105264	16.171791	Exact
		4.188790	7.068583	10.053096	13.089969	16.156762	Eq. (4) (N–V)
		0.80%	0.00%	−0.13%	−0.12%	−0.09%	Error
		4.243082	7.117450	10.092109	13.120973	16.181800	Eq. (10) (Present)
		2.10%	0.70%	0.26%	0.12%	0.06%	Error
0	100	3.889185	7.003227	10.118546	13.235413	16.353724	Exact
		3.816990	6.960660	10.084634	13.204659	16.324194	Eq. (4) (N–V)
		−1.86%	−0.61%	−0.34%	−0.23%	−0.18%	Error
		3.896541	7.009535	10.124258	13.240594	16.358431	Eq. (10) (Present)
		0.19%	0.09%	0.06%	0.04%	0.03%	Error
1	10 000	4.041438	7.133133	10.255610	13.386423	16.521079	Exact
		4.004427	7.103484	10.231712	13.366776	16.504431	Eq. (4) (N–V)
		−0.92%	−0.42%	−0.23%	−0.15%	−0.10%	Error
		4.063126	7.141534	10.259571	13.388761	16.522632	Eq. (10) (Present)
		0.54%	0.12%	0.04%	0.02%	0.01%	Error
10 000	10 000	4.729095	7.851636	10.993412	14.134343	17.275311	Exact
		4.711604	7.852412	10.993222	14.134032	17.274842	Eq. (4) (N–V)
		−0.37%	0.01%	0.00%	0.00%	0.00%	Error
		4.711761	7.852726	10.993691	14.134657	17.275623	Eq. (10) (Present)
		−0.37%	0.01%	0.00%	0.00%	0.00%	Error
∞	∞	4.730041	7.853205	10.995608	14.137165	17.278760	Exact
		4.712389	7.853982	10.995574	14.137167	17.278760	Eq. (4) (N–V)
		−0.37%	0.01%	0.00%	0.00%	0.00%	Error
		4.712389	7.853982	10.995574	14.137167	17.278760	Eq. (10) (Present)
		−0.37%	0.01%	0.00%	0.00%	0.00%	Error

now occur when both β_1 and β_2 are large (i.e., close to the clamped–clamped condition). The errors are around 0.4% and 0.01% for the first and second natural frequencies, respectively. This error cannot be reduced easily and is due to the assumption that $\exp(\lambda_n)$ is much larger than unity.

Table 2
Comparison of the frequency parameters λ between the exact values and the present method with iterations

β_1	β_2	Mode 1				Mode 2			
		Exact solution	Iteration	Present method	Error (%)	Exact solution	Iteration	Present method	Error (%)
0.01	10	3.666010	0	3.693926	0.76	6.688156	0	6.701113	0.19
			1	3.664864	−0.03		1	6.687756	−0.01
			2	3.666323	0.01		2	6.688171	0.00
10	10	4.155664	0	4.243082	2.10	7.068249	0	7.117450	0.70
			1	4.133322	−0.54		1	7.065875	−0.03
			2	4.143353	−0.30		2	7.068888	0.01
0	100	3.889185	0	3.896541	0.19	7.003227	0	7.009535	0.09
			1	3.889504	0.01		1	7.003173	0.00
			2	3.889569	0.01		2	7.003228	0.00
1	10 000	4.041438	0	4.063126	0.54	7.133133	0	7.141534	0.12
			1	4.035723	−0.14		1	7.133209	0.00
			2	4.036380	−0.13		2	7.133280	0.00
10 000	10 000	4.729095	0	4.711761	−0.37	7.851636	0	7.852726	0.01
			1	4.711447	−0.37		1	7.852412	0.01
			2	4.711447	−0.37		2	7.852412	0.01

4. Conclusions

In this letter, an approximate formula for the natural frequencies of simply supported Bernoulli–Euler beams with rotational restrains at the ends is given. The accuracy is better than the one given by Newmark and Veletsos in general. Compared to the exact values, the relative errors of the present formula are less than 0.3% for the third and higher natural frequencies. The relative errors of the first two natural frequencies are slightly higher and are less than 0.4% in general, except when the β values are around 10. When the β values are around 10, say $1 \leqslant \beta \leqslant 100$, it is recommended to use one or two iterations to reduce the error to 0.4%.

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