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A stiffness equation transfer method for transient dynamic response analysis of structures

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Abstract

A stiffness equation transfer method is proposed for transient dynamic response analysis of structures under various excitations. This method is a development and refinement of the combined finite element-transfer matrix (FE-TM) method. In the present method, the transfer of state vectors from left to right in the FE-TM method is changed into the transfer of general stiffness equations of every section from left to right. This method has the advantages of reducing the order of the ordinary transfer equation systems and minimizing the propagation of round-off errors occurring in recursive multiplication of transfer and point matrices. Furthermore, the drawback that the number of degrees of freedom on the left boundary must be the same as that on the right boundary in the ordinary FE-TM method, has now been avoided. The Newmark generalized acceleration formulation for time discretization is employed for a solution of the time problem. At the end, numerical examples are presented to demonstrate the accuracy as well as the potential of the proposed method for transient dynamic response analysis of structures.

1. Introduction

The development of accurate and computationally efficient algorithms that predict the transient dynamic response of general structures continues to be the subject of ongoing research. In dynamic analysis of structures, the finite element (FE) method is the most widely used and powerful tool. However, the disadvantage of the FE method is that, in the case of complex and large structures, it is necessary to use a large number of nodes, resulting in very large matrices which require large computers for their management and regulation. Furthermore, in the transient dynamic analysis of the structures subjected to random excitations by a direct

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integration method such as the *Newmark-β*, Houbolt, *Wilson-θ* and central difference method, these disadvantages of the FE method will become more serious, because in these methods the sequence of calculations must be repeated many times, i.e. small time steps must be used to obtain response of the structure accurately.

The purpose of this paper is to propose an improved finite element-transfer matrix (FE-TM) method of analyzing the transient dynamic response of structures under various random excitations and we may call it later on the stiffness equation transfer (SET) method.

A combined use of finite element and transfer matrix was proposed by Dokanish for the free vibration problems of plates [1]. In this approach, a finite element formulation was used to obtain the stiffness and mass matrices for a strip of elements whose boundaries were successively connected and whose end boundaries were characterized by state vectors, as defined in the standard transfer matrix method. As the size of stiffness and mass matrices was equal to the number of degrees of freedom in only one strip, this approach had the advantage of reducing the size of a matrix to less than that obtained by the ordinary FE method. Since then, several authors have proposed refinements and extensions of this method for various linear and non-linear structural problems [2–11]. However, it is pointed out that, one drawback of the standard transfer matrix method is that it is often beset with numerical instabilities [12]. In the ordinary FE-TM method, recursive multiplication of the transfer and point matrices are main sources of round-off errors. Particularly, in calculating high resonant frequencies or the response of a long structure, the numerical instability would appear and lead to an unwanted solution. Several techniques are available to overcome such numerical difficulties [12]. One of the better methods is to use Riccati transformation or the Riccati transfer matrix method [6–8,13–15]. Chu and Pilkey have successfully proposed a Riccati transfer matrix method for transient analysis of structural members [14,15]. Besides, the technique of exchanging the unknown state vectors has been used for solving this problem [5]. Another drawback of the ordinary FE-TM method is that the derivation of the transfer matrix from the dynamic stiffness matrix $[G]_i$ for strip i requires the inversion of sub-matrix $[G_{12}]_i$ and the inversion can be possible only if $[G_{12}]_i$ is a square matrix [1–3,5–9]. But, $[G_{12}]_i$ is a square matrix only if there are equal numbers of nodes on the right boundary and on the left boundary of strip i. Therefore, most of the previous formulations of the combined FE-TM method are only applicable to the models which have the same number of nodes on all the substructure boundaries.

In order to overcome simultaneously both these disadvantages in the ordinary FE-TM method, the author proposed a stiffness equation transfer (SET) method for the steady state vibration response and eigenvalue analysis of structures [16,17]. This study is an extension of this SET method to the transient analysis of structures subjected to various excitations. The *Newmark-β* method is, in this paper, used for time integration, but other integration methods such as the Houbolt, *Wilson-θ* and central difference methods may also be used. In the present method, the transfer of state vectors from left to right in the FE-TM method is transformed into a transfer of general stiffness equations in every section from left to right, and then the inverse matrix of sub-matrix $[G_{12}]_i$ of the FE-TM method becomes the inverse matrix of sub-matrix $[G_{11}]_i$. It is well known that $[G_{11}]_i$ is always a square matrix whether the structures are rectangular or not. The drawback that the number of degrees of freedom on the left boundary must be the same on the right boundary in the ordinary FE-TM method, is now avoided. On the other hand, since the numerical solution of a two-point boundary value problem in ordinary FE-TM method has been converted into a numerical solution of an initial value problem, the propagation of round-off errors occurring in recursive multiplication of the transfer and point matrices can also be avoided.

A program SETTDR based on this method for use on an IBMPC586 microcomputer is developed. Some numerical examples of transient dynamic response problems are proposed and their results are compared with those obtained with the ordinary FE method.

2. Direct integration method

Without losing generality, we consider the plate shown in Fig. 1. It is divided into n strips and each strip is further subdivided into finite elements. The vertical sides dividing or bordering the strips are called sections. It is apparent that the right of section i is also the left of strip i.

The discretized finite element equations for substructure i at time $t + \Delta t$ takes the form

$$[M]_i\{\ddot{U}_{t+\Delta t}\}_i + [C]_i\{\dot{U}_{t+\Delta t}\}_i + [K]_i\{U_{t+\Delta t}\}_i = \{N_{t+\Delta t}\}_i + \{Q_{t+\Delta t}\}_i, \tag{1}$$

where $[M]_i$, $[C]_i$ and $[K]_i$ are the mass, damping and stiffness matrices; $\{U_{t+\Delta t}\}_i$, $\{\dot{U}_{t+\Delta t}\}_i$, $\{\ddot{U}_{t+\Delta t}\}_i$, $\{N_{t+\Delta t}\}_i$ and $\{Q_{t+\Delta t}\}_i$ are the displacement, velocity, acceleration, internal force and external force vectors at time $t + \Delta t$, respectively. If the structure is subjected to a ground acceleration $\{\ddot{U}\}_0$, not a force, then $\{Q_{t+\Delta t}\}_i = -[M]_i\{\ddot{U}_{t+\Delta t}\}_0$. $\{U_{t+\Delta t}\}_i$, $\{\dot{U}_{t+\Delta t}\}_i$ and $\{\ddot{U}_{t+\Delta t}\}_i$ are the relative displacement, velocity and acceleration vectors at time $t + \Delta t$, respectively.

The transient dynamic response of structures is usually determined by numerically integrating the discretized equations in time. Direct integration methods are commonly used and they are usually categorized into explicit and implicit ones. Explicit techniques, such as the central difference method, are most computationally efficient. However, these methods are conditionally stable. The time step must be less than some critical time step. In contrast, implicit methods are unconditionally stable. The *Newmark-β* and *Wilson-θ* methods are the most popular implicit techniques. For these methods, the accuracy of the response depends on the selection of an appropriate time step size. The unconditional stability and ease of use of implicit schemes have resulted in their widespread use for general transient response problems.

In this paper, we use the *Newmark-β* method for time integration [18]. We assume variations for the displacement $\{U\}_i$ and velocity $\{\dot{U}\}_i$ in the time interval Δt to be such that the values at the beginning and end of the time step are related by equations of the form

$$\{\ddot{U}_{t+\Delta t}\}_i = \{\dot{U}_t\}_i + (1-\gamma)\Delta t\{\ddot{U}_t\}_i + \gamma\Delta t\{\ddot{U}_{t+\Delta t}\}_i, \tag{2}$$

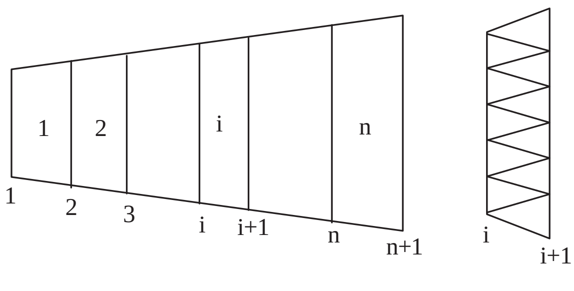

Fig. 1. Subdivision of structure into strips and finite elements.

$$\{U_{t+\Delta t}\}_i = \{U_t\}_i + \{\dot{U}_t\}_i\Delta t + [(\tfrac{1}{2} - \beta)\{\ddot{U}_t\}_i + \beta\{\ddot{U}_{t+\Delta t}\}_i]\Delta t^2, \tag{3}$$

where β and γ are parameters that can be determined to obtain integration accuracy and stability. When $\beta = \frac{1}{6}$ and $\gamma = \frac{1}{2}$, this method reduces to the linear acceleration method, and when $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, to the constant average acceleration method.

Eliminating $\{\dot{U}_{t+\Delta t}\}_i$, $\{\ddot{U}_{t+\Delta t}\}_i$ from Eq. (1), one obtains an equation with the unknown variables $\{U_{t+\Delta t}\}_i$ only,

$$[G]_i\{U_{t+\Delta t}\}_i = \{\bar{Q}_{t+\Delta t}\}_i + \{N_{t+\Delta t}\}_i, \tag{4}$$

where $[G]_i$ is the effective dynamic stiffness matrix for the strip i and $\{\bar{Q}_{t+\Delta t}\}_i$ is the generalized external force vector, which are given as follows:

$$[G]_i = [K]_i + (1/\beta\Delta t^2[M]_i + (\gamma/\beta\Delta t)[C]_i \tag{5}$$

$$\begin{aligned}\{\bar{Q}_{t+\Delta t}\}_i = \{Q_{t+\Delta t}\}_i &+ [M]_i(1/\beta\Delta t^2\{U_t\}_i + 1/\beta\Delta t\{\dot{U}_t\}_i + (1/2\beta - 1)\{\ddot{U}_t\}_i) \\ &+ [C]_i(\gamma/\beta\Delta t\{U_t\}_i + (\gamma/\beta - 1)\{\dot{U}_t\}_i + (\gamma/\beta - 2)\Delta t/2\{\ddot{U}_t\}_i).\end{aligned} \tag{6}$$

Eq. (4) is an equation with unknown variables $\{U_{t+\Delta t}\}_i$ as well as $\{N_{t+\Delta t}\}_i$ and can be solved by the SET method described in the following. Once the displacement $\{U_{t+\Delta t}\}$ of the total structure is obtained, the velocities and accelerations at time $t + \Delta t$ are evaluated from Eqs. (7) and (8), respectively:

$$\{\ddot{U}_{t+\Delta t}\}_i = 1/\beta\Delta t^2(\{U_{t+\Delta t}\}_i - \{U_t\}_i) - 1/\beta\Delta t\{\dot{U}_t\}_i - (1/2\beta - 1)\{\ddot{U}_t\}_i, \tag{7}$$

$$\{\dot{U}_{t+\Delta t}\}_i = \{\dot{U}_t\}_i + (1 - \gamma)\Delta t\{\ddot{U}_t\}_i + \gamma\Delta t\{\ddot{U}_{t+\Delta t}\}_i. \tag{8}$$

3. A stiffness equation transfer (SET) method

The following derivation is for solving Eq. (4) at time $t + \Delta t$. For simplicity, we omit the subscript $t + \Delta t$.

3.1. An ordinary finite element-transfer matrix (FE-TM) method

Let $\{U\}_i^R$, $\{N\}_i^R$ and $\{\bar{Q}\}_i^R$ be the right displacement, internal force and generalized external force vectors of section i, $\{U\}_{i+1}^L$, $\{N\}_{i+1}^L$ and $\{\bar{Q}\}_{i+1}^L$ be the left corresponding vectors of section $i + 1$, so that we have

$$\begin{aligned}\{U\}_i &= [\{U\}_i^R, \{U\}_{i+1}^L]^{\mathrm{T}}, \\ \{N\}_i &= [\{N\}_i^R, \{N\}_{i+1}^L]^{\mathrm{T}}, \\ \{\bar{Q}\}_i &= [\{\bar{Q}\}_i^R, \{\bar{Q}\}_{i+1}^L]^{\mathrm{T}}.\end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (4), the latter can be written as

$$[G]_i\begin{bmatrix}\{U\}_i^R \\ \{U\}_{i+1}^L\end{bmatrix} = \begin{Bmatrix}\{N\}_i^R \\ \{N\}_{i+1}^L\end{Bmatrix} + \begin{Bmatrix}\{\bar{Q}\}_i^R \\ \{\bar{Q}\}_{i+1}^L\end{Bmatrix}. \tag{10}$$

Matrix $[G]_i$ is the dynamic stiffness matrix for the strip i and it may be partitioned into four sub-matrices and Eq. (10) may be rewritten as

$$\begin{bmatrix} [G_{11}] & [G_{12}] \\ [G_{21}] & [G_{22}] \end{bmatrix}_i \begin{Bmatrix} \{U\}_i^R \\ \{U\}_{i+1}^L \end{Bmatrix} = \begin{Bmatrix} \{N\}_i^R \\ \{N\}_{i+1}^L \end{Bmatrix} + \begin{Bmatrix} \{\bar{Q}\}_i^R \\ \{\bar{Q}\}_{i+1}^L \end{Bmatrix}. \tag{11}$$

The displacements are continuous across section i, so that we obtain

$$\{U\}_i^R = \{U\}_i^L. \tag{12}$$

Without losing generality, we suppose that there is no concentrated external load acting on section i (concentrated external load acting on section i may be treated as a generalized external force on the left of strip i), due to the continuity of force at section i, we obtain

$$\{N\}_i^R = -\{N\}_i^L. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11) and with a little algebraic manipulation, Eq. (11) can be rearranged in the form

$$\begin{Bmatrix} \{U\}_{i+1}^L \\ \{N\}_{i+1}^L \\ 1 \end{Bmatrix} = \begin{bmatrix} [T_{11}] & [T_{12}] & \{Q_1\} \\ [T_{21}] & [T_{22}] & \{Q_2\} \\ 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} \{U\}_i^L \\ \{N\}_i^L \\ 1 \end{Bmatrix} = [T]_i \begin{Bmatrix} \{U\}_i^L \\ \{N\}_i^L \\ 1 \end{Bmatrix}, \tag{14}$$

where

$$\begin{aligned} &[T_{11}]_i = -[G_{12}]_i^{-1}[G_{11}]_i, \\ &[T_{12}]_i = -[G_{12}]_i^{-1}, \\ &[T_{21}]_i = [G_{21}]_i - [G_{22}]_i[G_{12}]_i^{-1}[G_{11}]_i, \\ &[T_{22}]_i = -[G_{22}]_i[G_{12}]_i^{-1}, \quad \{Q_1\}_i = [G_{12}]_i^{-1}\{\bar{Q}\}_i^R, \quad \{Q_2\}_i = [G_{22}]_i[G_{12}]_i^{-1}\{\bar{Q}\}_i^R - \{\bar{Q}\}_{i+1}^L. \end{aligned} \tag{15}$$

Proceeding as in Refs. [1,5], we obtain the transfer matrix of the state vectors for the total structure:

$$\begin{Bmatrix} \{U\}_{n+1}^L \\ \{N\}_{n+1}^L \\ 1 \end{Bmatrix} = [P] \begin{Bmatrix} \{U\}_1^L \\ \{N\}_1^L \\ 1 \end{Bmatrix} \tag{16}$$

in which

$$[P] = [T]_n[T]_{n-1}\cdots[T]_1. \tag{17}$$

Eq. (16) relates the section variables of the left boundary of the structure to those of its right boundary. The boundary conditions of the left edge of the structure would require some components of the state vectors to be zeros. Similarly, the boundary conditions of the right edge of the structure would also require certain components of the state vectors to be zeros. The known state variables at the right boundary are substituted into the above relationship to determine the unknown state variables at the left boundary. After the initial state vector at the left boundary is known, the state vector at the section can be obtained by recursively applying Eq. (14) until all the state vectors are known. In this method, it is obvious that the sub-matrix $[G_{12}]_i$ must be a square matrix in order to obtain $[T]_i$. In addition to this, propagation of round-off errors occurs due to recursive multiplications of transfer matrix $[T]_i$ in Eq. (17).

3.2. Transfer matrix for stiffness equations

In order to overcome the drawback in the ordinary FE-TM method, the present method makes a change in the transfer of state vectors from left to right in the ordinary FE-TM method to the transfer of stiffness equations of every section from left to right. At the same time, the recursive multiplications of the transfer matrix $[T]_i$ are then avoided.

Similarly as in generalized Riccati transformation of state vectors [13], we assume that the generalized stiffness equations which relate the force vectors to the displacement vectors on the left of section i are given by

$$\{N\}_i^L = [S]_i\{U\}_i^L + \{E\}_i \quad (i \geqslant 2), \tag{18}$$

where $[S]_i$, the coefficient matrix of the stiffness equation for section i, and $\{E\}_i$, the equivalent external force vectors on section i.

Substituting Eqs. (12) and (13) into Eq. (18), we obtain

$$\{N\}_i^R = -[S]_i\{U\}_i^R - \{E\}_i. \tag{19}$$

Eq. (19) describes the relation between the internal force vectors and the displacement vectors on the right of section i.

By expanding Eq. (11), we obtain

$$[G_{11}]_i\{U\}_i^R + [G_{12}]_i\{U\}_{i+1}^L = \{N\}_i^R + \{\bar{Q}\}_i^R, \tag{20}$$

$$[G_{21}]_i\{U\}_i^R + [G_{22}]_i\{U\}_{i+1}^L = \{N\}_{i+1}^L + \{\bar{Q}\}_{i+1}^L. \tag{21}$$

Substituting Eq. (19) into Eq. (20), we obtain

$$\{U\}_i^R = -([G_{11}] + [S])_i^{-1}[G_{12}]_i\{U\}_{i+1}^L + ([G_{11}] + [S])_i^{-1}(-\{E\}_i + \{\bar{Q}\}_i^R). \tag{22}$$

Substituting Eq. (22) into Eq. (21), we have

$$\{N\}_{i+1}^L = [S]_{i+1}\{U\}_{i+1}^L + \{E\}_{i+1}, \tag{23}$$

where

$$[S]_{i+1} = [G_{22}]_i - [G_{21}]_i([G_{11}] + [S])_i^{-1}[G_{12}]_i, \tag{24}$$

$$\{E\}_{i+1} = [G_{21}]_i([G_{11}] + [S])_i^{-1}(\{\bar{Q}\}_i^R - \{E\}_i) - \{\bar{Q}\}_{i+1}^L. \tag{25}$$

Eq. (23) represents the relationships between the internal force vectors and the displacement vectors on the left of section $i+1$.

3.3. Transfer of entire structure

Supposing $[S]_2$ and $\{E\}_2$ are known, using Eqs. (24) and (25), $[S]$ and $\{E\}$ are transferred from the left of the second section to the right of the total structure. Hence we have

$$\{N\}_{n+1}^L = [S]_{n+1}\{U\}_{n+1}^L + \{E\}_{n+1}. \tag{26}$$

By considering boundary conditions, the known force or displacement variables on right boundary of the total structure are substituted into Eq. (26) to determine the unknown force or displacement variables. After the force and displacement vectors on the right boundary of the total structure are solved, the force and displacement vectors at any section i are calculated from right to left by Eqs. (22) and (19).

It is noteworthy that the transfer matrix $[P]$ for the ordinary FE-TM method [9,11] is replaced by the transfer matrix $[S]_{n+1}$ in Eq. (26) for the SET method. The dimension of the matrix $[S]_{n+1}$ is only half that of the matrix $[P]$. In the SET method, the storage requirements would only be about half of the FE-TM method. In addition, the transfer matrix $[S]_{n+1}$ is obtained by recursively using Eqs. (24) and (25), and not by recursive multiplication of transfer and point matrices, so the propagation of round-off errors occurred in recursive multiplication of transfer and point matrices is thus avoided.

3.4. The method of determining $[S]_2$ and $\{E\}_2$

In Eq. (18), let i be 2, we obtain

$$\{N\}_2^L = [S]_2\{U\}_2^L + \{E\}_2. \tag{27}$$

For strip 1, by expanding Eq. (11), we have

$$[G_{11}]_1\{U\}_1^R + [G_{12}]_1\{U\}_2^L = \{N\}_1^R + \{\bar{Q}\}_1^R, \tag{28}$$

$$[G_{21}]_1\{U\}_1^R + [G_{22}]_1\{U\}_2^L = \{N\}_2^L + \{\bar{Q}\}_2^L. \tag{29}$$

It is obvious that $\{U\}_1^R$ and $\{N\}_1^R$ may be determined by using left boundary conditions of the total structure.

3.4.1. Displacement boundary condition

It is obvious that $\{U\}_1^R$ is known in a displacement boundary condition, hence by Eq. (29), we obtain

$$\{N\}_2^L = [G_{22}]_1\{U\}_2^L + [G_{21}]_1\{U\}_1^R - \{\bar{Q}\}_2^L. \tag{30}$$

Comparing with Eq. (27), we have

$$[S]_2 = [G_{22}]_1, \tag{31}$$

$$\{E\}_2 = [G_{21}]_1\{U\}_1^R - \{\bar{Q}\}_2^L. \tag{32}$$

3.4.2. Force boundary condition

It is obvious that $\{N\}_1^R$ is known in a force boundary condition, hence $\{U\}_1^R$ is obtained from Eq. (28).

$$\{U\}_1^R = -[G_{11}]_1^{-1}[G_{12}]_1\{U\}_2^L + [G_{11}]_1^{-1}(\{N\}_1^R + \{\bar{Q}\}_1^R). \tag{33}$$

Substituting the $\{U\}_1^R$ into Eq. (29), we have

$$\{N\}_2^L = ([G_{22}]_1 - [G_{21}]_1[G_{11}]_1^{-1}[G_{12}]_1)\{U\}_2^L + [G_{21}]_1[G_{11}]_1^{-1}(\{N\}_1^R + \{\bar{Q}\}_1^R) - \{\bar{Q}\}_2^L. \tag{34}$$

Comparing with Eq. (27), we have

$$[S]_2 = [G_{22}]_1 - [G_{21}]_1[G_{11}]_1^{-1}[G_{12}]_1, \tag{35}$$

$$\{E\}_2 = [G_{21}]_1[G_{11}]_1^{-1}(\{N\}_1^R + \{\bar{Q}\}_1^R) - \{\bar{Q}\}_2^L. \tag{36}$$

3.4.3. Mixture boundary condition

In mixture boundary condition, we suppose $\{U\}_1^R = [\{U'\}_1^R, \{U''\}_1^R]^{\mathrm{T}}$ and the corresponding $\{N\}_1^R = [\{N'\}_1^R, \{N''\}_1^R]^{\mathrm{T}}$. If $\{U'\}_1^R$ is unknown and $\{U''\}_1^R$ is known, the corresponding $\{N'\}_1^R$ is known and $\{N''\}_1^R$ is unknown. For strip 1, Eq. (11) is rearranged and repartitioned, so we have

$$\begin{bmatrix} [H_{11}] & [H_{12}] & [H_{13}] \\ [H_{21}] & [H_{22}] & [H_{23}] \\ [H_{31}] & [H_{32}] & [H_{33}] \end{bmatrix} \begin{Bmatrix} \{U'\}_1^R \\ \{U''\}_1^R \\ \{U\}_2^L \end{Bmatrix} = \begin{Bmatrix} \{N'\}_1^R \\ \{N''\}_1^R \\ \{N\}_2^L \end{Bmatrix} + \begin{Bmatrix} \{\bar{Q}'\}_1^R \\ \{\bar{Q}''\}_1^R \\ \{\bar{Q}\}_2^L \end{Bmatrix}. \tag{37}$$

Expanding Eq. (37) and solving the relations between $\{N\}_2^L$ and $\{U\}_2^L$, we obtain

$$[T]_2 = [H_{33}] - [H_{31}][H_{11}]^{-1}[H_{13}], \tag{38}$$

$$\{E\}_2 = [H_{31}][H_{11}]^{-1}(\{N'\}_1^R + \{\bar{Q}'\}_1^R) + [H_{32}]\{U''\}_1^R - [H_{31}][H_{11}]^{-1}[H_{12}]\{U''\}_1^R - \{\bar{Q}\}_2^L. \tag{39}$$

4. Numerical examples

In order to investigate the accuracy and the computational efficiency of our method, we developed a program SETTDR based on this method for use on an IBMPC586 microcomputer and give four numerical examples for illustration.

As the first example, a simply supported beam was subjected to a step load P of 100 kg at its middle point, as shown in Fig. 2. For this sample problem, the physical parameters of this beam are as follows: length $L = 20$ m, flexure strength $EJ = 6.4 \times 10^7\ \mathrm{kg\,m^2}$ and $\rho F = 987\ \mathrm{kg\,s^2\,m^{-2}}$, here ρ is the mass density and F the area. The damping is neglected. The beam is divided into 20 elements and the number of nodes is 21. The total response time is 10 s.

The exact solution of this problem is as follows:

$$y(x,t) = \frac{2Pl^3}{\pi^4 EJ} \sum_{i=}^{1,3,5\cdots} (-1)^{(i-1)/2} \frac{1}{i^4} \sin\frac{i\pi x}{l}(1 - \cos\omega_i t),$$

where

$$\omega_i = i^2\pi^2\sqrt{EJ/\rho F l^4}.$$

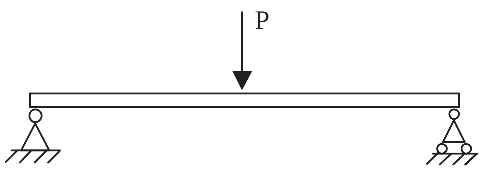

Fig. 2. A simply supported beam subjected to step function force at its middle point.

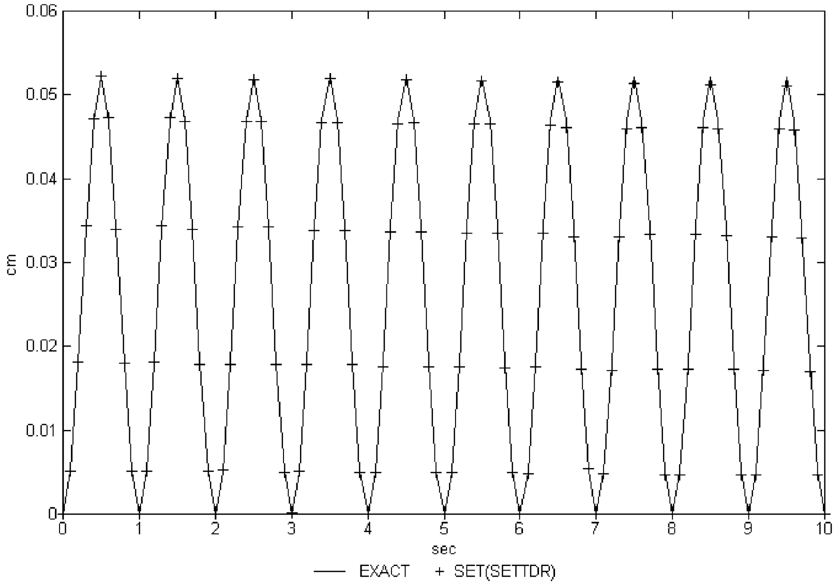

Fig. 3. Middle point displacement of a simply supported beam.

Fig. 3 shows the displacement response of the middle point of the beam for the first 10 cycles. The exact solution is shown as a solid line. Superimposed on the exact solution is the response obtained by using the SET method. For this example, a time step of 0.005 s was chosen for the time integration. The comparison indicates that the results from using the SET method are completely coincident with the exact solutions.

A plane frame subjected to the EL Centro earthquake wave at the base (the example selected from SAP5 manual), shown in Fig. 4, has been analyzed as the second example. Each node of the frame has two degrees of freedom u and θ_z, and for horizontal beams, the horizontal displacement of the node at two end points has the master and slave relation. The physical parameters of the beams forming the frame are as follows: modulus of elasticity $E = 30 \times 10^6$ psi, the Poisson ratio $\nu = 0.3$, a specific weight $\gamma = 0.286$ lb in^{-3}, modal damping parameter $\xi_i = 0.05$ (for all modes), other geometrical parameters, see SAP5 manual. For this example, a time step of 0.005 s was chosen for the FE method (SAP5 program) and the SET method (SETTDR program). Figs. 5–8 show the horizontal displacement response of nodes 21, 17, 11 and 7, respectively. The maximum displacement response for the nodes of the frame during the time of 10 s is shown in Table 1. The maximum internal force (flexure moment) response for the beams of the frame during the time of 10 s is shown in Table 2. Table 3 shows a comparison of computation time between the two methods in this example. As can be seen from Figs. 5–8, Tables 1 and 2, very little difference exists between the results for the first 10 s. The maximum value of the displacement and the flexure moment between the results are much the same. The SET method has the same accuracy with the FE method, but it has higher efficiency than the latter.

The third example is to obtain the dynamic response of a cantilever trapeziform plate under EL Centro earthquake wave as shown in Fig. 9 (vertical direction), where the physical parameters of the plate are as follows: length $l = 90$ cm, width $a = 90$ cm, $b = 60$ cm, thickness $t = 0.635$ cm, a

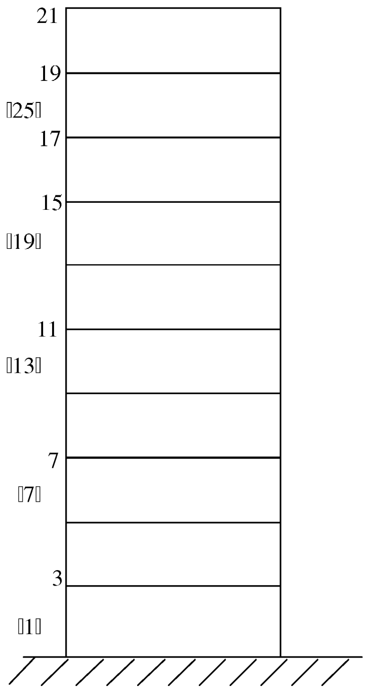

Fig. 4. A frame structure.

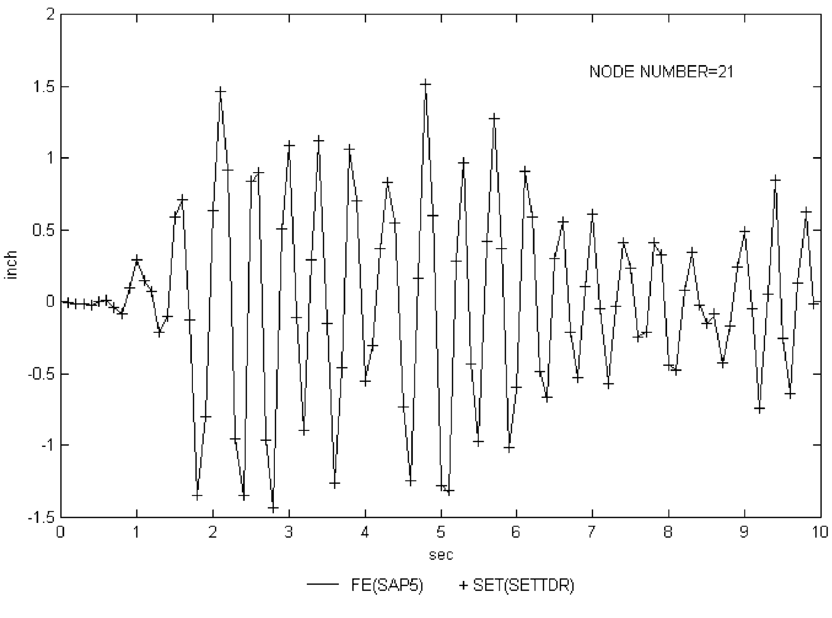

Fig. 5. Horizontal earthquake displacement of the frame node 21 under EL Centro ground excitation.

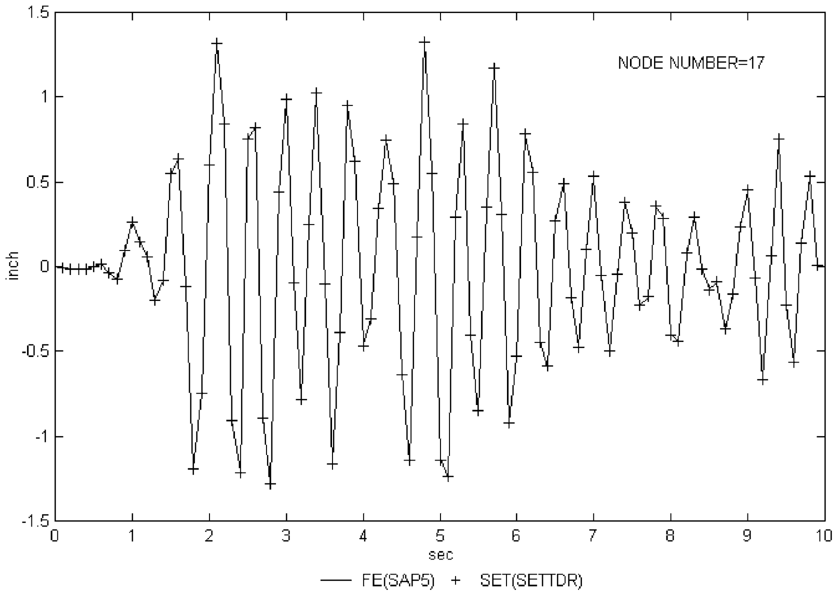

Fig. 6. Horizontal earthquake displacement of the frame node 17 under EL Centro ground excitation.

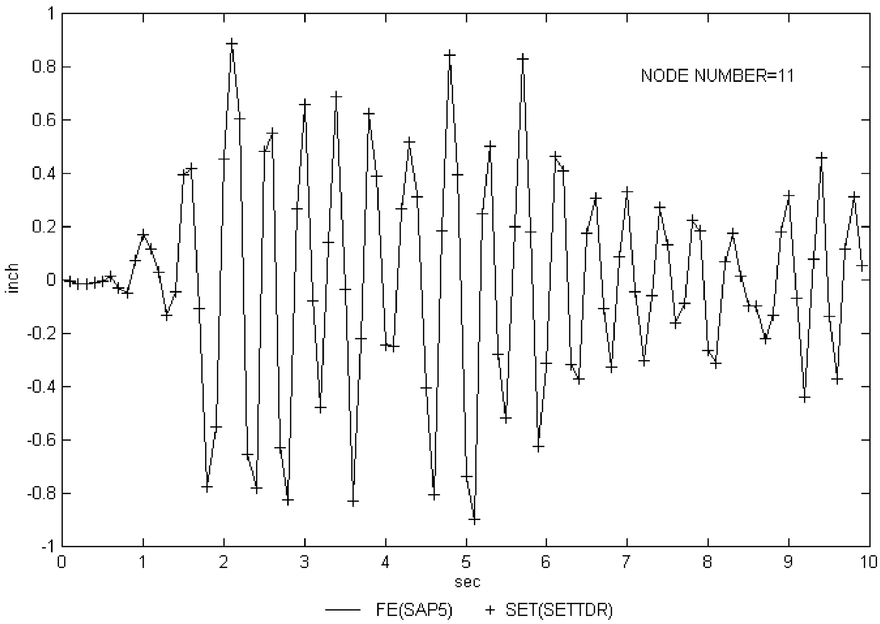

Fig. 7. Horizontal earthquake displacement of the frame node 11 under EL Centro ground excitation.

specific weight $\gamma = 78\ \mathrm{kN\,m^{-3}}$, the Poisson ratio $v = 0.3$, modules of elasticity $E = 2.0 \times 10^5$ MPa, Rayleigh damping constant $\alpha = 0.1$, $\beta = 0.05$. In the numerical calculation, the plate is divided into 9 substructures which are further divided into many triangular plate elements and time step $\Delta t = 0.005$ s is used. Fig. 10 shows the dynamic response displacement for node 4 of the plate.

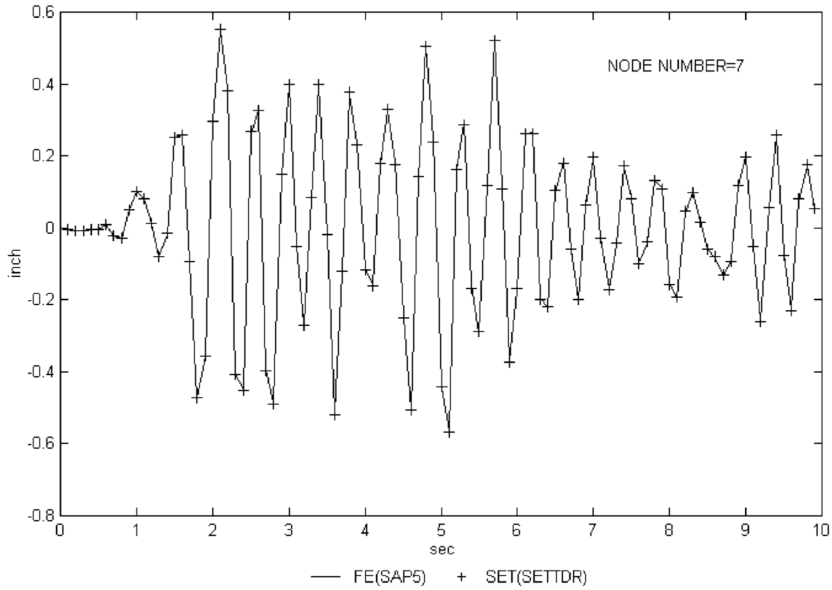

Fig. 8. Horizontal earthquake displacement of the frame node 7 under EL Centro ground excitation.

Table 1
The maximum displacement response for the nodes of the frame under EL Centro ground excitation (in)

Node number	3	7	11	15	17	19	21
FE (SAP5)	0.1789	0.5676	0.9533	1.355	1.534	1.677	1.776
SET (SETTDR)	0.1791	0.5678	0.9508	1.351	1.526	1.676	1.773

Table 2
The maximum flexure moment for the beams of the frame under EL Centro ground excitation (lb in)

Beam number	1	7	13	19	25
FE (SAP5)	1.33×10^6	6.62×10^5	5.70×10^5	4.51×10^5	2.08×10^5
SET (SETTDR)	1.33×10^6	6.63×10^5	5.69×10^5	4.52×10^5	2.06×10^5

Table 3
Comparison of computation time for the frame

Method by applying	Computation time (s)
FE (SAP5)	11.5
SET (SETTDR)	6.2

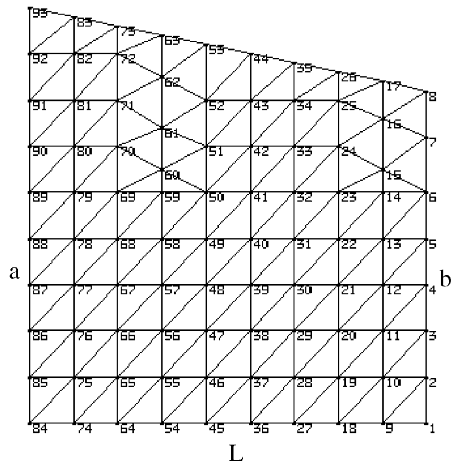

Fig. 9. Cantilever trapeziform plate model.

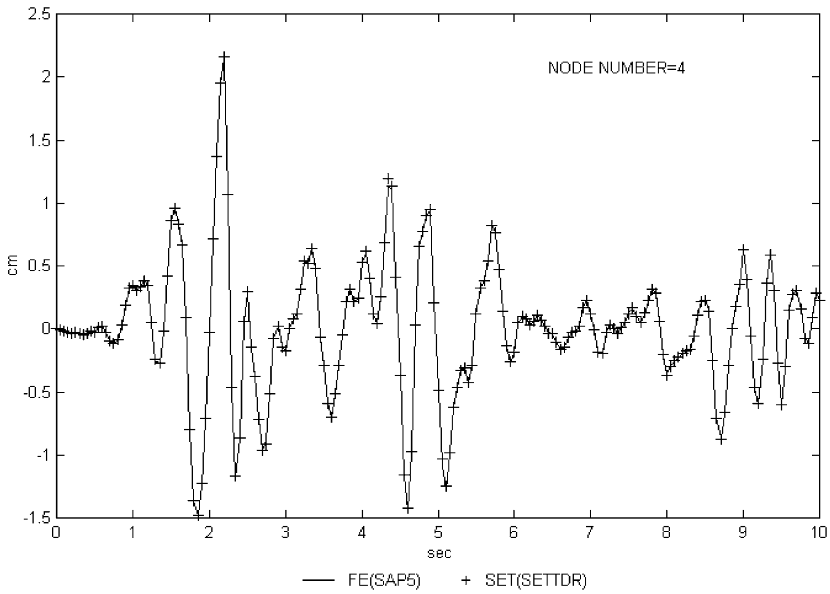

Fig. 10. Vertical earthquake displacement for the node 4 of the trapeziform plate under EL Centro ground excitation.

From the above results, it can be seen that the computed results by the present method are almost the same as those obtained by the FE method. A comparison of computation time shown in Table 4 indicates that the computation efficiency of the present method is higher than that of the FE method. In this example, the number of nodes on the left boundary is 10, and that on the right boundary is 8. Most of the ordinary FE-TM methods can only be applied to the chain-like structure with equal number of degrees of freedom on the boundaries, so the ordinary FE-TM method [5,11] cannot be used in this case. The present method has potentially wider application than the ordinary FE-TM method.

Table 4
Comparison of computation time for the plate

Method by applying	Computation time (s)
FE (SAP5)	102.5
SET (SETTDR)	57.8

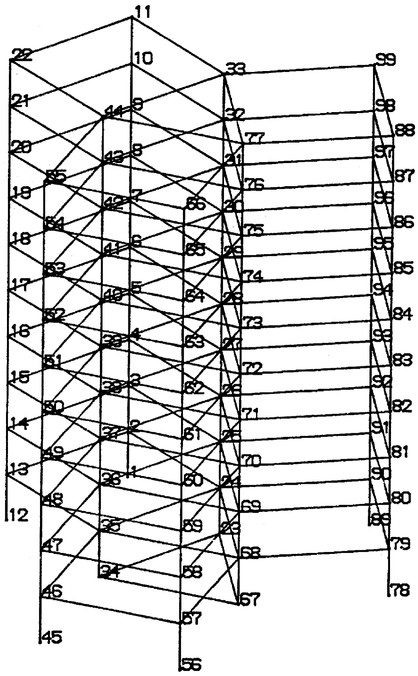

Fig. 11. A shear-wall structure with supporting frames.

In the fourth example, we analyzed a shear-wall structure with supporting frames to demonstrate the wider applicability of the present method. The present SET method can not only be applied to simple structures, but also to the more complicated structures. The shear-wall structure as shown in Fig. 11 was subjected to the EL Centro earthquake wave at the base. Each node has six degrees of freedom $u, v, w, \theta_x, \theta_y, \theta_z$. The shear-wall consists of 30 rectangular plate elements, where the physical and geometric parameters of each plate element are as follows: $E = 3 \times 10^{10}$ Pa, $\mu = 0.1667$, $\rho = 3000\ \mathrm{kg\,m^{-3}}$, thickness $t = 0.2$ m, length $l = 8$ m, and width $b = 3$ m. The total number of beam and column elements forming the supporting frames is 150. Their physical and geometric parameters are as follows: $E = 3 \times 10^{12}$ Pa, $\rho = 2500\ \mathrm{kg\,m^{-3}}$. For horizontal beams, the dimension is 8 m $\times$ 0.28 m $\times$ 0.28 m. For longitudinal columns, it is 3 m $\times$ 0.4 m $\times$ 0.4 m. Rayleigh damping constant $\alpha = 0.02$, $\beta = 0.01$ and time step $\Delta t = 0.005$ s. Fig. 12 shows the horizontal dynamic response displacement of node 77 computed by the SET method and the FE method, respectively. A comparison of computation time is shown in Table 5. Similar results as in Examples 2 and 3 are obtained. The size of the matrix in the FE method is much

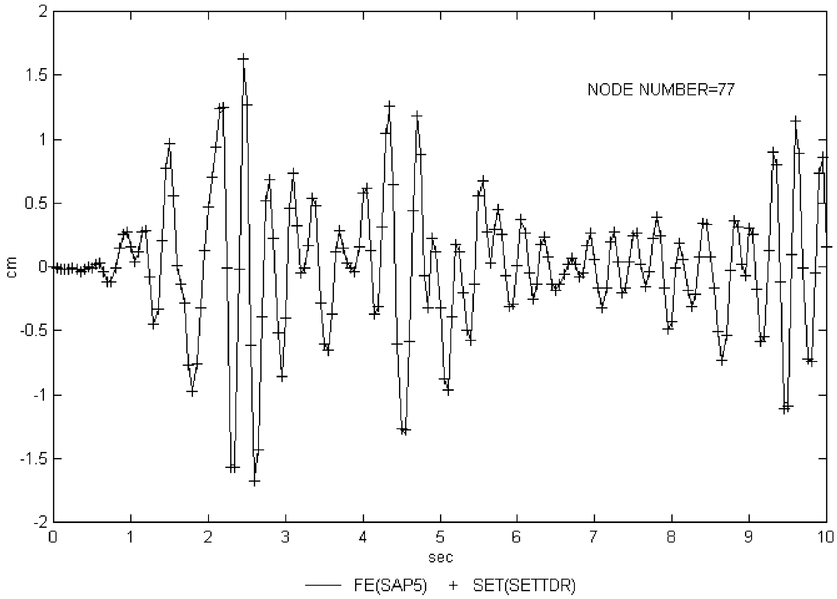

Fig. 12. Horizontal earthquake displacement for the node 77 of the shear-wall structure under EL Centro ground excitation.

Table 5
Comparison of computation time for the shear-wall structure

Method by applying	Computation time (s)
FE (SAP5)	284.8
SET (SETTDR)	112.3

larger than that in the present method. The computational efficiency of our method is higher than that of the FE method.

5. Conclusion

A stiffness equation transfer method for solving the transient dynamic response problems of structures is proposed and illustrated by four examples. An SETTDR microcomputer program based on this method is developed. Some numerical examples presented in this paper show that the proposed method can be successfully applied to the transient dynamic response analysis of structures under various excitations. In the present method, the transfer of state vectors from left to right in the ordinary FE-TM method is changed into the transfer of general stiffness equations of every section from left to right. This method has the advantages of reducing the order of standard transfer equation systems and minimizing the propagation of round-off errors occurring in recursive multiplication of transfer and point matrices. Furthermore, the drawback that the number of degrees of freedom on the left boundary must be the same on the right boundary in the ordinary FE-TM method, has now been avoided. Therefore, the present method has potentially wider application than the ordinary FE-TM method.

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