Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Sound and Vibration 273 (2004) 1118–1126

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Letter to the Editor

On the symmetry of solutions in non-smooth dynamical systems with two constraints

A.C.J. Luo*

Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 62026-1805, USA

Received 21 July 2003; accepted 4 September 2003

1. Introduction

It is quite difficult to give analytical solutions of dynamical responses in non-linear dynamical systems. With computers expansively used in science, numerical simulations as a useful tool play a very important role on obtaining dynamical responses in non-linear dynamics, which help one understand complexity in nature. However, the current digital computation is very passive, and the approximation-based algorithms cannot provide all possible complicated responses existing in dynamical systems, such as regular and chaotic motions caused by bifurcation and grazing etc. This is also partially because of the singularity of solutions for the complicated dynamical responses. Numerical simulations may find one of all possible solutions, but this solution may not belong to the same solution branch because the singularity will lead to jumping or catastrophe phenomena. The objective of this technical note is to find the symmetrical structure of solutions for regular and chaotic motions in non-linear dynamical systems through the symmetry of mapping structures. Once one of solutions in such non-smooth dynamical systems is obtained by a numerical or analytical approach, another symmetrical solution can be directly predicted through the solution symmetry property given in this technical note. In 1970, Masri [1] observed the asymmetrical motion in the impact damper system and the rigorous stability analysis was conducted as well. In 1990, Li et al. [2] used a numerical approach to get one of the asymmetrical solutions for the impacting oscillator. In 2002, Luo [3] introduced a time-interval approach to obtain two asymmetrical solutions analytically, and it was observed that the symmetry of two solutions exists. However, the time-interval approach cannot be very efficient for higher-order periodic motions. Once the motion becomes more complicated, it is absolutely necessary to investigate the symmetry of solutions in such non-smooth dynamical systems for obtaining all

*Tel.: +1-618-650-5389; fax: +1-618-650-2555.

E-mail address: aluo@siue.edu (A.C.J. Luo).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.09.011

possible stable and unstable motions. It is found that an invariant transformation exists in regular and chaotic motions relative to skew-symmetrical mapping pairs in symmetrical systems with harmonic excitations. Only the main results are presented without any proofs in this technical note.

2. Problem statement

Consider a two-dimensional dynamic system consisting of *three* sub-systems in a domain $\Omega \subset \Re^2$ that is divided into three sub-domains Ω_p ($p = 1, 2, 3$) by a symmetrical constraint $x_q = \pm E$, $q = 1$ or 2, and $\Omega = \bigcup_{p=1}^{3} \Omega_p$ as shown in Fig. 1. For the pth domain, there is a continuous system in form of

$$\dot{\mathbf{x}} = \mathbf{f}^{(p)}(\mathbf{x}, \boldsymbol{\mu}_p) + \mathbf{g}(\mathbf{x}, \varphi, \boldsymbol{\pi}), \quad \mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \Omega_p, \tag{1}$$

where $\mathbf{g} = (g_1, g_2)^{\mathrm{T}}$ are bounded, periodic functions with phase variable $\varphi = \Omega t$ and a parameter vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)^{\mathrm{T}} \in \Re^m$ and the corresponding period is $T = 2\pi/\Omega$. The $\mathbf{f}^{(p)} = (f_1^{(p)}, f_2^{(p)})^{\mathrm{T}} \in \Re^2$ with system parameter vector $\boldsymbol{\mu}_p = (\mu_{p1}, \mu_{p2}, \ldots, \mu_{pn})^{\mathrm{T}} \in \Re^n$ are C^r-continuous ($r \geqslant 2$). In all sub-domains Ω_p ($p = 1, 2, 3$), the dynamical system in Eq. (1) is continuous and there is a continuous flow expressed by

$$\mathbf{x}^{(p)}(t) = \boldsymbol{\Phi}^{(p)}(\mathbf{x}^{(p)}(t_0), t, \boldsymbol{\mu}_p, \boldsymbol{\pi}) \quad \text{and} \quad \mathbf{x}^{(p)}(t_0) = \boldsymbol{\Phi}^{(p)}(\mathbf{x}^{(p)}(t_0), t_0, \boldsymbol{\mu}_p, \boldsymbol{\pi}). \tag{2}$$

In this note, the following assumptions will be considered:

A1: This system possesses time-continuity.

A2: For a unbounded domain Ω_p, the vector field and the corresponding flow are bounded, i.e.,

$$\|\mathbf{f}^{(p)}\| + \|\mathbf{g}\| \leqslant K_1 \text{ (const) on } \Omega_p \quad \text{and} \quad \|\boldsymbol{\Phi}^{(p)}\| \leqslant K_2 \text{ (const)} \quad \text{for } t \in [0, \infty). \tag{3}$$

A3: For a bounded domain Ω_p, the vector field and the corresponding flow are bounded at finite time, i.e.,

$$\|\mathbf{f}^{(p)}\| + \|\mathbf{g}\| \leqslant K_1 \text{(const) on } \Omega_p \quad \text{and} \quad \|\boldsymbol{\Phi}^{(p)}\| < \infty \quad \text{for } t \in [0, \infty). \tag{4}$$

A4: The dynamical system is symmetrical in the symmetrical domains.

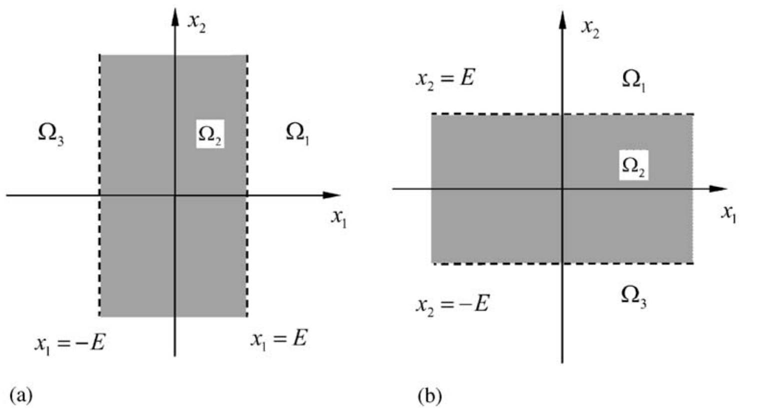

Fig. 1. Two symmetric domains in phase plane for two cases: (a) $x_1 = \pm E$ and (b) $x_2 = \pm E$.

A5: The entire flow at least is C^1-continuous or there is a transport law to connect two flows in two different domains.

3. Switching planes and mappings

For description of motion in Eq. (1), two switching sections (or sets) are

$$\Sigma^{\pm} = \{(t_i, y_i) | x_k = \pm E, y \equiv x_j, j \neq k \in \{1,2\}\} \tag{5}$$

and two singular points are

$$\Gamma^{\pm} = \{(t_i, 0) | x_k = \pm E, y \equiv x_j = 0, j \neq k \in \{1,2\}\}. \tag{6}$$

The two sets are decomposed as

$$\Sigma^{+} = \Sigma^{+}_{+} \cup \Sigma^{+}_{-} \cup \Gamma^{+} \quad \text{and} \quad \Sigma^{-} = \Sigma^{-}_{+} \cup \Sigma^{-}_{-} \cup \Gamma^{-}, \tag{7}$$

where four subsets are defined as

$$\begin{aligned} \Sigma^{+}_{+} &= \{(t_i, y_i) | x_k = E, y \equiv x_j > 0, j \neq k \in \{1,2\}\}, \\ \Sigma^{+}_{-} &= \{(t_i, y_i) | x_k = E, y \equiv x_j < 0, j \neq k \in \{1,2\}\}, \end{aligned} \tag{8}$$

$$\begin{aligned} \Sigma^{-}_{+} &= \{(t_i, y_i) | x_k = -E, y \equiv x_j > 0, j \neq k \in \{1,2\}\}, \\ \Sigma^{-}_{-} &= \{(t_i, y_i) | x_k = -E, y \equiv x_j < 0, j \neq k \in \{1,2\}\}. \end{aligned} \tag{9}$$

From four subsets, six basic mappings are

$$\begin{aligned} &P_1 : \Sigma^{+}_{+} \to \Sigma^{+}_{-}, \quad P_2 : \Sigma^{+}_{-} \to \Sigma^{+}_{+}, \quad P_3 : \Sigma^{+}_{-} \to \Sigma^{-}_{-}, \\ &P_4 : \Sigma^{-}_{-} \to \Sigma^{+}_{-}, \quad P_5 : \Sigma^{+}_{-} \to \Sigma^{-}_{-}, \quad P_6 : \Sigma^{+}_{-} \to \Sigma^{+}_{+}, \end{aligned} \tag{10}$$

for $x_1 = \pm E$, and

$$\begin{aligned} &P_1 : \Sigma^{+}_{-} \to \Sigma^{+}_{+}, \quad P_2 : \Sigma^{+}_{+} \to \Sigma^{+}_{-}, \quad P_3 : \Sigma^{+}_{+} \to \Sigma^{-}_{+}, \\ &P_4 : \Sigma^{-}_{+} \to \Sigma^{-}_{-}, \quad P_5 : \Sigma^{-}_{-} \to \Sigma^{-}_{+}, \quad P_6 : \Sigma^{-}_{-} \to \Sigma^{+}_{-} \end{aligned} \tag{11}$$

for $x_2 = \pm E$.

From the definition of mappings, the mappings P_q $(q = 1, 2, 4, 5)$ relative to one switching section are termed the *local* mapping, and the mappings P_q $(q = 3, 6)$ relative to two switching sections are termed the *global* mapping. The global mapping transports the motion from one switching section into another switching section. The local mapping is the self-mapping in the corresponding switching section. Six mappings are illustrated in Fig. 2.

To describe the complicated motion, the mapping structure of dynamical systems in Eq. (1) is used herein. For simplicity, the notation for mapping is introduced as

$$P_{n_1 n_2 \cdots n_k} \equiv P_{n_1} \circ P_{n_2} \circ \cdots \circ P_{n_k}, \tag{12}$$

where $P_{n_i} \in \{P_q | q = 1, 2, \ldots, 6\}$ and $n_i = \{1, 2, \ldots, 6\}$. Note that the rotation of the mapping of periodic motion in order gives the same motion (i.e., $P_{n_1 n_2 \cdots n_k}, P_{n_2 \cdots n_k n_1}, \ldots, P_{n_k n_1 \cdots n_{k-1}}$), and only the selected Poincaré mapping section is different. The motion of the *m-time* repeating of mapping

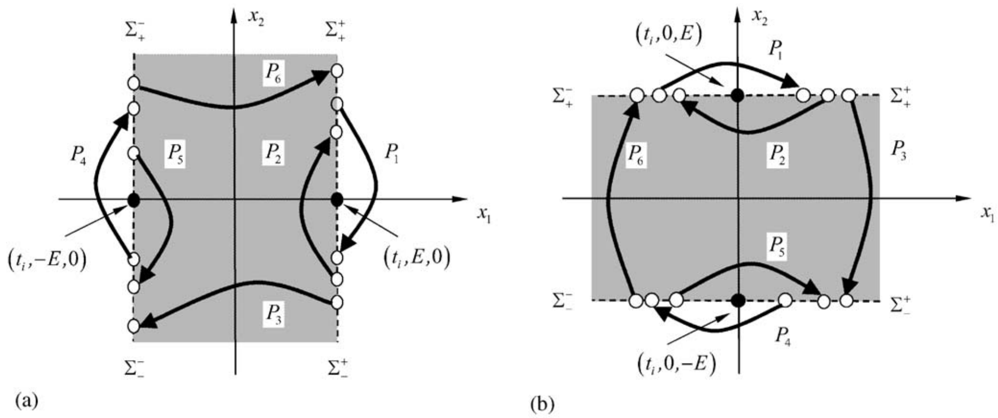

Fig. 2. Switching sections and basic mappings in phase plane: (a) $x_1 = \pm E$ and (b) $x_2 = \pm E$.

$P_{n_1 n_2 \cdots n_k}$ is defined as

$$P^m_{n_1 n_2 \cdots n_k} \equiv \underbrace{(P_{n_1} \circ P_{n_2} \circ \cdots \circ P_{n_k}) \circ \cdots \circ (P_{n_1} \circ P_{n_2} \circ \cdots \circ P_{n_k})}_{m\text{-sets}}. \tag{13}$$

To extend this concept to the local mapping, define

$$P^m_{15} \equiv \underbrace{(P_1 \circ P_5) \circ \cdots \circ (P_1 \circ P_5)}_{m\text{-sets}} \quad \text{and} \quad P^m_{36} \equiv \underbrace{(P_3 \circ P_6) \circ \cdots \circ (P_3 \circ P_6)}_{m\text{-sets}}. \tag{14}$$

For the special combination of global and local mapping, introduce a mapping structure

$$\begin{aligned} P_{n_1 n_2 \cdots (n_i n_l)^m \cdots n_k} &\equiv P_{n_1} \circ P_{n_2} \circ \cdots \circ P^m_{n_i n_l} \circ \cdots \circ P_{n_k} \\ &= P_{n_1} \circ P_{n_2} \circ \cdots \circ \underbrace{(P_{n_i} \circ P_{n_l}) \circ \cdots \circ (P_{n_i} \circ P_{n_l})}_{m\text{-sets}} \circ \cdots \circ P_{n_k}. \end{aligned} \tag{15}$$

From the definition, the motion for Eq. (1) can be very easily described through its mapping structure accordingly.

4. Main results

The initial and final times (t_i and t_{i+1}) are used for all the mappings in Eqs. (10) or (11), and the corresponding phases are $\varphi_i = \Omega t_i$ and $\varphi_{i+1} = \Omega t_{i+1}$. Eq. (2) gives

$$\mathbf{x}^{(p)}(t_{i+1}) = \mathbf{\Phi}^{(p)}(\mathbf{x}^{(p)}(t_i), t_{i+1}, \boldsymbol{\mu}_p, \boldsymbol{\pi}) \quad \text{or} \quad \mathbf{x}^{(p)}(\varphi_{i+1}) = \mathbf{\Phi}_1^{(p)}(\mathbf{x}^{(p)}(\varphi_i), \varphi_{i+1}, \boldsymbol{\mu}_p, \boldsymbol{\pi}). \tag{16}$$

From the foregoing equation, with a notation $\mathbf{y}_i \equiv (\varphi_i, y_i)^{\mathrm{T}}$, the governing equations for mappings from P_1 to P_6 can be written down as

$$\mathbf{y}_{i+1} = P_q \mathbf{y}_i \quad \Leftrightarrow \quad \mathbf{F}^{(q)}(\varphi_i, y_i, \varphi_{i+1}, y_{i+1}, \boldsymbol{\mu}_p, \boldsymbol{\pi}) = 0, \tag{17}$$

where $p = 1$ (or 3) for $q = 1$ (or 4) and $p = 2$ for $q \in \{2, 3, 5, 6\}$. From Assumption A4, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_3$ since the system is symmetric.

Definition 1. Under a transformation $T_P: P_q \to P_{\text{mod}(q+2,6)+1}$ during $(2M+1)$-periods with

$$\varphi_i^{(\text{mod}(q+2,6)+1)} = \varphi_i^{(q)} + (2M+1)\pi \quad \text{and} \quad y_i^{(\text{mod}(q+2,6)+1)} = -y_i^{(q)}, \tag{18}$$

if a relation

$$\begin{aligned}&\mathbf{F}^{(\text{mod}(q+2,6)+1)}(\varphi_i^{(\text{mod}(q+2,6)+1)}, y_i^{(\text{mod}(q+2,6)+1)}, \varphi_{i+1}^{(\text{mod}(q+2,6)+1)}, y_{i+1}^{(\text{mod}(q+2,6)+1)}, \boldsymbol{\mu}_p, \boldsymbol{\pi})\\ &\quad = -\mathbf{F}^{(q)}(\varphi_i^{(q)}, y_i^{(q)}, \varphi_{i+1}^{(q)}, y_{i+1}^{(q)}, \boldsymbol{\mu}_p, \boldsymbol{\pi})\end{aligned} \tag{19}$$

holds where $p=1$ for $q \in \{1,4\}$ and $p=2$ for $q \in \{2,3,5,6\}$, then the mapping pair $(P_q, P_{\text{mod}(q+2,6)+1})$ is skew-symmetric. If a mapping pair is relative to the *local* (or *global*) mapping, such a mapping pair is termed the *local* (or *global*) skew-symmetric mapping pair.

Note that integer $M \equiv 0, 1, 2, \ldots$ and $\text{mod}(\cdot,\cdot)$ is the modulus function.

Theorem 1. *The six mappings P_q $(q = 1, 2, \ldots, 6)$ for the dynamical system in Eq.* (1) *are invariant under the two actions of a transformation T_P, i.e., $T_P \circ T_P : P_q \to P_q$.*

When the six mappings go through the singular points, switching bifurcation may occur. Once switching exists, the motion models switches from an old motion to a new motion and the corresponding mapping structures are changed as well. The mapping structures for the post-switching of mapping P_q $(q = 1, 2, \ldots, 6)$ are

$$\begin{aligned}
&P_q \underset{\text{switching}}{\overset{\text{switching}}{\rightleftharpoons}} P_q \circ P_{q+1} \circ P_q \quad \text{for } (q = 1, 2, 4, 5),\\
&P_q \underset{\text{switching}}{\overset{\text{switching}}{\rightleftharpoons}} P_q \circ P_{\text{mod}(q+1,6)} \circ P_{\text{mod}(q+4,6)} \quad \text{for } (q = 2, 5),\\
&P_q \underset{\text{switching}}{\overset{\text{switching}}{\rightleftharpoons}} P_{q-1} \circ P_{q-2} \circ P_q \quad \text{for } (q = 3, 6),\\
&P_q \underset{\text{switching}}{\overset{\text{switching}}{\rightleftharpoons}} P_q \circ P_{\text{mod}(q+1,6)} \circ P_{\text{mod}(q+2,6)} \quad \text{for } (q = 3, 6).
\end{aligned} \tag{20}$$

From the above discussion, the invariance of the post-switching under the transformation T_P is of great interest. The grazing is a special phenomenon of the switching. Therefore, we have the following theorem.

Theorem 2. *For mappings P_q $(q = 1, 2, \ldots, 6)$ for the dynamical system in Eq.* (1), *if the mapping pair $(P_q, P_{\text{mod}(q+2,6)+1})$ is skew-symmetric with a transformation in Eq.* (18), *the post-switching mapping pair is still skew-symmetric with the same transformation.*

Since the symmetry invariance of the post-switching of mapping exists, it implies that the combination of the mapping P_q $(q = 1, 2, \ldots, 6)$ possesses a symmetry invariance under the transformation T_P in Eq. (18). Therefore, to determine such a symmetrical invariance, a theorem is presented as follows.

Theorem 3. *For mappings P_q $(q = 1, 2, \ldots, 6)$ for the dynamical system in Eq.* (1), *if the mapping pair $(P_q, P_{\mathrm{mod}(q+2,6)+1})$ under $(2M+1)$-periods is skew-symmetric with transformation in Eq.* (18), *then the following two mappings,*

$$\underbrace{P_{6(45)^{m_{k2}}43(12)^{m_{k1}}1} \circ \cdots \circ P_{6(45)^{m_{k2}}43(12)^{m_{k1}}1}}_{(k-1)\text{-actions}} \quad \textit{and} \quad \underbrace{P_{3(12)^{m_{k2}}16(45)^{m_{k1}}4} \circ \cdots \circ P_{3(12)^{m_{k2}}16(45)^{m_{k1}}4}}_{(k-1)\text{-actions}},$$

are a skew-symmetric mapping pair under the same transformation as in Eq. (18).

We have discussed the symmetry invariance of combined mapping structures. The following theorems will discuss the solution symmetrical structures. First of all, the symmetrical solution relative to a symmetrical mapping $P_{6(45)^m43(12)^m1}$ is discussed first, and then the corresponding, asymmetrical solution is investigated.

Theorem 4. *Consider a non-smooth dynamical system with two symmetrical constraints in Eq.* (1) *with six mappings P_q $(q = 1, 2, \ldots, 6)$. If the following two properties exist*: C1: *P_q $(q = 1, 2, 4, 5)$ are local mappings, and P_q $(q = 3$ and $6)$ are global mappings, and* C2: *the mapping pair $(P_q, P_{\mathrm{mod}(q+2,6)+1})$ is skew-symmetric, then the symmetrical solution relative to a mapping $P_{6(45)^m43(12)^m1}\mathbf{y} = \mathbf{y}$ under N-periods with a periodicity condition*

$$\mathbf{y}_{i+4m+4} = \mathbf{y}_i \quad \textit{or} \quad (\Omega t_{i+2m+4}, y_{i+2m+4})^{\mathrm{T}} \equiv (\Omega t_i + 2N\pi, y_i)^{\mathrm{T}} \tag{21}$$

possesses a solution structure

$$\begin{aligned} &\mathrm{mod}(\varphi_{i+j}, 2(2M+1)\pi) \\ &\quad = \mathrm{mod}((2M+1)\pi + \mathrm{mod}(\varphi_{i+\mathrm{mod}(2m+2+j,4m+4)}, 2(2M+1)\pi), 2(2M+1)\pi), \\ &y_{i+j} = -y_{i+\mathrm{mod}(2m+2+j,4m+4)} \end{aligned} \tag{22}$$

for $j = \{0, 1, \ldots, 4m+3\}$.

The foregoing theorem discussed the symmetrical solutions of period-1 motion associated with the mapping $P_{6(45)^m43(12)^m1}$. This structure is quite stable. For instance if one investigated the symmetrical period-1 motion of impacting oscillators (e.g., [4–7]), then one may think that this motion may have period-doubling bifurcation. In fact, no period-doubling bifurcation exists (e.g., Ref. [3,8]). The symmetrical motion will convert into the asymmetrical period-1 motion with the same mapping structures through the first saddle-node bifurcation and an unstable region. The solution symmetry for such an asymmetrical period-1 motion is presented in the following theorem.

Theorem 5. *Consider a non-smooth dynamical system with two symmetrical constraints in Eq.* (1) *with six mappings P_q $(q = 1, 2, \ldots, 6)$. If the following two properties exist*: C1: *P_q $(q = 1, 2, 4, 5)$ are local mappings, and P_q $(q = 3$ and $6)$ are global mappings, and* C2: *the mapping pair $(P_q, P_{\mathrm{mod}(q+2,6)+1})$ is skew-symmetric, then the two branches of solutions for asymmetrical, periodic and chaotic motions relative to a mapping*

$$\underbrace{P_{6(45)^m43(12)^m1} \circ \cdots \circ P_{6(45)^m43(12)^m1}}_{2^k} \mathbf{y} = \mathbf{y} \quad (k = 0, 1, \ldots, \infty)$$

under N-periods with periodicity condition (*i.e.*, $\mathbf{y}_{i+2^{k+2}(m+1)} = \mathbf{y}_i$) *possess a solution relation*

$$\begin{aligned}&\mathrm{mod}(\varphi^{\mathrm{I}}_{i+4r(m+1)+j}, 2(2M+1)\pi)\\&\quad = \mathrm{mod}((2M+1)\pi + \mathrm{mod}(\varphi^{\mathrm{II}}_{i+r(m+1)+\mathrm{mod}(2m+2+j,4m+4)}, 2(2M+1)\pi), 2(2M+1)\pi),\\&y^{\mathrm{I}}_{i+r(4m+4)+j} = -y^{\mathrm{II}}_{i+4r(m+)+\mathrm{mod}(2m+2+j,4m+4)}\end{aligned} \tag{23}$$

for all $r = \{1, \ldots, 2^k\}$ *and* $j = \{0, 1, \ldots, 4m+3\}$. *Superscripts* (I, II) *denote two asymmetrical solutions.*

Theorem 6. *Consider a non-smooth dynamical system with two symmetrical constraints in Eq.* (1) *with six mappings* P_q $(q = 1, 2, \ldots, 6)$. *If the following two properties exist*: C1: P_q $(q = 1, 2, 4, 5)$ *are local mappings, and* P_q $(q = 3$ *and* $6)$ *are global mappings, and* C2: *the mapping pair* $(P_q, P_{\mathrm{mod}(q+2,6)+1})$ *is skew-symmetric, then the solutions for regular and chaotic motion relative to two mapping equations*

$$\underbrace{P_{6(45)^{m_2}43(12)^{m_1}1} \circ \cdots \circ P_{6(45)^{m_2}43(12)^{m_1}1}}_{2^k} \mathbf{y} = \mathbf{y}$$

and

$$\underbrace{P_{6(45)^{m_1}43(12)^{m_2}1} \circ \cdots \circ P_{6(45)^{m_1}43(12)^{m_2}1}}_{2^k} \mathbf{y} = \mathbf{y} \quad (k = 0, 1, \ldots, \infty)$$

under N-periods with periodicity condition $\mathbf{y}_{i+2^{k+1}(m_2+m_1+2)} = \mathbf{y}_i$ *satisfy the following relations*:

$$\begin{aligned}&\mathrm{mod}(\varphi^{\mathrm{I}}_{i+2r(m_1+m_2+2)+j}, 2(2M+1)\pi)\\&\quad = \mathrm{mod}((2M+1)\pi + \mathrm{mod}(\varphi^{\mathrm{II}}_{i+2r(m_1+m_2+2)+\mathrm{mod}(2m_1+2+j,2(m_1+m_2+2))}, 2(2M+1)\pi), 2(2M+1)\pi),\\&y^{\mathrm{I}}_{i+2r(m_1+m_2+2)+j} = -y^{\mathrm{II}}_{i+2r(m_1+m_2+2)+\mathrm{mod}(2m_1+2+j,2(m_1+m_2+2))},\end{aligned} \tag{24}$$

or

$$\begin{aligned}&\mathrm{mod}(\varphi^{\mathrm{II}}_{i+2r(m_1+m_2+2)+j}, 2(2M+1)\pi)\\&\quad = \mathrm{mod}((2M+1)\pi + \mathrm{mod}(\varphi^{\mathrm{I}}_{i+2r(m_1+m_2+2)+\mathrm{mod}(2m_1+2+j,2(m_1+m_2+2))}, 2(2M+1)\pi), 2(2M+1)\pi),\\&y^{\mathrm{II}}_{i+2r(m_1+m_2+2)+j} = -y^{\mathrm{I}}_{i+2r(m_1+m_2+2)+\mathrm{mod}(2m_1+2+j,2(m_1+m_2+2))}\end{aligned} \tag{25}$$

for all $r = \{1, \ldots, 2^k\}$ *and* $j = \{0, 1, \ldots, 4m+3\}$. *The superscripts* (I, II) *represent the two mapping structures.*

The above results can be generalized in the following theorem.

Theorem 7. *Consider a non-smooth dynamical system with two symmetrical constraints in Eq.* (1) *with six mappings* P_q $(q = 1, 2, \ldots, 6)$. *If the following two properties exists*: C1: P_q $(q = 1, 2, 4, 5)$ *are local mappings, and* P_q $(q = 3$ *and* $6)$ *are global mappings, and* C2: *the mapping pair* $(P_q, P_{\mathrm{mod}(q+2,6)+1})$ *are skew-symmetric, then the solutions for regular and chaotic motion relative to* *two mappings*

$$\underbrace{P_{6(45)^{m_{k2}}43(12)^{m_{k1}}1} \circ \cdots \circ P_{6(45)^{m_{i2}}43(12)^{m_{i1}}1} \circ \cdots \circ P_{6(45)^{m_{12}}43(12)^{m_{11}}1}}_{k} \mathbf{y} = \mathbf{y}$$

and

$$\underbrace{P_{6(45)^{m_{k1}}43(12)^{m_{k2}}1} \circ \cdots \circ P_{6(45)^{m_{i1}}43(12)^{m_{i2}}1} \circ \cdots \circ P_{6(45)^{m_{11}}43(12)^{m_{12}}1}}_{k} \mathbf{y} = \mathbf{y} \quad (k = 0, 1, \ldots, \infty)$$

under N-periods with periodicity condition $\mathbf{y}_{i+\sum_{z=1}^{k}(2m_{s2}+2m_{s2}+4)} = \mathbf{y}_i$ *for a specific* $r = \{1, 2, \ldots, k\}$, *are*

$$\begin{aligned}
&\operatorname{mod}\left(\varphi^{\mathrm{I}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+j}, 2(2M+1)\pi\right) \\
&\quad = \operatorname{mod}\left((2M+1)\pi + \operatorname{mod}\left(\varphi^{\mathrm{II}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+\operatorname{mod}(2m_{r1}+2+j,2(m_{r1}+m_{r2}+2))}, 2(2M+1)\pi\right), 2(2M+1)\pi\right), \\
&y^{\mathrm{I}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+j} = -y^{\mathrm{II}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+\operatorname{mod}(2m_{r1}+2+j,2(m_{r1}+m_{r2}+2))},
\end{aligned} \tag{26}$$

or

$$\begin{aligned}
&\operatorname{mod}\left(\varphi^{\mathrm{II}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+j}, 2(2M+1)\pi\right) \\
&\quad = \operatorname{mod}\left((2M+1)\pi + \operatorname{mod}\left(\varphi^{\mathrm{I}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+\operatorname{mod}(2m_{r1}+2+j,2(m_{r1}+m_{r2}+2))}, 2(2M+1)\pi\right), 2(2M+1)\pi\right), \\
&y^{\mathrm{II}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+j} = -y^{\mathrm{I}}_{i+2\sum_{s=1}^{r-1}(m_{s2}+m_{s2}+2)+\operatorname{mod}(2m_{r1}+2+j,2(m_{r1}+m_{r2}+2))},
\end{aligned} \tag{27}$$

where $j = \{0, 1, \ldots, 4m+3\}$.

5. Conclusion

In this technical note, the symmetry of solutions in non-smooth dynamical systems with two symmetrical constraints is discussed. The grazing does not change the symmetry invariance of mapping structures in such dynamical systems, and the periodic and chaotic motions in such a dynamical system possess the same symmetry invariance as the basic mappings. Based on this investigation, the group structure of mapping combination exists. Thus, further investigations of this issue should be carried out. This theory can be applied to piecewise, linear and non-linear systems, impacting oscillator systems, friction-induced vibration systems, etc.

References

[1] S.F. Masri, General motion of impact dampers, *Journal of the Acoustical Society of America* 47 (1970) 229–237.

[2] G.X. Li, R.H. Rand, F.C. Moon, Bifurcation and Chaos in a forced zero-stiffness impact oscillator, *International Journal of Nonlinear Mechanics* 25 (4) (1990) 414–432.

[3] A.C.J. Luo, An unsymmetrical motion in a horizontal impact oscillator, *American Society of Mechanical Engineers, Journal of Vibrations and Acoustics* 124 (2002) 420–426.

[4] S.F. Masri, T.D. Caughey, On the stability of the impact damper, *American Society of Mechanical Engineers, Journal of Applied Mechanics* 33 (1966) 586–592.

[5] M. Senator, Existence and stability of periodic motions of a harmonically forced impacting system, *Journal of Acoustics Society of America* 47 (1970) 1390–1397.

[6] C.N. Bapat, N. Popplewell, K. McLachlan, Stable periodic motion of an impact pair, *Journal of Sound and Vibration* 87 (1983) 19–40.

[7] R.P.S. Han, A.C.J. Luo, W. Deng, Chaotic motion of a horizontal impact pair, *Journal of Sound and Vibration* 181 (1995) 231–250.

[8] A.C.J. Luo, S. Menon, Global chaos in a periodically forced, linear system with a dead-zone restoring force, *Chaos, Solitons and Fractals* 19 (2004) 1139–1199.