Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 274 (2004) 455–459
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Free vibration of a linked rod

C.Y. Wang*

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

Received 25 August 2003; accepted 5 September 2003

1. Introduction

The study of linked or segmented structures is important in the modelling of many practical multi-body systems such as robots and space structures (see e.g. Refs. [1–3]). The joints between links are often stiffened by rotational springs to maintain the final shape. Previous works in this area are few. Wang [4] studied the deformations of an axially rotated linked rod. Wang and Du [5] considered the non-linear statics of a linked rod loaded at one end. Wittenburg [6] formulated the vibrations of a linked cantilever, but no details are given. The present paper investigates in depth the basic problem of the free vibrations of a linked rod.

2. Formulation

Consider a rod of length L composed of N rigid links joined together (Fig. 1(a)). The links are uniform and of length $l = L/N$. There are rotational springs at the joints to maintain the straightness of the rod. Fig. 1(b) shows the forces on the nth link. Let (x'_{n-1}, y'_{n-1}) and (x'_n, y'_n) be the co-ordinates of the left and right ends of the link, respectively. A dynamic force balance gives

$$F'_n - F'_{n-1} = m(\ddot{x}'_n + \ddot{x}'_{n-1})/2, \tag{1}$$

$$G'_n - G'_{n-1} = m(\ddot{y}'_n + \ddot{y}'_{n-1})/2, \tag{2}$$

where F and G are horizontal and vertical forces, m is the mass of the link, and the two dots represent second derivatives in time. A dynamic moment balance about the mid-point gives

$$M_n - M_{n-1} + (G'_n + G'_{n-1})l\cos\theta_n/2 - (F'_n + F'_{n-1})l\sin\theta_n/2 = I\ddot{\theta}_n, \tag{3}$$

where M is the moment at the ends, θ is the angle of inclination, and I is the rotational moment of inertia. For uniform bars, $I = ml^2/12$. The rotational springs are assumed to be linear, such that

*Fax: +1-517-432-1562.
E-mail address: cywang@mth.msu.edu (C.Y. Wang).

doi:10.1016/j.jsv.2003.09.019

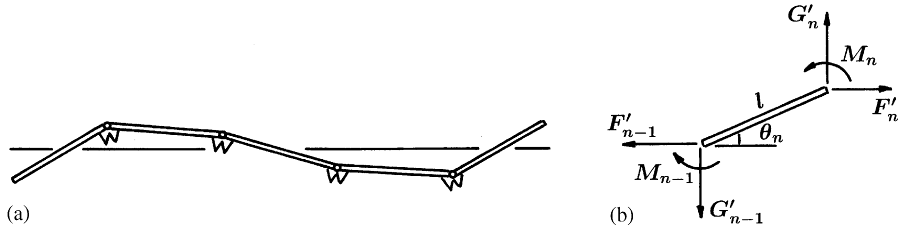

Fig 1. (a) The vibrating linked rod and (b) the nth link.

the end moment is proportional to the angle difference between two links,

$$M_n = k(\theta_{n+1} - \theta_n), \tag{4}$$

where k is the spring constant. Geometry also dictates

$$x'_n - x'_{n-1} = l\cos\theta_n, \quad y'_n - y'_{n-1} = l\sin\theta_n. \tag{5}$$

Normalize all lengths by L, forces by k/L, time by $L\sqrt{m/k}$ and drop primes. Eqs. (1)–(5) become

$$F_n - F_{n-1} = (\ddot{x}_n + \ddot{x}_{n-1})/2, \quad G_n - G_{n-1} = (\ddot{y}_n + \ddot{y}_{n-1})/2, \tag{6,7}$$

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} + (G_n + G_{n-1})\cos\theta_n/(2N) - (F_n + F_{n-1})\sin\theta_n/(2N) = \hat{I}\ddot{\theta}_n, \tag{8}$$

$$x_n - x_{n-1} = \cos\theta_n/N, \quad y_n - y_{n-1} = \sin\theta_n/N, \tag{9,10}$$

where

$$\hat{I} = \frac{I}{L^2 m} = \frac{1}{12N^2}. \tag{11}$$

The boundary conditions are that the rod has zero force and zero moment at the ends

$$F_0 = G_0 = F_N = G_N = 0, \tag{12}$$

$$M_0 = M_N = 0. \tag{13}$$

3. Vibration frequencies

Of interest are the natural frequencies of such a linked rod. Perturb from the horizontal state where $x_n = n/N$ and all other variables are zero.

$$x_n = n/N + \xi_n e^{i\omega t}, \quad y_n = \eta_n e^{i\omega t}, \tag{14}$$

$$F_n = f_n e^{i\omega t}, \quad G_n = g_n e^{i\omega t}, \tag{15}$$

$$\theta_n = \phi_n e^{i\omega t}, \tag{16}$$

where ξ, η, f, g, ϕ are small and ω is the vibration frequency normalized by the inverse time scale. The governing equations linearize to

$$f_n - f_{n-1} = -\omega^2(\xi_n + \xi_{n-1})/2, \quad g_n - g_{n-1} = -\omega^2(\eta_n + \eta_{n-1})/2, \tag{17, 18}$$

$$\phi_{n+1} - 2\phi_n + \phi_{n-1} + (g_n + g_{n-1})/(2N) = -\omega^2 \hat{I}\phi_n, \tag{19}$$

$$\xi_n - \xi_{n-1} = 0, \quad \eta_n - \eta_{n-1} = \phi_n/N. \tag{20, 21}$$

The boundary conditions are

$$f_0 = f_N = 0, \quad g_0 = g_N = 0, \quad \phi_0 = \phi_1, \quad \phi_N = \phi_{N+1}. \tag{22–25}$$

Here fictitious links $n = 0$ and $N + 1$ are added to facilitate the boundary conditions. It is evident that

$$f_n = 0, \quad \xi_n = 0. \tag{26}$$

The vibration problem is then obtained from Eqs. (18), (19), (21) and (23)–(25). There are $3N + 4$ equations and an equal number of unknowns. For non-trivial solutions, the eigenvalues or frequencies ω are found from the determinant of coefficients. Given the number of links N, there are $N - 1$ distinct non-trivial eigenvalues and modes.

The one link rigid rod does not vibrate. For $N = 2$, the frequency is found to be $4\sqrt{6}$. For $N = 3$, the frequencies are $3\sqrt{6}$ and $27\sqrt{2/5}$. For $N = 4$ the frequencies are $\sqrt{312 - 24\sqrt{137}}$, $16\sqrt{6/7}$, $\sqrt{312 + 24\sqrt{137}}$. A computer is used to find the numerical values of the frequencies for $N > 4$. Table 1 shows the first four frequencies.

As N approach infinity, the results for the linked rod should tend to those of a continuous elastic beam. Vibrations of a continuous beam have been considered previously (e.g. Ref. [6]). Without going through the details, the linearized beam equation is

$$\frac{d^4y}{dx^4} - \lambda^4 y = 0, \tag{27}$$

where

$$\lambda^4 = \frac{\rho\omega'^2 L^4}{EI}. \tag{28}$$

Here ρ is the mass per length, ω' is the frequency and EI is the flexural rigidity. The boundary conditions are zero moment and zero force at the ends

$$\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0 \quad \text{at } x = 0, 1. \tag{29}$$

Table 1
The first four natural frequencies ω

$N = 2$	3	4	5	6	8	10	15	20
9.798	7.348	5.576	4.470	3.727	2.796	2.237	1.492	1.119
—	17.076	14.81	12.19	10.23	7.701	6.165	4.111	3.084
—	—	24.34	22.62	19.67	15.00	12.07	8.058	6.045
—	—	—	31.56	30.42	24.53	19.87	13.31	9.991

Table 2
First four values of ωN compared to λ^2 of the continuous case

$N=6$	$N=8$	$N=10$	$N=15$	$N=20$	λ^2
22.362	22.371	22.372	22.373	22.373	22.373
61.398	61.607	61.650	61.669	61.672	61.673
118.01	120.01	120.70	120.87	120.90	120.90
182.53	196.22	198.73	199.70	199.82	199.86

The solution of y is in terms of $\cos(\lambda x)$, $\sin(\lambda x)$, $\cosh(\lambda x)$, $\sinh(\lambda x)$. For non-trivial solutions the boundary conditions yield the characteristic equation

$$1 - \cosh\lambda\cos\lambda = 0. \tag{30}$$

The eigenvalues are at 4.730, 7.853, 10.996, 14.137 etc. The jth eigenvalue is well approximated by $(j+0.5)\pi$ when $j>4$. The connection between discrete links and the continuous rod was delineated by Wang [3], that for large N

$$EI \approx \frac{kL}{N}. \tag{31}$$

Using this relation, we find

$$\lambda^2 \approx NL\omega'\sqrt{m/k} = \omega N. \tag{32}$$

Table 2 shows a comparison of ωN with λ^2.

It is seen that the discrete case does converge to the continuous case when the number of links become large. In fact for $N=20$, the first 12 values of ωN are within 3% of λ^2 computed from Eq. (30).

4. Vibration modes

The vibration modes can be found after the natural frequencies are obtained. Since the amplitude for small vibrations is arbitrary, given any ϕ_0 Eqs. (18), (19), (21), (23) and (24) are solved for the displacement η. For example, the five frequencies for the six-link rod are 3.727, 10.233, 19.669, 30.421, and 38.730. The corresponding modes are shown in Fig. 2. In general, the jth frequency has $j+1$ sign changes in displacement. The mode shapes are even with respect to the mid-point if j is odd, and the mode shapes are odd if j is even. The largest displacement always occurs at the ends. Notice that if N is even and the mode shape is odd, half of the rod represents a vibrating linked cantilever.

5. Discussion

The basic free–free vibration of a linked rod is studied for the first time. Eqs. (6)–(13) are the exact differential-difference dynamical equations, which, given the initial conditions, can be numerically integrated with some effort. In this paper the small, natural vibrations are studied and thus the linearized difference equations suffice.

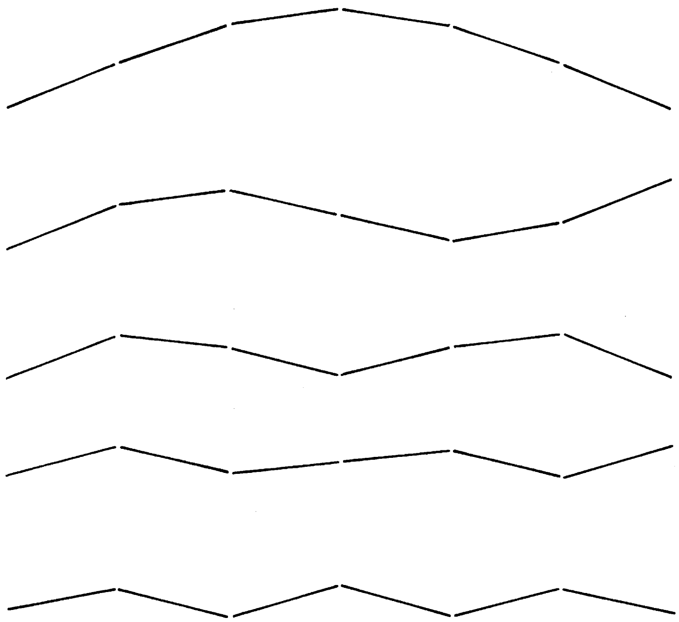

Fig. 2. The vibrational modes for $N = 6$. Eigen frequencies increase from top.

The vibration frequencies are found to be heavily dependent on the number of links. For small number of links, the frequencies are higher and fewer in number. For large number of links, the frequencies approach those of a continuous elastic rod through Eq. (32).

The rotational spring may be non-linear. For example Eq. (4) may include a term which is a cube of the angle difference. This non-linearity may influence the large deformations of the rod, but does not affect the natural frequencies and vibrational modes studied in this paper.

References

[1] M. Geradin, M.A. Cardona, *Flexible Multibody Dynamics*, Wiley, New York, 2001.

[2] R.L. Huston, *Multibody Dynamics*, Butterworth-Heinemann, Boston, 1990.

[3] C.Y. Wang, The axially rotating segmented rod, *International Journal of Solids and Structures* 30 (1993) 2437–2444.

[4] C.Y. Wang, Q. Du, Study of an inclined, segmented cantilever with tip load, *Quarterly of Applied Mathematics* 53 (1995) 81–93.

[5] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, Teubner, Stuttgart, 1997.

[6] E.B. Magrab, *Vibrations of Elastic Structural Members*, Sijthoff and Noordhoff, Alphen a/d Rijn, Netherlands, 1979.