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Letter to the Editor

On initial condition transformation for response calculation using modal analysis method with application to wheel/rail impact

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1. Introduction

The equation of motion for a linear dynamic system is given by

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}, \tag{1}$$

where **M** and **K** are symmetric mass and stiffness matrices, respectively, **X** is the displacement vector and **F** is the external excitation. For simplicity, damping is not considered here. If the natural frequencies and mode shapes of the system are known, using the linear transformation

$$\mathbf{X} = \mathbf{P}\mathbf{Y}, \tag{2}$$

where **P** is the mode shape matrix whose columns are the normalized modal vectors:

$$\mathbf{P} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_M \end{bmatrix}, \tag{3}$$

replacing the physical displacement vector **X** in Eq. (1) and pre-multiplying Eq. (1) by the transpose of the mode shape matrix result in

$$\ddot{\mathbf{Y}} + [\omega^2]\mathbf{Y} = \mathbf{P}^{\mathrm{T}}\mathbf{F}, \tag{4}$$

where **Y** is the modal displacement vector and $[\omega^2]$ is a diagonal matrix composed of the squared natural frequencies of the system:

$$[\omega^2] = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_M^2 \end{bmatrix}. \tag{5}$$

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Eq. (4) is decoupled in the modal co-ordinates and can be written as

$$\ddot{y}_n + \omega_n^2 y_n = f_n, \quad n = 1, 2, \ldots, M. \tag{6}$$

Working out each modal displacement, y_n, and using the linear transformation by Eq. (2), the physical response $\mathbf{X}$ to the external excitation $\mathbf{F}$ is obtained. The above modal analysis method is widely used for response calculations.

If the system's initial velocity or displacement is non-zero, they have to be transformed from the physical co-ordinates $\mathbf{X}$ into the modal co-ordinates $\mathbf{Y}$ in order to give the initial conditions for y_n calculation. If the number of the modal vectors in the mode shape matrix $\mathbf{P}$ is equal to the order of the system, i.e., $M = N$, the initial velocity and displacement in the modal co-ordinates can be determined by formulae

$$\dot{\mathbf{Y}}_0 = \mathbf{P}^{-1}\dot{\mathbf{X}}_0 \quad \text{and} \quad \mathbf{Y}_0 = \mathbf{P}^{-1}\mathbf{X}_0. \tag{7, 8}$$

However, if the modal vectors in matrix $\mathbf{P}$ are incomplete, i.e., $M < N$, Eqs. (7) and (8) cannot be used for the initial condition transformation because the mode shape matrix $\mathbf{P}$ now is not an $N \times N$ matrix, instead it is an $N \times M$ matrix, and its inverse matrix does not exist.

In practice, not all modes of a dynamic system need to or can be calculated, especially for large structures. In most cases, only the vibration modes in a specific frequency region are calculated and used for response calculation. Thus, the initial condition transformation from the physical co-ordinates to the modal co-ordinates by use of incomplete modes needs to be studied.

2. Initial condition transformation between two systems

The modal analysis method actually substitutes a modal system for the physical system to calculate vibration response. If the mode shapes are fully calculated and used for the system transformation between the physical and modal co-ordinates, the modal system is exactly equal to the physical system. If the mode shapes are partially calculated and used for the system transformation, the modal system is an approximation of the physical system. The key point is how to reduce the errors caused by the approximation when transforming the initial conditions between the two co-ordinates. One methodology for doing this is to minimize the distance between the two velocities and between the two displacements from the physical and modal co-ordinates, i.e., between $\dot{\mathbf{X}}_0$ and $\mathbf{P}\dot{\mathbf{Y}}_0$ and between $\mathbf{X}_0$ and $\mathbf{P}\mathbf{Y}_0$, respectively. Through minimizing the distances between the velocities and between the displacements, the formulae for the initial condition transformation from the physical co-ordinates to the modal co-ordinates can be derived.

2.1. Initial velocity transformation

The relationship between the physical and modal co-ordinates is given by Eq. (2) and the difference in the initial velocity between the two co-ordinates is given by

$$\Delta\dot{\mathbf{X}}_0 = \dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0, \tag{9}$$

where $\mathbf{P}$ is the mode shape matrix composed of the normalized modal vectors calculated in the frequency region considered, $\dot{\mathbf{X}}_0$ is the initial velocity in the physical co-ordinates and $\dot{\mathbf{Y}}_0$ is the initial velocity in the modal co-ordinates and is transformed from the physical co-ordinates. If $\mathbf{P}$ is composed of complete modal vectors of the system, Eq. (7) holds and $\Delta\dot{\mathbf{X}}_0 = 0$, otherwise $\Delta\dot{\mathbf{X}}_0 \neq 0$.

To reduce the errors caused by approximation, it is more reasonable to minimize the mass matrix weighted velocity distance, so that the velocity difference related to larger mass elements between the two systems is expected to be smaller. The mass matrix weighted distance ΔS_V between the initial velocities from the two systems is given by

$$\Delta S_V = (\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0)^{\mathrm{T}}\mathbf{M}(\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0), \tag{10}$$

where $\mathbf{P}$ is the incomplete mode shape matrix. At this stage, it is not yet known how the initial velocity in the modal co-ordinates is transformed from the physical co-ordinates. The essential condition to make ΔS_V minimum can be derived by

$$\frac{\partial \Delta S_V}{\partial \dot{\mathbf{Y}}_0} = \frac{\partial}{\partial \dot{\mathbf{Y}}_0}\left(\dot{\mathbf{X}}_0^{\mathrm{T}}\mathbf{M}\dot{\mathbf{X}}_0 - 2\dot{\mathbf{X}}_0^{\mathrm{T}}\mathbf{M}\mathbf{P}\dot{\mathbf{Y}}_0 + \dot{\mathbf{Y}}_0^{\mathrm{T}}\dot{\mathbf{Y}}_0\right) = 2\dot{\mathbf{Y}}_0 - 2\mathbf{P}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{X}}_0 = 0. \tag{11}$$

Thus, the initial velocity transformation from the physical co-ordinates to the modal co-ordinates can be performed using

$$\dot{\mathbf{Y}}_0 = \mathbf{P}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{X}}_0. \tag{12}$$

In fact, as

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{I}, \tag{13}$$

$$\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{M}. \tag{14}$$

Here $\mathbf{P}^{-1}$ is the inverse matrix of $\mathbf{P}$ in a general sense, because $\mathbf{P}^{-1}$ is an $M \times N$ and $\mathbf{P}$ an $N \times M$ matrix, $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$ and $\mathbf{P}\mathbf{P}^{-1} \neq \mathbf{I}$.

It can be verified that the mass matrix weighted distance between the initial velocities from the physical and modal systems is minimum when Eq. (12) is used, compared with using any other method, for transforming the initial velocity from the physical to the modal co-ordinates. Assuming that $\hat{\dot{\mathbf{Y}}}_0$ is the initial modal velocity transformed using any other method, $\Delta\hat{S}_V$ is the corresponding initial velocity distance between the physical and modal systems and can be represented by

$$\begin{aligned}\Delta\hat{S}_V &= (\dot{\mathbf{X}}_0 - \mathbf{P}\hat{\dot{\mathbf{Y}}}_0)^{\mathrm{T}}\mathbf{M}(\dot{\mathbf{X}}_0 - \mathbf{P}\hat{\dot{\mathbf{Y}}}_0) \\ &= [(\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0) + \mathbf{P}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)]^{\mathrm{T}}\mathbf{M}[(\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0) + \mathbf{P}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)] \\ &= \Delta S_V + 2(\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0)^{\mathrm{T}}\mathbf{M}\mathbf{P}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0) + (\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)^{\mathrm{T}}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0),\end{aligned} \tag{15}$$

where ΔS_V is the initial velocity distance between the physical system and the modal system where the initial modal velocity $\dot{\mathbf{Y}}_0$ is calculated using Eq. (12). Examining the second term on the right side of Eq. (15), it can be written as

$$2(\dot{\mathbf{X}}_0 - \mathbf{P}\dot{\mathbf{Y}}_0)^{\mathrm{T}}\mathbf{M}\mathbf{P}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0) = 2(\dot{\mathbf{X}}_0^{\mathrm{T}}\mathbf{M}\mathbf{P} - \dot{\mathbf{Y}}_0^{\mathrm{T}})(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0) = 0 \tag{16}$$

because of Eq. (12). Thus, Eq. (15) becomes

$$\Delta\hat{S}_V = \Delta S_V + (\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)^{\mathrm{T}}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0) \tag{17}$$

This implies

$$\Delta\hat{S}_V \geqslant \Delta S_V \tag{18}$$

because $(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)^{\mathrm{T}}(\dot{\mathbf{Y}}_0 - \hat{\dot{\mathbf{Y}}}_0)$ is non-negative. Eq. (18) means that the mass matrix weighted velocity distance between the two systems is minimum when Eq. (12) is used to transform the initial velocity from the physical co-ordinates to the modal co-ordinates.

2.2. Initial displacement transformation

The formula for the initial displacement transformation can be derived by minimizing the stiffness matrix weighted displacement distance between the physical and modal systems. The stiffness matrix weighted distance between the initial displacements from the two systems is given by

$$\Delta S_D = (\mathbf{X}_0 - \mathbf{P}\mathbf{Y}_0)^{\mathrm{T}}\mathbf{K}(\mathbf{X}_0 - \mathbf{P}\mathbf{Y}_0). \tag{19}$$

The essential condition to make ΔS_D minimum can be derived by

$$\frac{\partial \Delta S_D}{\partial \mathbf{Y}_0} = \frac{\partial}{\partial \mathbf{Y}_0}\left(\mathbf{X}_0^{\mathrm{T}}\mathbf{K}\mathbf{X}_0 - 2\mathbf{X}_0^{\mathrm{T}}\mathbf{K}\mathbf{P}\mathbf{Y}_0 + \mathbf{Y}_0^{\mathrm{T}}[\omega^2]\mathbf{Y}_0\right) = 2[\omega^2]\mathbf{Y}_0 - 2\mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{X}_0 = 0. \tag{20}$$

Thus, the initial displacement transformation from the physical co-ordinates to the modal co-ordinates can be performed using

$$\mathbf{Y}_0 = [\omega^2]^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{X}_0, \tag{21}$$

where $[\omega^2]^{-1}$ is diagonal:

$$[\omega^2]^{-1} = \begin{bmatrix} \frac{1}{\omega_1^2} & & & \\ & \frac{1}{\omega_2^2} & & \\ & & \ddots & \\ & & & \frac{1}{\omega_M^2} \end{bmatrix}. \tag{22}$$

From the relationship between the natural frequency ω_n and modal vector $\mathbf{\Phi}_n$,

$$\mathbf{M}\mathbf{\Phi}_n = \frac{1}{\omega_n^2}\mathbf{K}\mathbf{\Phi}_n, \tag{23}$$

it can be derived that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M} = [\omega^2]^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{K}. \tag{24}$$

Thus, Eq. (21) can be given as

$$\mathbf{Y}_0 = \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{X}_0. \tag{25}$$

This is consistent with Eq. (12) for the initial velocity transformation. It can also be verified that the stiffness matrix weighted distance between the initial displacements from the physical and modal systems is minimum when Eq. (25) is used to transform the initial displacement.

3. Calculation examples

3.1. Example 1: A three degrees of freedom (d.o.f.) system without damping

A 3-d.o.f. system shown in Fig. 1 is chosen to calculate vibration response to the initial velocities at m_1 and m_2 using the modal analysis method. The initial conditions of the system are assumed to be

$$\mathbf{X_0} = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{X}}_\mathbf{0} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\mathrm{T} \mathrm{m/s}.$$

There is no external excitation to the system. The system's parameters are: $m_1 = m_3 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$, $k_1 = k_4 = 1\,\mathrm{N/m}$, $k_2 = k_3 = 2\,\mathrm{N/m}$. The system's natural frequencies are calculated to be

$$\omega_1 = 0.66, \quad \omega_2 = 1.73, \quad \omega_3 = 2.14.$$

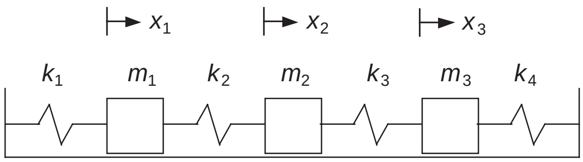

Fig. 1. A 3-d.o.f. system.

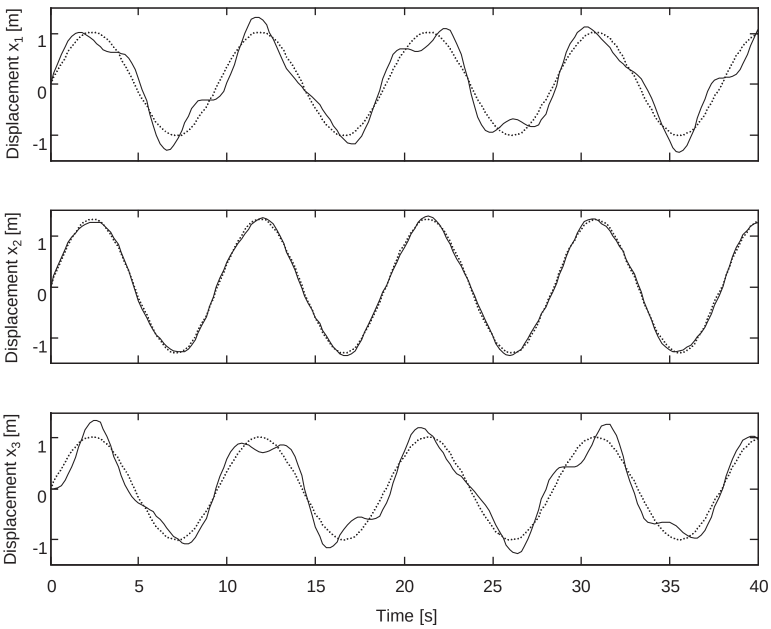

Fig. 2. Vibration displacements of the 3-d.o.f. system without damping due to unit initial velocities at m_1 and m_2: —, exact solutions; $\cdots$, approximated solutions.

In the approximated calculations using the modal analysis method, only the first vibration mode is taken account of. The initial velocity in the modal co-ordinates is calculated using Eq. (12). The results are shown in Fig. 2 and they are compared with the exact solutions. The approximated solutions can be seen to be quite close to the exact solutions, especially for the vibration at m_2.

3.2. Example 2: A 3-d.o.f. system with damping

The same system and initial conditions are used as in example 1, but damping is added to the modal system. The initial velocity in the modal co-ordinates is also calculated using Eq. (12). The modal damping ratio is chosen to be 0.1 for each mode, although only the first vibration mode is taken account in the approximated calculations. The results are shown in Fig. 3. It is seen that the differences between the approximated and exact solutions are larger at the beginning, but they become smaller and smaller when vibration decays with time.

Concerning the difference in vibration energy over a time interval due to the initial displacement or velocity transformation between the physical and modal systems, it remains unchanged in an undamped system because there are no energy input and dissipation. In a damped system, however, the vibration energy difference between the physical and modal systems becomes smaller and smaller, as the contribution from the residual modes which are excluded from the response decreases due to the energy dissipation. As a result, the difference between the approximated and exact response becomes smaller and smaller. This can be observed from Fig. 3.

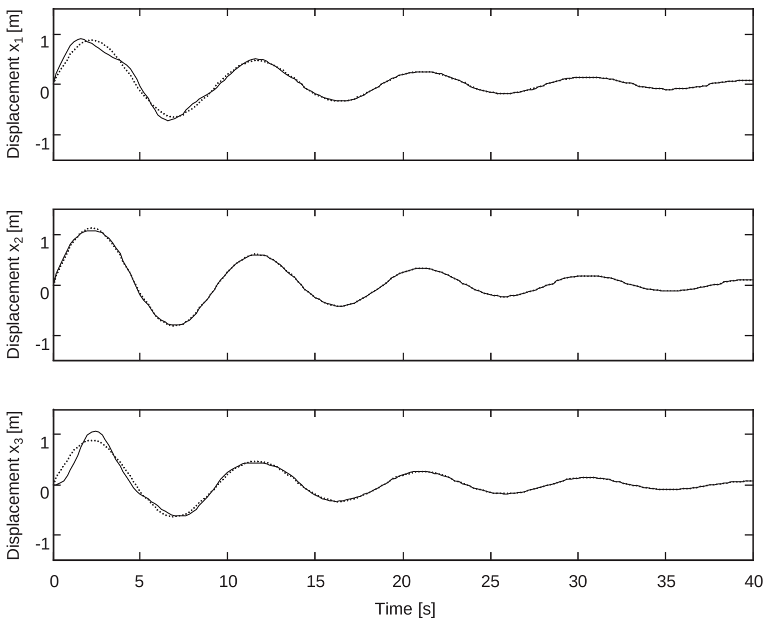

Fig. 3. Vibration displacements of the 3-d.o.f. system with damping due to unit initial velocities at m_1 and m_2: —, exact solutions; ···, approximated solutions.

3.3. Example 3: Wheel/rail impact simulation

This example is from the practice of railway engineering. Wheel and rail running surfaces are not perfectly smooth but contain discontinuities, such as rail joints, switches and wheel flats. These discontinuities on the wheel and rail can generate large impact forces between the wheel and track when wheels with flats subsequently rotate or wheels roll over a rail joint. As a consequence, a transient impact noise is produced.

A relative displacement excitation model [1] schematically shown in Fig. 4 is used to calculate wheel/track impact. In such a model, the wheel remains stationary on the rail and the discontinuities on the wheel or rail rolling surfaces are effectively moved at the train speed V between the wheel and rail as an excitation. The wheel interacts with the rail through a non-linear Hertzian contact spring and loss of contact between the wheel and rail is allowed.

It is known from studies of rolling noise that the wheel modes containing a significant radial component of motion at the contact zone dominate the noise radiation of the wheel/rail system in the frequency region above about 2 kHz [2]. Fig. 5 shows the radial receptance (displacement divided by force) of a railway wheel at the wheel/rail contact point, which is calculated using a finite element model. It can be seen that the wheel modes in the frequency range up to 5 kHz are significant.

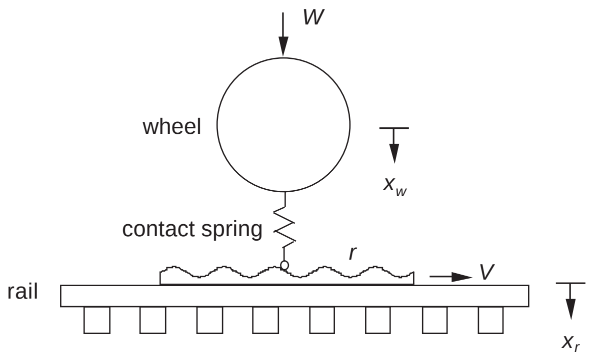

Fig. 4. Relative displacement excitation model for wheel/rail interaction.

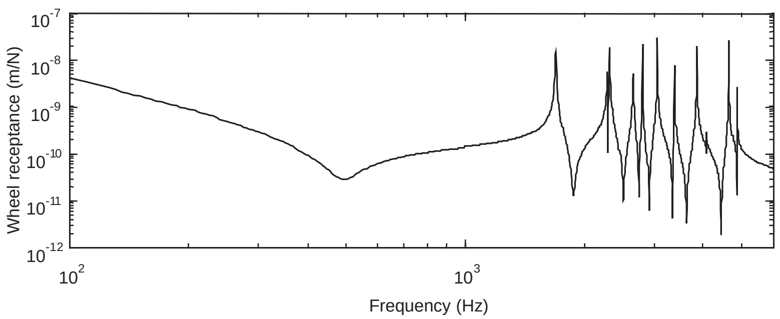

Fig. 5. Radial receptance of a railway wheel at the wheel/rail contact point.

Based on the modal analysis, the radial receptance of a wheel at the contact point can be given as

$$\alpha^W = \sum_{n=1}^{M} \frac{\phi_n^2}{\omega_n^2 - \omega^2 + \mathrm{i}2\zeta_n\omega_n\omega}, \tag{26}$$

where, for the nth mode, ϕ_n is the normalized mode shape at the contact point, ω_n is the natural frequency, ζ_n is the damping ratio and M is the number of the modes in the frequency region considered.

With the normal contact force being applied to the wheel, the differential equation of motion corresponding to each mode can be given as

$$\ddot{y}_n + 2\zeta_n\omega_n\dot{y}_n + \omega_n^2 y_n = \phi_n(W - f_c), \quad n = 1, 2, \ldots, M, \tag{27}$$

where y_n is the modal displacement of the wheel, W is the static load due to the vehicle weight (W is assumed to act at the wheel/rail contact point instead of the wheel center for simplicity) and f_c is the normal contact force between the wheel and rail. The wheel displacement x_w at the contact point is therefore composed of a superposition of all the modal displacement y_n:

$$x_w = \sum_{n=1}^{M} \phi_n y_n. \tag{28}$$

The normal contact force f_c is non-linear against the contact deflection and it follows that

$$f_c = \begin{cases} C_H(x_w - x_r - r)^{3/2}, & x_w - x_r - r > 0, \\ 0, & x_w - x_r - r \leqslant 0, \end{cases} \tag{29}$$

where x_r is the rail displacement at the contact point, r is the relative displacement excitation and C_H is the Hertzian constant. The rail displacement x_r is calculated using an equivalent track model represented by a fourth order differential equation [3]. The relative displacement excitation r is chosen to be the center trajectory of a railway wheel rolling over a dipped rail joint [4], see Fig. 6.

The initial velocities of the wheel and rail are assumed to be zero. The initial displacements are the static displacements calculated under a vehicle load $W = 100\,\text{kN}$. The initial displacement

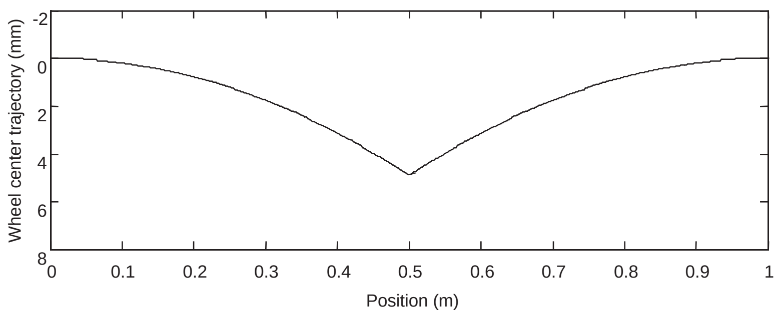

Fig. 6. Trajectory of a railway wheel center when it rolls over a dipped rail joint.

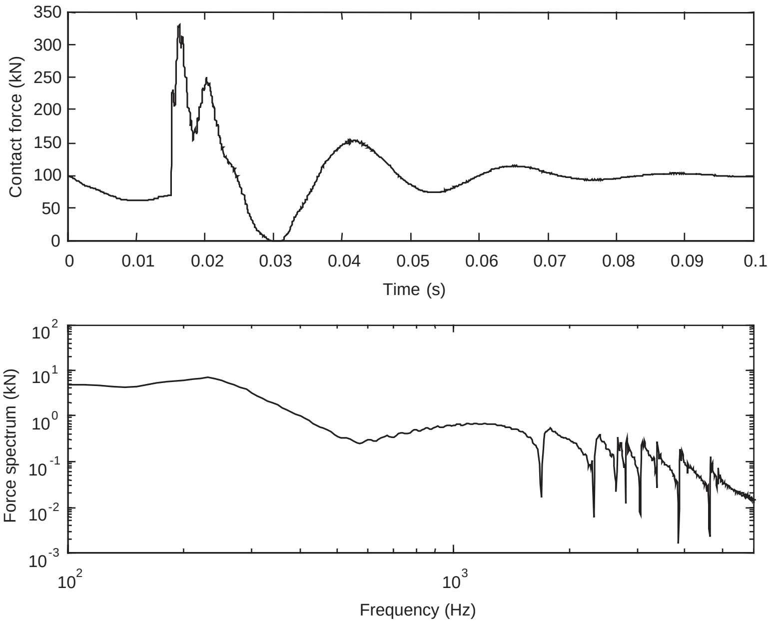

Fig. 7. Wheel/rail interaction force when the wheel rolls over a dipped rail joint at 120 km/h.

of the wheel needs to be transformed from the physical co-ordinates into the modal co-ordinates. As the modal vector and the mass matrix of the wheel are of very large volume, for simplicity the initial potential energy of the wheel is assumed to be dominated by the rigid body motion mode. This is because the rigid body displacement of the wheel caused by the vehicle load W is much larger than the elastic deformations of the wheel and thus the latter can be ignored. The initial modal displacement of the wheel is therefore given as

$$y_{10} = \frac{x_{w0}}{\phi_1} \quad \text{and} \quad y_{n0} = 0, \qquad n = 2, 3, \ldots, M,$$

where x_{w0} is the initial displacement of the wheel in the physical co-ordinates and ϕ_1 is the normalized rigid body mode of the wheel at the contact point.

The fourth order Runge–Kutta method is used to simulate the wheel/rail non-linear impact, although the wheel and track are linear. The wheel mass is 600 kg and its rolling speed is 120 km/h. The rail type is UIC60 (about 60 kg/m) with medium stiffness supports [4]. Twelve flexible modes up to 5 kHz and one rigid body mode of the wheel are taken account in the calculations. The simulation results are given in Fig. 7 in terms of the wheel/rail impact force in both the time domain and the frequency domain. The high-frequency components of the contact force due to the high-frequency modes of the wheel can be observed from the force spectrum, where the troughs correspond to the peaks in the wheel receptance in Fig. 5.

4. Conclusions

A vibration system's response can be calculated using the modal analysis method. If the system's initial velocity or displacement is non-zero, they have to be transformed from the physical co-ordinates into the modal co-ordinates using the mode shape matrix in order to work out the modal responses. The formulae for initial velocity and displacement transformation using incomplete modal vectors have been derived based on the methodology of minimizing the mass and stiffness matrix weighted distance, respectively, between the velocities and between the displacements from the physical and modal systems. Two simple calculation examples of a 3-d.o.f. system with and without damping have been presented in which the initial conditions are transformed using the derived formulae. The methodology of minimizing the velocity distance and the displacement distance between the two systems for initial condition transformation is verified to be effective by the calculation examples.

Although the formulae for initial velocity and displacement transformation are derived based on an undamped system, they can also be used for damped systems. For damped vibration caused by initial velocity or displacement, the energy difference between the physical and modal systems becomes smaller and smaller due to the energy dissipation, and thus the difference between the approximated and exact responses becomes smaller and smaller.

Finally, the initial condition transformation methodology is effectively applied to a wheel/rail impact problem, in which a modal railway wheel is used to take account of the high-frequency components of the wheel vibration.

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