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Letter to the Editor

Equivalent non-linear system method for stochastically excited and dissipated integrable Hamiltonian systems-resonant case

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1. Introduction

A natural way to predict the response of a non-linear stochastic system is to replace the system with an equivalent system whose response is exactly predictable, and to take the exact response probability density or statistics of the equivalent system as the approximate one of the original system. This idea leads to several different types of statistically equivalent system methods, depending upon types of equivalent systems and equivalence criteria. Among them, the most simple and broadly applicable one is the statistical linearization or equivalent linearization, or stochastic linearization. In this method, the equivalent system is linear with external Guassian excitation and the equivalence criterion is usually the minimum mean square discrepancy between the original and equivalent systems. This method was proposed in early 1950s and now is quite mature. For the details, readers are referred to monograph [1] and review papers [2–4]. However, the mostly used Gaussian linearization is not justified mathematically [5,6]. Furthermore, by using this method, only the first and second moments of the response can be predicted, which define uniquely a Gaussian distribution. So the statistical linearization is not applicable to the non-linear stochastic systems with intrinsic non-linearity and (or) with parametric (multiplicative) excitations.

To improve the statistical linearization, the so-called equivalent non-linear system method was developed. In this method, the equivalent system is non-linear one subject to external (added) and (or) parametric (multiplicative) excitations of Gaussian white noises and equivalent criterion have several different options. In 1980s, several versions of this method were proposed [7–10] mainly for single degree-of-freedom non-linear stochastic systems. Later, the method was extended to multi-degree-of-freedom stochastically excited and dissipated Hamiltonian systems, including non-integrable, intergrable, and partially integrable ones [11–13]. An important feature of this

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version of equivalent non-linear system method is that the stiffness and stochastic excitations are kept the same for the original and equivalent systems and only damping terms are replaced. Thus, part of the non-linear characteristics of the original system can be carried over to the equivalent one and the method is applicable to the non-linear stochastic systems with intrinsic non-linearity and (or) with parametric excitations. Furthermore, the stationary probability density rather than a few moments can be predicted by using this method.

In the equivalent non-linear system method for stochastically excited and dissipated integrable Hamiltonian systems proposed previously [12], to obtain the approximate stationary probability density of the original system, $n(n+1)/2$ damping coefficients of the equivalent system are first determined. In fact, however, it is not necessary. To obtain the approximate stationary solution, only n derivatives of the probability potential with respect to first integrals of the associated Hamiltonian system in non-resonant case are needed to be determined. Besides, the method was developed only for non-resonant case.

In the present paper, the equivalent non-linear system method for MDOF stochastically excited and dissipated integrable Hamiltonian systems in resonant case is developed. The approximate probability density of a given non-linear stochastic system is obtained via finding n derivatives of probability potential with respect to first integrals and α derivatives of probability potential with respect to combinations of angle variables. The application and effectiveness of the new procedure are illustrated with an example.

2. Exact stationary solution

The prerequisite for an equivalent non-linear system method for certain class of non-linear stochastic systems is that there are sufficient exactly stationary solutions for that class of systems. Before developing the equivalent system method, consider the exact stationary solutions of n d.o.f. stochastically excited and dissipated integrable Hamiltonian systems. The equations of motion of such a system are of the form:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i},$$
$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - c_{ij}\frac{\partial H'}{\partial P_j} + f_{ik}\xi_k(t), \quad i,j = 1,2,\ldots,n, \;\; k = 1,2,\ldots,m, \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H' = H'(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian; $c_{ij} = c_{ij}(\mathbf{Q},\mathbf{P})$ represent coefficients of quasi-linear dampings; $f_{ik} = f_{ik}(\mathbf{Q},\mathbf{P})$ are differentiable functions representing intensities of stochastic excitations; $\xi_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[\xi_k(t)\xi_l(t+\tau)] = 2D_{kl}\delta(\tau)$.

System (1) can be modelled as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equations by adding Wong–Zakai correction terms. Splitting the Wong–Zakai correction terms into conservative and dissipative parts and combining them with $-\partial H'/\partial Q_i$ and $-c_{ij}\partial H'/\partial P_j$, respectively, the following Itô stochastic differential equations are obtained:

$$\mathrm{d}Q_i = \frac{\partial H}{\partial P_i}\mathrm{d}t,$$
$$\mathrm{d}P_i = -\left(\frac{\partial H}{\partial Q_i} + m_{ij}\frac{\partial H}{\partial P_j}\right)\mathrm{d}t + \sigma_{ik}B_k(t), \quad i,j = 1,2,\ldots,n, \;\; k = 1,2,\ldots,m, \tag{2}$$

where $H = H(\mathbf{Q},\mathbf{P})$ and $m_{ij} = m_{ij}(\mathbf{Q},\mathbf{P})$ are, respectively, the Hamiltonian and damping coefficients modified by the Wong–Zakai correction terms; $\sigma_{ij} = \sigma_{ij}(\mathbf{Q},\mathbf{P})$ with $\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}} = 2\mathbf{fDf}^{\mathrm{T}}$; $B_k(t)$ are standard Wiener processes.

It has been shown that the exact-stationary solution of Eq. (2) depends upon the integrability and resonance of the associated Hamiltonian system with Hamiltonian H [14,15].

Suppose that the modified Hamiltonian system with Hamiltonian H is integrable with n first integrals $H_1, H_2, \ldots, H_n$ (one of H_i can be replaced by H) and internally resonant with α resonant relations

$$k_i^u\omega_i = 0, \quad u = 1,2,\ldots,\alpha, \tag{3}$$

where k_i^u are small integers and $\omega_i = \omega_i(\mathbf{I}) = \partial H/\partial I_i$ are the n frequencies of the Hamiltonian system. Action variables I_i are related to H_s by $I_i = f_i(\mathbf{H})$. Introduce angle variables $\theta_i = \omega_i(\mathbf{I})t + \delta_i$ and combinations of angle variables $\psi_u = k_i^u\theta_i$, $u = 1,2,\ldots,\alpha$. The exact stationary solution in resonant case is of the form [14]

$$p(\mathbf{q},\mathbf{p}) = C\exp[-\lambda(\mathbf{H},\boldsymbol{\psi})]|_{\mathbf{H}=\mathbf{H}(\mathbf{q},\mathbf{p}),\boldsymbol{\psi}=\boldsymbol{\psi}(\mathbf{q},\mathbf{p})}, \tag{4}$$

where $\boldsymbol{\psi} = [\psi_1\psi_2\ldots\psi_\alpha]^{\mathrm{T}}$ and probability potential λ satisfy the following n first order linear partial differential equations:

$$2m_{ij}\frac{\partial H}{\partial p_j} + \frac{\partial b_{ij}}{\partial p_j} - b_{ij}\left(\frac{\partial H_s}{\partial p_j}\frac{\partial \lambda}{\partial H_s} + \frac{\partial \psi_u}{\partial p_j}\frac{\partial \lambda}{\partial \psi_u}\right) = 0,$$
$$i,j,s = 1,2,\ldots,n, \;\; u = 1,2,\ldots,\alpha. \tag{5}$$

If the derivatives of probability potential with respect to first integrals, $\partial\lambda/\partial H_s$, and with respect to the combinations of angle variables, $\partial\lambda/\partial\psi_u$, can be obtained from Eq. (5) and they satisfy the following compatibility conditions:

$$\frac{\partial^2\lambda}{\partial H_{s_1}\partial H_{s_2}} = \frac{\partial^2\lambda}{\partial H_{s_2}\partial H_{s_1}}, \quad \frac{\partial^2\lambda}{\partial \psi_{u_1}\partial \psi_{u_2}} = \frac{\partial^2\lambda}{\partial \psi_{u_2}\partial \psi_{u_1}}, \quad \frac{\partial^2\lambda}{\partial H_s\partial \psi_u} = \frac{\partial^2\lambda}{\partial \psi_u\partial H_s}, \tag{6}$$

then the solution to Eq. (5) is

$$\lambda = \int^{H_s}\frac{\partial\lambda}{\partial H_s}\mathrm{d}H_s + \int^{\psi_u}\frac{\partial\lambda}{\partial\psi_u}\mathrm{d}\psi_u. \tag{7}$$

The right-hand side of Eq. (7) is two line integrals with integrands summing over $s = 1,2,\ldots,n$ and $u = 1,2,\ldots,\alpha$, respectively. The exact stationary probability density of the response of system (1) is obtained by substituting Eq. (7) into Eq. (4).

It is noted that the key to find the exact stationary solution of system (1) is to find both $\partial\lambda/\partial H_s$ and $\partial\lambda/\partial\psi_u$ satisfying Eqs. (5) and (6). It will be shown in the following that these derivatives are also the key for finding the approximate stationary solution by using the equivalent non-linear system method.

If the action-angle variables I_s, θ_s for the modified Hamiltonian system with Hamiltonian H can be found, then Eqs. (4)–(7) still hold if H_s are replaced by I_s.

3. Equivalent non-linear system method

Suppose that we are given an n d.o.f. stochastically excited and dissipated integrable Hamiltonian system whose Itô equations are of the form

$$\begin{aligned} dQ_i &= \frac{\partial H}{\partial P_i} dt, \\ dP_i &= -\left(\frac{\partial H}{\partial Q_i} + M_{ij}\frac{\partial H}{\partial P_j}\right) dt + \sigma_{ik} B_k(t), \quad i,j = 1,2,\ldots,n, \ \ k = 1,2,\ldots,m, \end{aligned} \tag{8}$$

where all notations are the same as those in Eq. (2) except $M_{ij} = M_{ij}(\mathbf{Q},\mathbf{P})$, which are the coefficients of quasi-linear dampings. Assume that the exact stationary solution to system (8) is not obtainable. The objective of our study is to find the approximate stationary solution of system (8) by using the equivalent non-linear system method. The equivalent system is of the form of Eq. (2). Systems (8) and (2) have the same Hamiltonian and stochastic excitations and they differ only in damping coefficients. It is seen from Eq. (5) that m_{ij} contribute the exact stationary solution jointly rather than individually. Thus, as for finding the approximate stationary solution of system (8), it is not necessary to determine all equivalent damping coefficients individually as did in [12]. Instead, it is sufficient to find the equivalent $\partial\lambda/\partial H_s$ and $\partial\lambda/\partial\psi_u$.

The equivalent system has exact stationary solution of the form of Eq. (4) with λ satisfying Eqs. (5) and (6). Let

$$\begin{aligned} g_s = g_s(\mathbf{q},\mathbf{p}) = \partial\lambda/\partial H_s, \quad h_u = h_u(\mathbf{q},\mathbf{p}) = \partial\lambda/\partial\psi_u, \\ s = 1,2,\ldots,n, \ \ u = 1,2,\ldots,\alpha. \end{aligned} \tag{9}$$

The difference between the given and equivalent systems is

$$\varepsilon_i = (M_{ij} - m_{ij})\frac{\partial H}{\partial P_j} = M_{ij}\frac{\partial H}{\partial P_j} + \frac{1}{2}\frac{\partial b_{ij}}{\partial P_j} - \tfrac{1}{2} b_{ij}\left(\frac{\partial H_s}{\partial P_j} g_s + \frac{\partial \psi_u}{\partial P_j} h_u\right). \tag{10}$$

Once the equivalent g_s and h_u are found for given M_{ij} based on some criterion and they satisfy the compatibility conditions in Eq. (6), the approximate stationary solution of given system (8) is of the form of Eq. (4) with λ defined by Eq. (7). The following three equivalence criteria are used in the following.

The first criterion for obtaining g_s and h_u is minimizing the mean square deficiency in damping forces with respect to g_s and h_u, i.e.,

$$\min_{g_s,h_u} E[\varepsilon_i\varepsilon_i]. \tag{11}$$

The necessary conditions for Eq. (11) are

$$\delta E[\varepsilon_i\varepsilon_i]/\delta g_s = 0, \quad s = 1,2,\ldots,n, \tag{12a}$$

$$\delta E[\varepsilon_i\varepsilon_i]/\delta h_u = 0, \quad u = 1, 2, \ldots, \alpha, \tag{12b}$$

or

$$\int_{-\infty}^{\infty} \varepsilon_i(\delta\varepsilon_i/\delta g_s)p(\mathbf{q}, \mathbf{p})\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p} = 0, \quad s = 1, 2, \ldots, n, \tag{13a}$$

$$\int_{-\infty}^{\infty} \varepsilon_i(\delta\varepsilon_i/\delta h_u)p(\mathbf{q}, \mathbf{p})\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p} = 0, \quad u = 1, 2, \ldots, \alpha. \tag{13b}$$

Introducing transformation from $\mathbf{q}, \mathbf{p}$ to $\mathbf{H}, \boldsymbol{\psi}$, $\boldsymbol{\theta}_1 = [\theta_1\theta_2\cdots\theta_{n-\alpha}]^{\mathrm{T}}$, Eqs. (13a) and (13b) can be rewritten as

$$\int_0^{2\pi}\int_0^{\infty} p(\mathbf{H}, \boldsymbol{\psi})\int_0^{2\pi}\left(\varepsilon_i\frac{\delta\varepsilon_i}{\delta g_s}\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right)\mathrm{d}\boldsymbol{\theta}_1\,\mathrm{d}\mathbf{H}\,\mathrm{d}\boldsymbol{\psi} = 0,$$
$$s = 1, 2, \ldots, n, \tag{14a}$$

$$\int_0^{2\pi}\int_0^{\infty} p(\mathbf{H}, \boldsymbol{\psi})\int_0^{2\pi}\left(\varepsilon_i\frac{\delta\varepsilon_i}{\delta h_u}\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right)\mathrm{d}\boldsymbol{\theta}_1\,\mathrm{d}\mathbf{H}\,\mathrm{d}\boldsymbol{\psi} = 0,$$
$$u = 1, 2, \ldots, \alpha, \tag{14b}$$

where $|\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)/\partial(\mathbf{q}, \mathbf{p})|$ is the absolute value of the Jacobian determinant for the transformation. Since $p(\mathbf{H}, \boldsymbol{\psi})$ is unknown, to proceed further, Eqs. (14a) and (14b) are replaced by the following more restrictive sufficient conditions:

$$\int_0^{2\pi}\left[\left(\varepsilon_i\frac{\delta\varepsilon_i}{\delta g_s}\right)\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right]\mathrm{d}\boldsymbol{\theta}_1 = 0, \quad s = 1, 2, \ldots, n, \tag{15a}$$

$$\int_0^{2\pi}\left[\left(\varepsilon_i\frac{\delta\varepsilon_i}{\delta h_u}\right)\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right]\mathrm{d}\boldsymbol{\theta}_1 = 0, \quad u = 1, 2, \ldots, \alpha. \tag{15b}$$

Inserting Eq. (10) into Eqs. (15a) and (15b) leads to

$$\int_0^{2\pi}\left\{\left[M_{ij}\frac{\partial H}{\partial p_j} + \frac{1}{2}\frac{\partial b_{ij}}{\partial p_j} - \frac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s + \frac{\partial \psi_u}{\partial p_j}h_u\right)\right]b_{ij}\frac{\partial H_s}{\partial p_j}\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1 = 0,$$
$$s = 1, 2, \ldots, n, \tag{16a}$$

$$\int_0^{2\pi}\left\{\left[M_{ij}\frac{\partial H}{\partial p_j} + \frac{1}{2}\frac{\partial b_{ij}}{\partial p_j} - \frac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s + \frac{\partial \psi_u}{\partial p_j}h_u\right)\right]b_{ij}\frac{\partial \psi_u}{\partial p_j}\bigg/\left|\frac{\partial(\mathbf{H}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1 = 0,$$
$$u = 1, 2, \ldots, \alpha. \tag{16b}$$

The second criterion for obtaining g_s and h_u is minimizing the mean square deficiency in the two energies dissipated by the damping forces in the original and equivalent systems with respect to g_s and h_u, i.e.,

$$\min_{g_s, h_u} E\left[\left(\varepsilon_i\frac{\partial H}{\partial p_j}\right)^2\right]. \tag{17}$$

Then, similar derivation as that from Eqs. (13a) to (16b) yields

$$\int_0^{2\pi}\left\{\frac{\partial H}{\partial p_i}\left[M_{ij}\frac{\partial H}{\partial p_j}+\frac{1}{2}\frac{\partial b_{ij}}{\partial p_j}-\tfrac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s+\frac{\partial \psi_u}{\partial p_j}h_u\right)\right]\times\left(\frac{\partial H}{\partial p_i}b_{ij}\frac{\partial H_s}{\partial p_j}\right)\Big/\left|\frac{\partial(\mathbf{H},\boldsymbol{\psi},\boldsymbol{\theta}_1)}{\partial(\mathbf{q},\mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1=0,\quad s=1,2,\ldots,n, \tag{18a}$$

$$\int_0^{2\pi}\left\{\frac{\partial H}{\partial p_i}\left[M_{ij}\frac{\partial H}{\partial p_j}+\frac{1}{2}\frac{\partial b_{ij}}{\partial p_j}-\tfrac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s+\frac{\partial \psi_u}{\partial p_j}h_u\right)\right]\times\left(\frac{\partial H}{\partial p_i}b_{ij}\frac{\partial \psi_u}{\partial p_j}\right)\Big/\left|\frac{\partial(\mathbf{H},\boldsymbol{\psi},\boldsymbol{\theta}_1)}{\partial(\mathbf{q},\mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1=0,\quad u=1,2,\ldots,\alpha. \tag{18b}$$

The third criterion for obtaining g_s and h_u is equality of the expected time rates of H_s (or I_s) and ψ_u associated with the given and equivalent systems, i.e.,

$$E\left[\frac{\mathrm{d}H_s}{\mathrm{d}t}\right]_{giv}=E\left[\frac{\mathrm{d}H_s}{\mathrm{d}t}\right]_{equ},\quad s=1,2,\ldots,n, \tag{19a}$$

$$E\left[\frac{\mathrm{d}\psi_u}{\mathrm{d}t}\right]_{giv}=E\left[\frac{\mathrm{d}\psi_u}{\mathrm{d}t}\right]_{equ},\quad u=1,2,\ldots,\alpha. \tag{19b}$$

The Itô equations for $\mathrm{d}H_s/\mathrm{d}t$, and $\mathrm{d}\psi_u/\mathrm{d}t$ can be derived from Eqs. (8) and (2) by using Itô differential rule. Then, following the derivation from Eqs. (13a)–(16b), the following equations for determining g_s and h_u can be obtained:

$$\int_0^{2\pi}\left\{\frac{\partial H_s}{\partial p_i}\left[M_{ij}\frac{\partial H}{\partial p_j}+\frac{1}{2}\frac{\partial b_{ij}}{\partial p_j}-\tfrac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s+\frac{\partial \psi_u}{\partial p_j}h_u\right)\right]\Big/\left|\frac{\partial(\mathbf{H},\boldsymbol{\psi},\boldsymbol{\theta}_1)}{\partial(\mathbf{q},\mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1=0,\quad s=1,2,\ldots,n, \tag{20a}$$

$$\int_0^{2\pi}\left\{\frac{\partial \psi_u}{\partial p_i}\left[M_{ij}\frac{\partial H}{\partial p_j}+\frac{1}{2}\frac{\partial b_{ij}}{\partial p_j}-\tfrac{1}{2}b_{ij}\left(\frac{\partial H_s}{\partial p_j}g_s+\frac{\partial \psi_u}{\partial p_j}h_u\right)\right]\Big/\left|\frac{\partial(\mathbf{H},\boldsymbol{\psi},\boldsymbol{\theta}_1)}{\partial(\mathbf{q},\mathbf{p})}\right|\right\}\mathrm{d}\boldsymbol{\theta}_1=0,\quad u=1,2,\ldots,\alpha. \tag{20b}$$

Eqs. (16a) and (16b), (18a) and (18b), (20a) and (20b), after completing the integration, are a set of $n+\alpha$ linear algebric equations for g_s and h_u. If g_s and h_u can be obtained from solving one set of these equations and they satisfy the compatibility conditions in Eq. (6), then the approximate stationary probability density of given system (8) is obtained by substituting g_s and h_u into Eq. (7) then into Eq. (4).

If the action-angle vector $\mathbf{I}$, $\boldsymbol{\theta}$ can be found for the Hamiltonian systems, then formulas similar to Eqs. (16a) and (16b), (18a) and (18b), (20a) and (20b) can be derived for determining the probability potential. In this case the Jacobian determinant $|\partial(\mathbf{I},\boldsymbol{\psi},\boldsymbol{\theta}_1)/\partial(\mathbf{q},\mathbf{p})|$ can be omitted since it is a linear combination of Jacobian determinant $|\partial(\mathbf{I},\boldsymbol{\theta})/\partial(\mathbf{q},\mathbf{p})|$ of a canonical transformation with integers as coefficients and thus an integer.

4. Example

Consider two non-linear damping oscillators coupled with linear dampings and subject to Guassian white noise excitations. The equations of motion of the system are of the form

$$\begin{aligned}
\dot{Q}_1 &= P_1, \\
\dot{P}_1 &= -\omega_1^2 Q_1 - (\alpha_{11} + \alpha_{12} P_1^2) P_1 - \beta_1 P_2 + \xi_1(t), \\
\dot{Q}_2 &= P_2, \\
\dot{P}_2 &= -\omega_2^2 Q_2 - (\alpha_{21} + \alpha_{22} P_2^2) P_2 - \beta_2 P_1 + \xi_2(t),
\end{aligned} \tag{21}$$

where ω_i, β_i, α_{ij} are constants, $\xi_k(t)$ are independent Gaussian white noises with intensities $2D_k$. There is no Wong–Zakai correction terms for this system. The Hamiltonian associated with this system is

$$H = H_1 + H_2, \quad H_i = (p_i^2 + \omega_i^2 q_i^2)/2. \tag{22}$$

System (21) can be modelled as Itô equations of the form of Eq. (8) with

$$\begin{aligned}
&M_{11} = \alpha_{11} + \alpha_{12} P_1^2, \quad M_{12} = \beta_1, \quad M_{21} = \beta_2, \quad M_{22} = \alpha_{21} + \alpha_{22} P_2^2, \\
&b_{11} = 2D_1, \quad b_{22} = 2D_2, \quad b_{12} = b_{21} = 0.
\end{aligned} \tag{23}$$

Suppose that $\omega_1 = \omega_2$. Then, the Hamiltonian system associated with system (21) is in primary internal resonance. Introduce angle variables

$$\theta_i = tg^{-1}(p_i/\omega_i q_i), \quad i = 1, 2 \tag{24}$$

and the combination of angle variable

$$\psi = \theta_1 - \theta_2. \tag{25}$$

The inverse of Eqs. (24) and (25) is

$$\begin{aligned}
q_1 &= \frac{\sqrt{2H_1}}{\omega_1} \cos(\theta_2 + \psi), \quad p_1 = \sqrt{2H_1} \sin(\theta_2 + \psi), \\
q_2 &= \frac{\sqrt{2H_2}}{\omega_2} \cos\theta_2, \quad p_2 = \sqrt{2H_2} \sin\theta_2.
\end{aligned} \tag{26}$$

Thus

$$\begin{aligned}
\frac{\partial\psi}{\partial p_1} &= \frac{\partial\theta_1}{\partial p_1} = \frac{\omega_1 q_1}{2H_1}, \\
\frac{\partial\psi}{\partial p_2} &= \frac{-\partial\theta_2}{\partial p_2} = \frac{-\omega_2 q_2}{2H_2}.
\end{aligned} \tag{27}$$

The approximate stationary solution of system (21) is of the form of Eqs. (4) and (7), i.e.,

$$p(\mathbf{q}, \mathbf{p}) = C \exp\left[-\int^{H_1} \frac{\partial\lambda}{\partial H_1} \mathrm{d}H_1 - \int^{H_2} \frac{\partial\lambda}{\partial H_2} \mathrm{d}H_2 - \int^{\psi} \frac{\partial\lambda}{\partial\psi} \mathrm{d}\psi\right], \tag{28}$$

where H_i, ψ are functions of $\mathbf{q}, \mathbf{p}$.

Based on the first criterion, i.e., Eqs. (16a) and (16b), $g_s = \partial\lambda/\partial H_s$, $s = 1, 2$, and $h = \partial\lambda/\partial\psi$ satisfy the following equations:

$$\int_0^{2\pi}\left[\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\left(b_{11}\frac{p_1}{\omega_1}\right)\right]\mathrm{d}\theta_2 = 0,$$
$$\int_0^{2\pi}\left[\left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\left(b_{22}\frac{p_2}{\omega_2}\right)\right]\mathrm{d}\theta_2 = 0,$$
$$\int_0^{2\pi}\left[\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\left(b_{11}\frac{\partial\psi}{\partial p_1}\right)\right.$$
$$\left. + \left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\left(b_{22}\frac{\partial\psi}{\partial p_2}\right)\right]\mathrm{d}\theta_2 = 0. \tag{29}$$

Inserting Eqs. (26) and (27) into Eq. (29) and completing the integration lead to

$$g_1 = \frac{\alpha_{11}}{D_1} + \frac{3}{2}\frac{\alpha_{12}}{D_1}H_1 + \frac{\beta_1}{D_1}\sqrt{\frac{H_2}{H_1}}\cos\psi,$$
$$g_2 = \frac{\alpha_{21}}{D_2} + \frac{3}{2}\frac{\alpha_{22}}{D_2}H_2 + \frac{\beta_2}{D_2}\sqrt{\frac{H_1}{H_2}}\cos\psi,$$
$$h = \frac{-2(\beta_1D_1H_2 + \beta_2D_2H_1)}{D_2^2H_1 + D_1^2H_2}\sqrt{H_1H_2}\sin\psi. \tag{30}$$

The compatibility conditions in Eq. (6) require

$$\beta_1/D_1 = \beta_2/D_2. \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (28) yields

$$p(\mathbf{q},\mathbf{p}) = C\exp\left[-\left(\frac{\alpha_{11}}{D_1}H_1 + \frac{\alpha_{21}}{D_2}H_2 + \frac{3}{4}\frac{\alpha_{12}}{D_1}H_1^2 + \frac{3}{4}\frac{\alpha_{22}}{D_2}H_2^2 + \frac{6\beta_2}{D_2}\sqrt{H_1H_2}\cos\psi\right)\right]\Bigg|_{H_i=(p_i^2+\omega_i^2q_i^2)/2,\psi=tg^{-1}(p_1/\omega_1q_1)-tg^{-1}(p_2/\omega_2q_2)}. \tag{32}$$

Based on the second criterion, i.e., Eqs. (18a) and (18b), $g_s = \partial\lambda/\partial H_s$, $s = 1, 2$, and $h = \partial\lambda/\partial\psi$ satisfy the following equations:

$$\int_0^{2\pi}\left[p_1\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\left(b_{11}\frac{p_1^2}{\omega_1}\right)\right]\mathrm{d}\theta_2 = 0,$$
$$\int_0^{2\pi}\left[p_2\left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\left(b_{22}\frac{p_2^2}{\omega_2}\right)\right]\mathrm{d}\theta_2 = 0,$$
$$\int_0^{2\pi}\left[p_1\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\left(b_{11}p_1\frac{\partial\psi}{\partial p_1}\right)\right.$$
$$\left. + p_2\left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\left(b_{22}p_2\frac{\partial\psi}{\partial p_2}\right)\right]\mathrm{d}\theta_2 = 0. \tag{33}$$

Inserting Eqs. (26) and (27) into Eq. (33) and completing the integration yield

$$\begin{aligned} g_1 &= \frac{\alpha_{11}}{D_1} + \frac{5}{3}\frac{\alpha_{12}}{D_1} H_1 + \frac{\beta_1}{D_1}\sqrt{\frac{H_2}{H_1}}\cos\psi, \\ g_2 &= \frac{\alpha_{21}}{D_2} + \frac{5}{3}\frac{\alpha_{22}}{D_2} H_2 + \frac{\beta_2}{D_2}\sqrt{\frac{H_1}{H_2}}\cos\psi, \\ h &= \frac{-2(\beta_1 D_1 + \beta_2 D_2)}{D_2^2 + D_1^2}\sqrt{H_1 H_2}\sin\psi. \end{aligned} \tag{34}$$

The compatibility conditions in Eq. (6) require that Eq. (31) be satisfied. The approximate stationary probability density of system (21) is obtained by inserting Eqs. (34) and (31) into Eq. (28) and completing the integration. The result is

$$\begin{aligned} p(\mathbf{q},\mathbf{p}) = C\exp\bigg[-\bigg(&\frac{\alpha_{11}}{D_1}H_1 + \frac{\alpha_{21}}{D_2}H_2 + \frac{5}{6}\frac{\alpha_{12}}{D_1}H_1^2 + \frac{5}{6}\frac{\alpha_{22}}{D_2}H_2^2 \\ &+ \frac{6\beta_2}{D_2}\sqrt{H_1H_2}\cos\psi\bigg)\bigg]\bigg|_{H_i=(p_i^2+\omega_i^2 q_i^2)/2,\psi=tg^{-1}(p_1/\omega_1 q_1)-tg^{-1}(p_2/\omega_2 q_2)}. \end{aligned} \tag{35}$$

According to the third criterion, i.e., Eqs. (20a) and (20b), $g_s' = \partial\lambda/\partial H_s$, $s = 1, 2$, and $h = \partial\lambda/\partial\psi$ satisfy the following equations:

$$\begin{aligned} &\int_0^{2\pi}\left[\frac{p_1}{\omega_1}\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1 g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\right]\mathrm{d}\theta_2 = 0, \\ &\int_0^{2\pi}\left[\frac{p_2}{\omega_2}\left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2 g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\right]\mathrm{d}\theta_2 = 0, \\ &\int_0^{2\pi}\left[\frac{\partial\psi}{\partial p_1}\left(M_{11}p_1 + M_{12}p_2 - \tfrac{1}{2\omega_1}b_{11}p_1 g_1 - \tfrac{1}{2}b_{11}\frac{\partial\psi}{\partial p_1}h\right)\right. \\ &\qquad \left. + \frac{\partial\psi}{\partial p_2}\left(M_{21}p_1 + M_{22}p_2 - \tfrac{1}{2\omega_2}b_{22}p_2 g_2 - \tfrac{1}{2}b_{22}\frac{\partial\psi}{\partial p_2}h\right)\right]\mathrm{d}\theta_2 = 0. \end{aligned} \tag{36}$$

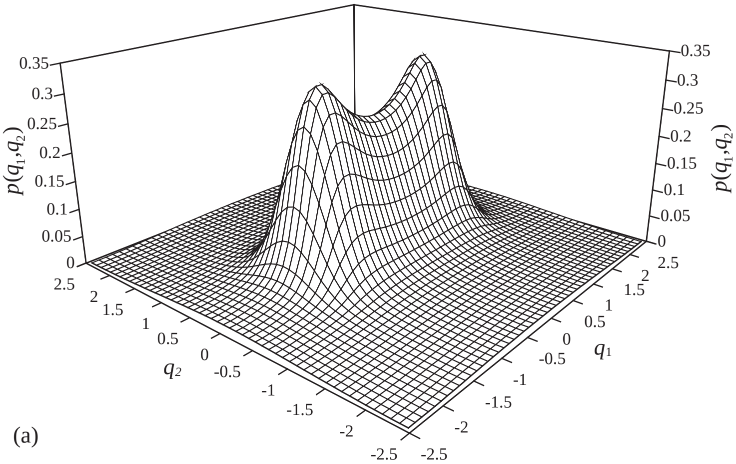
Inserting Eqs. (26) and (27) into Eq. (36) and completing the integration yield

$$\begin{aligned} g_1 &= \frac{\alpha_{11}}{D_1} + \frac{3}{2}\frac{\alpha_{12}}{D_1} H_1 + \frac{\beta_1}{D_1}\sqrt{\frac{H_2}{H_1}}\cos\psi, \\ g_2 &= \frac{\alpha_{21}}{D_2} + \frac{3}{2}\frac{\alpha_{22}}{D_2} H_2 + \frac{\beta_2}{D_2}\sqrt{\frac{H_1}{H_2}}\cos\psi, \\ h &= \frac{-2(\beta_1 H_2 + \beta_2 H_1)}{D_2 H_1 + D_1 H_2}\sqrt{H_1 H_2}\sin\psi. \end{aligned} \tag{37}$$

The compatibility conditions in Eq. (6) require that Eq. (31) be satisfied. Instituting Eqs. (37) and (31) into Eq. (28) leads to the approximate stationary probability density of system (21)

$$p(\mathbf{q},\mathbf{p}) = C\exp\left[-\left(\frac{\alpha_{11}}{D_1}H_1 + \frac{\alpha_{21}}{D_2}H_2 + \frac{3}{4}\frac{\alpha_{12}}{D_1}H_1^2 + \frac{3}{4}\frac{\alpha_{22}}{D_2}H_2^2 + \frac{6\beta_2}{D_2}\sqrt{H_1H_2}\cos\psi\right)\right]\Bigg|_{H_i=(p_i^2+\omega_i^2q_i^2)/2,\psi=tg^{-1}(p_1/\omega_1q_1)-tg^{-1}(p_2/\omega_2q_2)}. \tag{38}$$

Note that, Since b_{ii} are constants, the first two equations in Eq. (36) are identical to those in Eq. (29). The third equation in Eq. (36) is identical to that in Eq. (29) if $b_{11} = b_{22}$. So if $b_{11} = b_{22}$ are constants, then the first and third criteria yield the same g_s and h. Although h in Eq. (30) is different from that in Eq. (37), $\int^{\psi} h\,\mathrm{d}\psi$ is the same for the two criteria. So, the approximate



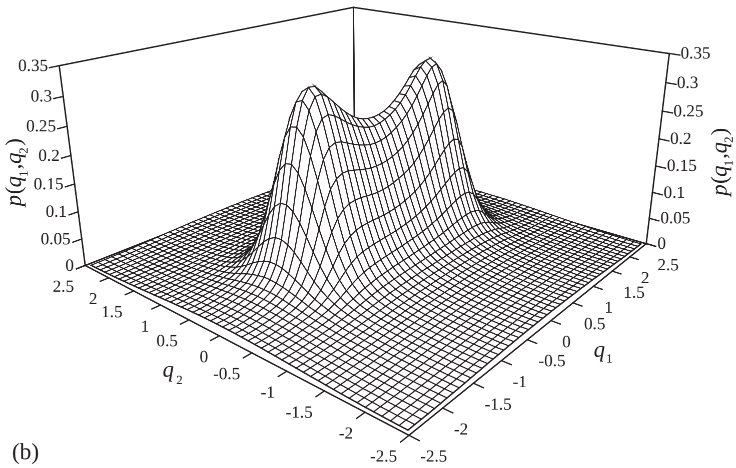


Fig. 1. Stationary marginal probability density $p(q_1, q_2)$ of system (21) in resonant case. $\omega_1 = \omega_2 = 1$, $\alpha_{11} = -0.08$, $\alpha_{12} = 0.08$, $\alpha_{21} = -0.1$, $\alpha_{22} = 0.1$, $\beta_1 = 0.008$, $\beta_2 = 0.01$, $D_1 = 0.008$, $D_2 = 0.01$. (a) from Eq. (32); (b) from digital simulation.

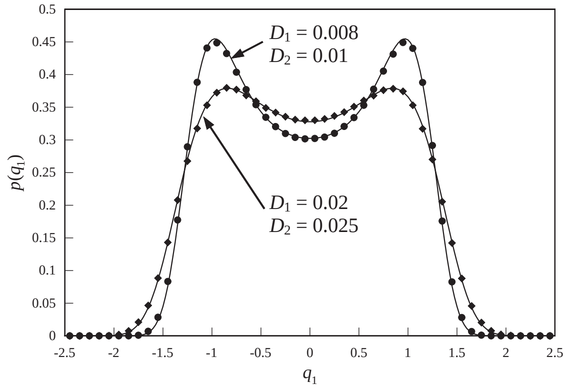


Fig. 2. Stationary marginal probability density $p(q_1)$ of system (21) in resonant case. The parameters are the same as those in Fig. 1, except D_1 and D_2 are variable. — analytical result; ● ◆, from digital simulation.

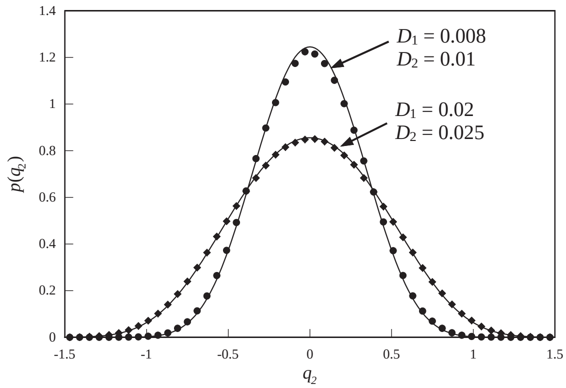


Fig. 3. Stationary marginal probability density $p(q_2)$ of system (21) in resonant case. The parameters and symbols are the same as those in Fig. 2.

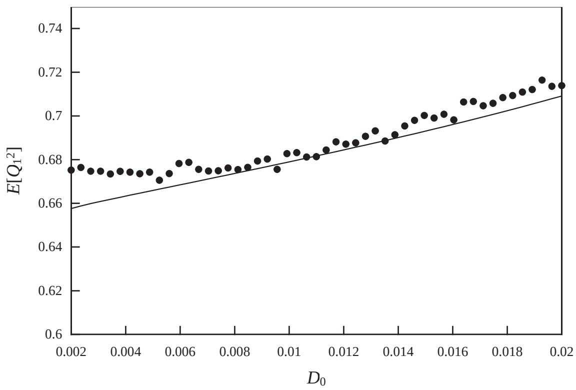


Fig. 4. Stationary mean-square value $E[Q_1^2]$ of system (21) in resonant case as function of excitation intensity $D_1 = D_0$ and $D_2 = 1.25D_0$. The other parameters and symbols are the same as those in Fig. 2.

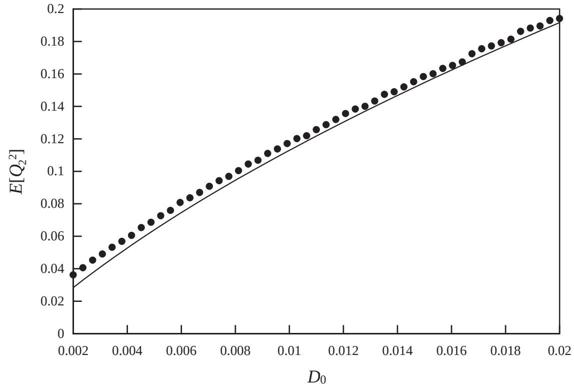


Fig. 5. Stationary mean square value $E[Q_2^2]$ of system (21) in resonant case as function of excitation intensity $D_1 = D_0$ and $D_2 = 1.25D_0$. The other parameters and symbols are the same as those in Fig. 2.

stationary probability densities of system (21) obtained from the first and third criteria are the same. The same circumstance occurs in the associated non-resonant case. This happens due to the special excitation. In the case of multiple excitations or parametric excitation in damping term, the first and third criteria will yield different results.

Some numerical results obtained by using the proposed method and from the digital simulation of original system are shown in Figs. 1–5. The stationary marginal probability density $p(q_1, q_2)$ of system (21) obtained from Eq. (32) is shown in Fig. 1(a). The corresponding result from digital simulation is shown in Fig. 1(b). The stationary marginal probability densities $p(q_1)$ and $p(q_2)$ of system (21) with two sets of different excitation intensities are shown in Figs. 2 and 3, respectively. The stationary mean square values $E[Q_1^2]$ and $E[Q_2^2]$ of system (21) as functions of excitation intensity are shown in Figs. 4 and 5, respectively. It is seen from these figures that the analytical results and corresponding results from digital simulation are in excellent agreement.

5. Concluding remarks

The equivalent non-linear system method for stochastically excited and dissipated integrable Hamiltonian systems in resonant case has been developed. Instead of finding the damping coefficients of equivalent system, the derivatives of probability potential with respect to first integrals and combinations of angle variables are determined for obtaining the approximate stationary solution of a given system. The new technique is much more simper than that in Ref. [12], especially for higher d.o.f. systems. Thus, this version of the equivalent non-linear system method is more widely applicable. The comparison between the two results obtained by using this method and from digital simulation for an example shows that this method may yield quite accurate result. The results obtained by using this method for the example also show that different criteria may lead to the same result in some special cases and that linear damping coupling plays its roll only in resonant case.

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