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An inverse strategy for relocation of eigenfrequencies in structural design. Part I: first order approximate solutions

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Abstract

This paper proposes an inverse strategy for relocation of structural natural frequencies using first order formulation and solution algorithm. In the proposed method a sensitivity analysis of the systems eigenvalues with respect to material or geometrical parameters of the structure is conducted initially. The required parameter variation to achieve a desired frequency shift for the structure is then computed. The proposed technique incorporates the design constraints or objective functions in the system equations in such a way that a square system of equations is always preserved. The formulations are general and applicable to all finite element structures because the sensitivity analysis is based on the stiffness and mass matrices regardless of the type of elements used. An algorithm suitable for implementation of the technique for practical purposes is developed. The accuracies of the proposed methods are tested conducting several case studies and the results are validated against exact solutions.

1. Introduction

In optimizing the dynamic behaviour of structures, it is usually required to shift selected natural frequencies of the system by making modifications to the mass, stiffness or damping characteristics of the structure. This is particularly important when the structure is susceptible to detrimental fatigue or dynamic resonance problems, which may violate the design specifications for strength, stability, or performance requirements. Computation of frequency sensitivities with respect to design parameters has, therefore, become the basis of many finite element vibration optimization algorithms.

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The common industrial practise for optimizing the vibration behaviour of structures is based on performing a series of modifications on the finite element models to achieve the required eigenfrequencies. This method known as the *forward variation* approach is time-consuming and rarely converges to a desired solution. The present scope of inverse eigenvalue formulations of the vibration problem are predominantly limited to simple linear springs, dampers and point masses. Little attention has been paid to formulating the inverse eigenvalue problem for two- and three-dimensional and higher order finite elements, commonly used in simulation of most real structures. Optimization of vibration characteristics is defined as an inverse eigenvalue problem to modify the design of the system in order to produce the desired response. The inverse approach may be used to estimate the required change in the design variables to achieve a desired change in the natural frequencies of the structure.

The eigenvalue sensitivities in structural design were treated in the past by Belle Van [1] and Vanhonacker [2] by utilizing the first order Taylor's series expansion. Others such as Chen and Garba [3] used the iterative method to modify structural systems. Balmes [4] concentrated on the error norms between the measured and predicted frequency response function of the structure. Baldwin and Hutton [5] presented a detailed review of structural modification techniques classified into categories of the techniques based on small modification, localized modification, and modal approximation.

Further research on structural modification carried out by Tsuei and Yee [6–8] proposed methods of shifting the desired eigenfrequencies using the forced response of the system and modal analysis. The method is based on modification of either the mass or stiffness matrix by treating the modification of the system matrices as an external forced response. This external forced response is formulated in terms of the modification parameters, thus creating a modified eigenvalue problem. Zhang and Kim [9] investigated the use of mass matrix modification to achieve desired natural frequencies. McMillan and Keane [10] investigated a method of shifting eigenfrequencies of a rectangular plate by adding concentrated mass elements. Sivan and Ram [11–13] extended further research on structural modification by studying the construction of a mass–spring system with prescribed natural frequencies and obtained stiffness and mass matrices by means of the orthogonality principles. They also developed an algorithm based on an earlier work by Joseph [14], which involves the solution of the inverse eigenvalue problem.

Also, in the last few years, the work of Gladwell [15] introduced an inverse approach for both the discrete and continuous structures. Mottershead [16] also considered the problem of resonance in the structures' forced vibrations by the design of physical modifications to achieve targeted natural frequencies. His technique of achieving the required system included structural modifications through adding concentrated mass or springs to the system. In Ref. [17], He and Li considered optimizing dynamic behaviour of a multi-body system through modifications in mass and stiffness matrices. Most of these techniques have thus far been applied to simple mass and spring systems. The inverse methods presented in this paper have significant advantages over classical optimization approaches in that there are no iterations involved other than those associated with numerical solution of equations. In terms of computational processing time the proposed inverse methods were found to take less than 10 per cent of the time taken when using the classical optimization algorithms implemented in some of the current FEA commercial codes.

The methods proposed in this paper are extension of earlier work by Djoudi and Bahai [18] and are based on the differential equation solution and the matrix procedure approach to modify stiffness and mass properties of a finite element structure. Both methods are based on the use of first order expansion approximation of eigenvalues with respect to design variables, and can be applied to any kind of finite elements including continuum elements. The proposed technique is implemented in a computer algorithm and applied to a number of case studies, the results of which reveal good accuracy in a range of engineering problems.

1.1. Preliminaries

Consider the general equation of motion of a structure:

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\} + [\mathbf{C}]\{\dot{\mathbf{U}}\} + [\mathbf{K}]\{\mathbf{U}\} = \{f(\mathbf{t})\}, \tag{1}$$

where $[\mathbf{K}]$, $[\mathbf{C}]$, and $[\mathbf{M}]$ are stiffness, damping and mass matrices respectively. Also, $\{f(\mathbf{t})\}$ and $\{\mathbf{U}\}$ are time-dependent load and displacement vectors respectively. The systems' natural frequencies are determined by solving the eigenvalue equation

$$\{[\mathbf{K}] - \zeta_m[\mathbf{M}]\}\{\mathbf{y}_m\} = \{\mathbf{0}\}, \tag{2}$$

where $\{\mathbf{y}_m\}$ is the mth eigenvector of the system and ζ_m is its corresponding eigenvalue which is related to the systems mth natural frequency f_m through

$$\zeta_m = (2\pi f_m)^2. \tag{3}$$

A modification in the systems' material or geometrical parameters will result in a variation of the dynamic characteristics of the structure represented by $\{\mathbf{y}_m\}$ and ζ_m. Therefore, the sensitivity of the dynamic characteristics of the structure with respect to a given design parameter b, should initially be established. The design parameter b can be any material or geometric variable of the structure. The sensitivity analysis is conducted initially by differentiating Eq. (2) with respect to b:

$$\{[\mathbf{K}]' - \zeta'_m[\mathbf{M}] - \zeta_m[\mathbf{M}]'\}\{\mathbf{y}_m\} + \{[\mathbf{K}] - \zeta_m[\mathbf{M}]\}\{\mathbf{y}'_m\} = \{\mathbf{0}\}, \tag{4}$$

where $(\ )' = \partial/\partial b$. It is assumed that the eigenvectors are normalized with respect to the mass matrix $[\mathbf{M}]$ such that

$$\{\mathbf{y}_i\}^{\mathrm{T}}[\mathbf{M}]\{\mathbf{y}_j\} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

Using Eq. (5) and pre-multiplication of Eq. (4) by $\{\mathbf{y}_m\}^{\mathrm{T}}$, the second term will take the form $(\{\mathbf{y}_m\}^{\mathrm{T}}.\ \{[\mathbf{K}] - \zeta_m[\mathbf{M}]\})\{\mathbf{y}'_m\}$ which will be zero, because the term in parentheses is actually the transpose of Eq. (2) which is zero. Re-arrangement of the terms results in

$$\zeta'_m = \{\mathbf{y}_m\}^{\mathrm{T}}\{[\mathbf{K}]' - \zeta_m[\mathbf{M}]'\}\{\mathbf{y}_m\}. \tag{6}$$

In general, there may be as many as k structural parameters used for dynamic optimization. Eq. (6) is, therefore, written as

$$\frac{\partial \zeta_m}{\partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}}\left\{\frac{\partial[\mathbf{K}]}{\partial b_j} - \zeta_m\frac{\partial[\mathbf{M}]}{\partial b_j}\right\}\{\mathbf{y}_m\}, \quad j = 1, \ldots, k. \tag{7}$$

It is assumed that the structure has no repeated or equal eigenvalues. In the case of repeated eigenvalues, problems arise as any linear combination of eigenvectors corresponding to the repeated eigenvalue is also a valid eigenvector. Therefore, Eq. (6) or (7) can no longer be used for the eigenvalue derivative. In this case, it can be shown that an eigenvalue derivative is the solution of another subeigenvalue problem [19,20]. Throughout this paper, the vector and matrix quantities are enclosed in { } and [ ] brackets, respectively, and barred parameters denote the initial value of the parameters in the initial status of the structure.

2. Approximate solution based on first order differential equation

Eq. (6) can be rearranged to form a first order differential equation:

$$\frac{\partial \zeta_m}{\partial b} + \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial b} \{\mathbf{y}_m\} \zeta_m - \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial b} \{\mathbf{y}_m\} = 0 \tag{8a}$$

with the initial conditions:

$$\zeta_m|_{b=\bar{b}} = \bar{\zeta}_m. \tag{8b}$$

Defining $A = \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial b} \{\mathbf{y}_m\}$, $B = \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial b} \{\mathbf{y}_m\}$, and assuming that A and B do not vary with b, the solution of Eq. (8) can be expressed as

$$\zeta_m(b) = \frac{B}{A} + \frac{\mathrm{e}^{(-Ab)}(-B + A\bar{\zeta}_m)}{A\ \mathrm{e}^{(-A\bar{b})}}, \quad A \neq 0. \tag{9}$$

Eq. (9) should be solved for b for a desired frequency shift, which is in general a non-linear equation in terms of b. The accuracy of Eq. (9) will be investigated later through some case studies. For $A = 0$, the solution of Eq. (8) is obviously a linear equation.

In the cases where the effect of several parameters on the natural frequency of the structure is sought, the resulting differential equations will be in the form of

$$\frac{\partial \zeta_m}{\partial b_j} + \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial b_j} \{\mathbf{y}_m\} \zeta_m - \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial b_j} \{\mathbf{y}_m\} = 0, \quad j = 1, \ldots, k \tag{10}$$

with their corresponding initial conditions. It is possible to impose a number of physical constraints on the structure to reduce the system of equations given in Eq. (10) into a single equation.

2.1. Method 1: Proportionality constraint for the first order system of differential equations

To reduce the number of differential equations (10) into a single equation, additional assumptions and constraints are used. For example, it can be assumed that the coefficients of the equations are constants, corresponding to the current status of the structure. The weighted relative variation of structural parameters b_j can all be proportional to a global variable α:

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha \omega_j, \quad j = 1, \ldots, k, \tag{11}$$

where the parameter α will become the only unknown. The constant weights, ω_j, are completely arbitrary and may be adopted from the results of a sensitivity analysis or by using engineering judgements. If the selection of ω_j is based on the results of the sensitivity analysis, Eq. (11) will take the form

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha \frac{\partial \zeta_m}{\partial b_j}, \quad j = 1, \ldots, k. \tag{12}$$

Therefore, since the only unknown is α, the resulting differential equation for the eigenvalue ζ_m will be

$$\frac{\partial \zeta_m}{\partial \alpha} + \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial \alpha} \{\mathbf{y}_m\} \zeta_m - \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial \alpha} \{\mathbf{y}_m\} = 0 \tag{13a}$$

with initial conditions at $\alpha = 0$ as

$$\zeta_m|_{\alpha=0} = \bar{\zeta}_m. \tag{13b}$$

In this case, since the only variable is α, the derivative of every scalar, vector, and matrix quantity X, with respect to α is

$$\frac{\partial X}{\partial \alpha} = \sum_{j=1}^{j=k} \frac{\partial X}{\partial b_j} \frac{\partial b_j}{\partial \alpha} = \sum_{j=1}^{j=k} \frac{\partial X}{\partial b_j} (\omega_j \bar{b}_j). \tag{14}$$

The above scheme of differentiation will, therefore, have to be used to establish Eq. (13) the solution of which will then assume the following form:

$$\zeta_m(\alpha) = \frac{B}{A} + \frac{\mathrm{e}^{(-A\alpha)}(-B + A\bar{\zeta}_m)}{A}, \quad A \neq 0, \tag{15}$$

where $A = \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial \alpha} \{\mathbf{y}_m\}$ and $B = \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial \alpha} \{\mathbf{y}_m\}$. For the case of $A = 0$ in Eq. (8) or (13), the solution will be linear in terms of b or α respectively.

3. Total differential form for the first order approximation

3.1. Method 2: Unconstrained method

As a first order approximate inverse formulation approach, the direct matrix application of Eq. (6) is used to optimize the dynamical behaviour of structures. The applicability and accuracy of this approach will be investigated by conducting several case studies. This method, defined as the first order differential approximation method, can be described as

$$\Delta \zeta_m = \sum_{j=1}^{j=k} \frac{\partial \zeta_m}{\partial b_j} \Delta b_j, \tag{16}$$

where $\partial \zeta_m / \partial b_j$ can be obtained from Eq. (7). As it can be seen, the number of parameters b_j to be modified may be more than the number of frequencies to be optimized. In order to overcome this problem, Eq. (7) is set up so that the number of frequency changes, $\Delta \zeta_j$, are equal to the number of parameters to be changed:

$$\Delta \zeta_m = \sum_{j=1}^{j=k} \frac{\partial \zeta_m}{\partial b_j} \Delta b_j, \quad m = 1, \ldots, k. \tag{17}$$

The values of $\Delta\zeta_m$ corresponding to the frequencies which are to remain unchanged will simply be set to zero in Eq. (17). The resulting system of linear equations can now be written as

$$\{\boldsymbol{\Delta}\zeta\} = [\mathbf{S}]\{\boldsymbol{\Delta}b\} \tag{18}$$

where $\{\boldsymbol{\Delta}\zeta\}$ and $\{\boldsymbol{\Delta}b\}$ are the vectors of the required eigenfrequency changes and the unknown parameter changes respectively. The coefficient matrix $[\mathbf{S}]$ is defined as

$$S_{ij} = \frac{\partial \zeta_i}{\partial b_j}, \quad i,j = 1, \ldots, k. \tag{19}$$

Therefore, a change in structural parameters for a desired change in natural frequencies can be obtained from Eq. (19) as

$$\{\boldsymbol{\Delta}b\} = [\mathbf{S}]^{-1}\{\boldsymbol{\Delta}\zeta\}. \tag{20}$$

It is easily seen that the coefficient matrix $[\mathbf{S}]$ is not symmetric. It should be noted that although a change in any structural parameter b_j may cause a change in an eigenvalue ζ_m, in practice only those parameters to which ζ_m is most sensitive are chosen. These parameters are initially identified in the sensitivity analysis before the above equations are solved. Naturally, the most sensitive parameters are those which exhibit the largest first derivatives in the pre-modified status of the structure given by Eq. (6).

3.2. Method 3: Constrained method based on the optimization of an objective function

It is also possible to solve Eq. (16) with just one frequency shift and several variables, using additional constraints. These constraints may be chosen, based on engineering judgement and feasibility of modification on the structure. An objective function may also be defined in terms of other structural parameters such as volume or mass of the structure.

As an example, if the function to be optimized is the mass of the structure, Eq. (16) must be solved such that the added mass to the structure is a minimum. Now, assume that Δm is a measure of the added mass to the structure:

$$\Delta m = m(b_1, b_2, \ldots, b_k). \tag{21}$$

Adopting the Lagrange Multipliers method for the problem, the functional to be minimized is

$$\Pi = m(b_1, b_2, \ldots, b_k) + \lambda \left\{ \sum_{i=1}^{k} \frac{\partial \zeta_m}{\partial b_i} \Delta b_i - \Delta\zeta_m \right\}. \tag{22}$$

Therefore, the following system of equations is arived at:

$$\frac{\partial \Pi}{\partial b_j} = \frac{\partial \Pi}{\partial \lambda} = 0, \quad j = 1, \ldots, k \tag{23}$$

or

$$\begin{cases} \dfrac{\partial m(b_1, b_2, \ldots, b_k)}{\partial b_j} + \lambda \dfrac{\partial \zeta_m}{\partial b_j} = 0, \quad j = 1, \ldots, k \\ \sum_{i=1}^{i=k} \dfrac{\partial \zeta_m}{\partial b_j} \Delta b_j = \Delta\zeta_m. \end{cases} \tag{24}$$

Solution of Eq. (24) will give the required amount of variation in the structural variables b_j for a desired eigenvalue shift $\Delta\zeta_m$, with minimum added mass.

3.3. Method 4: Constrained method based on proportionality

In some engineering problems, the only constraint is the feasibility of the structural configuration for dynamic optimization. Therefore, introducing additional constraints only for the purpose of having a sufficient number of equations, may make the problem unnecessarily more complicated. Instead, as discussed in Section 3.1, it is possible to choose a rational proportionality criterion to reduce the number of unknowns for a particular kind of problem. As an example, consider a problem in which b_j is the thickness of some finite elements. It is possible to add a weighted proportion $\alpha\omega_j$ to the thickness of all the corresponding elements:

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha\omega_j, \quad j = 1, \ldots, k \tag{25}$$

with α as the only unknown. This results in the single variable equation as

$$\Delta\zeta_m = \alpha \sum_{j=1}^{j=k} (\bar{b}_j\omega_j)\frac{\partial\zeta_m}{\partial b_j}. \tag{26}$$

Similarly, it is possible to designate a weight for each thickness proportional to their contribution to the frequency shift as

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha\frac{\partial\zeta_m}{\partial b_j}, \quad j = 1, \ldots, k \tag{27}$$

which again results in the single variable equation in terms of α:

$$\Delta\zeta_m = \alpha \sum_{j=1}^{j=k} \bar{b}_j\left(\frac{\partial\zeta_m}{\partial b_j}\right)^2. \tag{28}$$

In Eq. (28), other rational weights could also be used. This approach can be adopted whenever the number of unknowns is more than the number of equations which is discussed later.

3.4. Method 5: Partially constrained method based on proportionality

This method is used in situations where control over several eigenvalues is desired. The method is a combination of total differential form of Section 4.1 and the constrained method of Section 4.3. Since the number of variables may be more than the eigenvalues under consideration, they may be reduced through the proportionality constraints in order to retain the equality between the number of unknowns and the variables. Hence, the equations can be written as

$$\Delta\zeta_r = \sum_j \frac{\partial\zeta_r}{\partial b_j}\Delta b_j, \quad r = 1, \ldots, k, \tag{29}$$

$$\frac{b_j - \bar{b}_j}{\bar{b}_j} = \sum_{s=1}^{s=k} \alpha_s \omega_{sj}, \tag{30}$$

$$\frac{\partial \zeta_r}{\partial \alpha_s} = \sum_j \frac{\partial \zeta_r}{\partial b_j} \frac{\partial b_j}{\partial \alpha_s} = \sum_j \frac{\partial \zeta_r}{\partial b_j} (\omega_{sj} \bar{b}_j), \quad r, s = 1, \ldots, k, \tag{31}$$

where k is the number of unknowns α_s, and j can vary up to the number of parameters to be changed. Again, a decision on the magnitude of the weights, ω_{sj}, may be made upon engineering judgement, feasibility, and other limitations. Now, assuming a linear differential approximation with respect to α_s, for eigenvalue shifts, one has

$$\Delta \zeta_r = \sum_{j=1}^{j=k} \frac{\partial \zeta_r}{\partial \alpha_j} \alpha_j, \quad r = 1, \ldots, k. \tag{32}$$

This will again lead to a system of linear equations in the form of

$$\{\boldsymbol{\Delta \zeta}\} = [\mathbf{S}]\{\boldsymbol{\alpha}\}, \tag{33}$$

where $\{\boldsymbol{\Delta \zeta}\}$ and $\{\boldsymbol{\alpha}\}$ are, respectively, the vector of constant frequency shifts and the vector of the required parameter changes. The coefficient matrix $[\mathbf{S}]$ is defined as

$$S_{rs} = \frac{\partial \zeta_r}{\partial \alpha_s}, \quad r, s = 1, \ldots, k. \tag{34}$$

Therefore, the change in structural parameters for a desired change in natural frequencies can be obtained:

$$\{\boldsymbol{\alpha}\} = [\mathbf{S}]^{-1}\{\boldsymbol{\Delta \zeta}\}. \tag{35}$$

Finally, it should be noted that having computed the required changes in the structures' design variables, the modifications can be implemented in different ways. For example, the required additional thickness to a plated structure can be achieved by attaching equivalent stiffeners.

4. Additional members

The first step of the methods presented above involved a sensitivity analysis and also computation of Eq. (7) for the structure. Using Eq. (7) to find sensitivity is quite fast because it just requires finding mass and stiffness derivatives at the level of members. The sensitivity analysis may be extended to consider the addition of new members into the structure. The process is similar to the case of modifying existing members through the use of Eq. (7). The sensitivity analysis will establish the position of additional members in the stiffness matrix of the structure where they will have the maximum influence in achieving the required eigenvalue shift. In this context, the physical feasibility of the modified structure and geometric limitations are among the most important constraints.

5. Case studies

A number of case studies were conducted to demonstrate and validate the accuracy and applicability of the above techniques on various finite element models.

Example 1. Consider a simply supported two-dimensional (2-D) bridge modelled as a truss structure as shown in Fig. 1. The model comprises 17 bar elements. The material and geometric specifications are given in the figure. Three concentrated masses are placed at nodes 3,5, and 7 and the mass of the truss elements is neglected here. In order to shift the first frequency of the structure, the following steps are adopted:

(a) Compute the dynamic characteristics of the structure from an initial FEA run.
(b) Find the members to which the first frequency is most sensitive.
Then to shift the first frequency of the truss, change the properties of:
(c) the most sensitive member using the unconstrained total differential form for the first order approximation (Method 2);
(d) the most sensitive members using the total differential form and a constraint based on proportionality with equal weights (Method 4);
(e) the most sensitive members using the total differential form and a constraint based on proportionality with weights equal to their sensitivity (Method 4).

Solution. (a) To find the dynamic characteristics of the initial structure, the ANSYS program was used.

The first five natural frequencies of the structure are given in Table 1.

(b) To find the most sensitive members, the derivatives of the first frequency with respect to the cross-sectional areas of all the members are computed from Eq. (7). Since the structure is symmetric, the results shown in Table 2 relate to only one half of the structure.

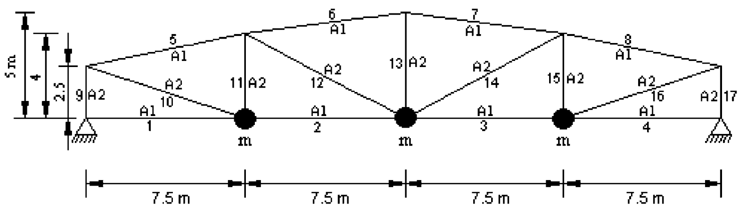

Fig. 1. 2-D truss with concentrated masses: $A_1 = 250\ \mathrm{cm}^2$, $A_2 = 82.5\ \mathrm{cm}^2$, $m = 5000$ kg, $E = 2.1E11\ \mathrm{N/m}^2$.

Table 1
Dynamic characteristics

	Mode i				
	1	2	3	4	5
ζ_i	15790	56707	127098	333202	1218025
f_i (Hz)	20.0	37.9	56.7	91.9	175.7

Table 2
First eigenvalue sensitivity

	Elem. j								
	1	2	5	6	9	10	11	12	13
A_j	250	250	250	250	82.5	82.5	82.5	82.5	82.5
$\mathrm{d}\zeta_1/\mathrm{d}A_j$	11961.0	11961.0	49322.1	57963.2	40490.1	500171.4	0.2	6442.3	24579.5

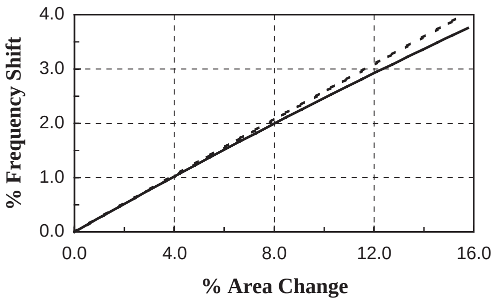

Fig. 2. Comparison of exact method and Method 2 for the first frequency shift by members 10 and 16: —, exact; - - -, approximate.

Therefore, elements 10 and 16 are the most sensitive elements.

(c) Since in this example the structure's mass is constant, then $\mathrm{d}[\mathbf{M}]/\mathrm{d}A_i = 0$ and Eq. (6) will yield a linear function of eigenvalue with respect to design variables. Since members 10, and symmetrically 16, have more influence on the first frequency, it is assumed that their cross-sectional areas will be varied by an equal amount. Hence,

$$\Delta\zeta_1 = 500171(\Delta A_{10} + \Delta A_{16}) = 500171(2 \times \Delta A_{10}) = 1000343\Delta A_{10} \tag{36}$$

or, according to the initial conditions shown in Table 1:

$$\zeta_1 = 1000343(A_{10} - 82.5 \times 10^{-4}) + 15790. \tag{37}$$

Fig. 2 shows percentage of frequency shift versus percentage of area change, computed exactly using ANSYS and approximately using Eq. (37). Fig. 3 shows the percentage error in computing frequency shift using Eq. (37).

(d) From Table 3, elements 5, 6, 10, 7, 8, and 16 are chosen to be modified using the proportionality rule:

$$\frac{(A_i - \bar{A}_i)}{\bar{A}_i} = \alpha, \quad i = 5, 6, 7, 8, 10, 16. \tag{38}$$

Since mass matrix is constant, from Eq. (13)

$$\frac{\partial \zeta_1}{\partial \alpha} = \{\mathbf{y}_1\}^{\mathrm{T}} \frac{\partial[\mathbf{K}]}{\partial \alpha}\{\mathbf{y}_1\}, \tag{39}$$

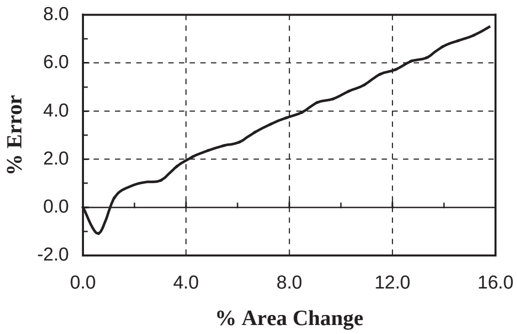

Fig. 3. Percentage error of Method 2 for the first frequency shift by members 10 and 16.

Table 3
Normalized proportionality weights

	Elem. j					
	5	6	7	8	10	16
ω_j	0.099	0.116	0.116	0.099	1	1

where

$$\frac{\partial[\mathbf{K}]}{\partial\alpha} = \sum_i \frac{\partial[\mathbf{K}]}{\partial A_i}\frac{\partial A_i}{\partial\alpha} = \sum_i \bar{A}_i \frac{\partial[\mathbf{K}]}{\partial A_i}. \tag{40}$$

The stiffness matrix $[\mathbf{K}^{(i)}]$ for the ith element is defined as

$$[\mathbf{K}^{(i)}] = \frac{E_i A_i}{L_i}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}, \tag{41}$$

where E, A, and L denote modulus of elasticity, cross-sectional area and the length of the elements respectively. C and S are, respectively, the cosine and sine of the angle of the member orientation with the global co-ordinate axis. Therefore, $\partial[\mathbf{K}]/\partial A_i$ and consequently $\partial[\mathbf{K}]/\partial\alpha$ may be easily computed from Eq. (41). Hence, using Eqs. (39)–(41) one arrives at

$$\zeta_1 = 13617\alpha + 15790. \tag{42}$$

Fig. 4 shows percentage of frequency shift versus percentage of area change (which is 100α), computed using ANSYS and from Eq. (42). Also Fig. 5 shows the error percentage in computing frequency shift using Eq. (42). It is noted that the percentage of error is considerably reduced in this case where the changes are less local.

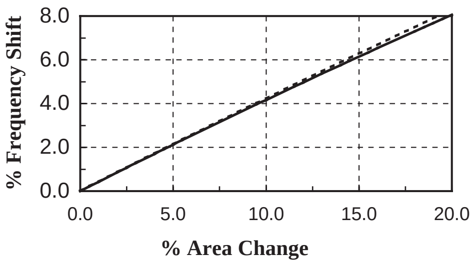

Fig. 4. Comparison of exact method and Method 4 with unit weights: ——, exact; - - -, approximate.

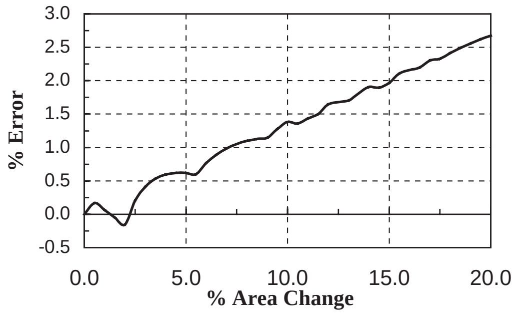

Fig. 5. Percentage error of Method 4 for the first frequency shift.

(e) In a similar way as in part (d), elements 5, 6, 10, 7, 8, and 16 are chosen to be changed proportionally according to their contribution to the frequency shift as

$$\frac{A_i - \bar{A}_i}{\bar{A}_i} = \frac{\partial \zeta_1}{\partial A_i}\alpha, \quad i = 5, 6, 7, 8, 10, 16, \tag{43}$$

$$\frac{\partial [\mathbf{K}]}{\partial \alpha} = \sum_i \frac{\partial [\mathbf{K}]}{\partial A_i}\frac{\partial A_i}{\partial \alpha} = \sum_i \bar{A}_i \omega_i \frac{\partial [\mathbf{K}]}{\partial A_i},$$

where the weights are normalized with respect to maximum sensitivity shown in Table 3:

$$\omega_i = \left(\frac{\partial \zeta_1}{\partial A_i}\right) \bigg/ \left(\frac{\partial \zeta_1}{\partial A_{10}}\right).$$

Therefore the following equation is arrived at in a similar manner to the previous part of the example:

$$\zeta_1 = 8832\alpha + 15790. \tag{44}$$

Comparison of this equation with the exact solution and its percentage error are shown in Figs. 6 and 7 respectively.

Example 2. Consider the same simply supported 2-D truss structure shown in Fig. 8. In this example the structure has no concentrated mass and the mass of the members are to be taken into

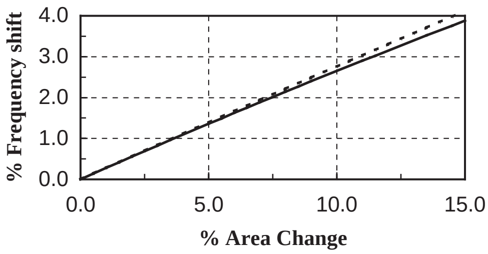

Fig. 6. Comparison of exact method and Method 4 with sensitivity weights for the first frequency shift: ——, exact; - - -, approximate.

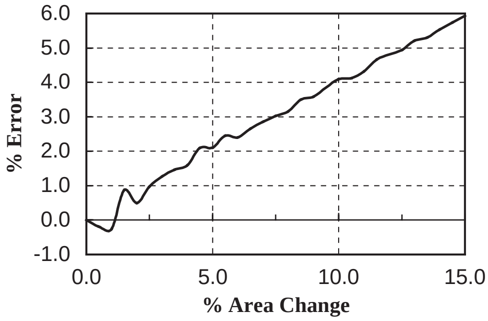

Fig. 7. Percentage error of Method 4 for the first frequency shift.

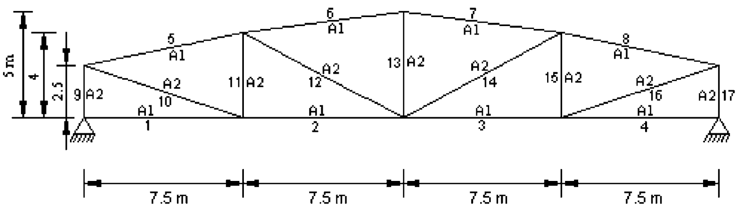

Fig. 8. 2-D truss with consistent mass: $A_1 = 250\ \text{cm}^2$, $A_2 = 100\ \text{cm}^2$, $\rho = 2880$ kg, $E = 2.0E11\ \text{N/m}^2$.

account. The material and geometric specifications are given in the figure. The first frequency of the structure will be shifted again.

The consistent mass matrix will be used for the truss members i:

$$[\mathbf{M}^{(i)}] = \frac{\rho_i A_i L_i}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{45}$$

from which $\partial[\mathbf{M}]/\partial A_i$ is known. To shift the first frequency the following steps are carried out:

(a) compute the dynamic characteristics of the structure;
(b) find the members most effective to a change in the first frequency.
Then to shift the first frequency of the truss, change the properties of:
(c) the most sensitive member using the first order differential equation approach (Method 1);
(d) the most sensitive members using the proportionality constraint for the first order system of differential equations (Method 1);
(e) the most sensitive member using the total differential form for the first order approximation (Method 2);
(f) the most sensitive members with equal proportions using the total differential form for the first order approximation (Method 4);
(g) the most sensitive members, constrained with minimum change in structural mass (Method 3).

Solution. (a) To find the dynamic characteristics of the structure, ANSYS program is used. The first five natural frequencies of the structure are presented in Table 4.

(b) To find the most sensitive members, the derivative of the first frequency with respect to all the members' cross-sectional areas, are computed from Eq. (7). Since the structure is symmetric the results for only one half of the structure are shown in Table 5.

It is observed that the elements 10 and 16 are the most sensitive elements. It is also noted that some of the derivatives are negative. This implies that in those members, the additional stiffness gained from increasing their cross-sectional area in order to increase the frequency, is less than the influence of their added mass in decreasing the frequency.

(c) Now focussing on members 10 and 16 and equating their respective cross-sectional areas, $A_{10} = A_{16}$, one has

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}A_{10}} = \frac{\partial\zeta_1}{\partial A_{10}} + \frac{\partial\zeta_1}{\partial A_{16}}\frac{\mathrm{d}A_{16}}{\mathrm{d}A_{10}} = 2\frac{\partial\zeta_1}{\partial A_{10}}. \tag{46}$$

Table 4
Dynamic characteristics

	Mode i				
	1	2	3	4	5
ζ_i	15796	83311	135637	270344	886489
f_i (Hz)	20.0	45.9	58.6	82.8	149.9

Table 5
Eigenvalue sensitivity

	Elem. j								
	1	2	5	6	9	10	11	12	13
A_j	250	250	250	250	100	100	100	100	100
$\mathrm{d}\zeta_1/\mathrm{d}A_j$	−10813	−77940	27647	−25032	34463	347683	−33112	−98508	−70429

According to Eq. (8)

$$\begin{cases} \dfrac{d\zeta_1}{dA_{10}} = 2\{\mathbf{y}_1\}^T \left\{ \dfrac{\partial[\mathbf{K}]}{\partial A_{10}} - \zeta_1 \dfrac{\partial[\mathbf{M}]}{\partial A_{10}} \right\} \{\mathbf{y}_1\} \quad \text{or} \quad \zeta_1' = 747779 - 3.318\zeta_1, \\ \zeta_1|_{A_{10}=0.01} = 15796, \end{cases} \tag{47}$$

which results in

$$\zeta_1 = 2.2536 \times 10^5 - 2.1664 \times 10^5 e^{-3.318 A_{10}}. \tag{48}$$

Figs. 9 and 10, respectively, show percentage of frequency shift and error versus percentage of area change, compared with the exact solution obtained from the ANSYS program.

(d) It is now assumed that elements 2, 3, 10, 12, 13, 14, and 16 are targeted for modification in order to increase the first eigenfrequency of the structure. Also, equal percentage area change for all the selected elements is assumed. Therefore,

$$\frac{A_i - \bar{A}_i}{\bar{A}_i} = \omega_i \alpha, \quad i = 2, 3, 10, 12, 13, 14, 16, \tag{49}$$

where $\omega_i = 1$ $(i = 10, 16)$, and $\omega_i = -1$ $(i = 2, 3, 12, 13, 14)$. These weights are chosen due to the fact that elements 10 and 16 have a positive effect on the first frequency change while the others have a negative effect. It is assumed that decreasing cross-sectional areas does not violate strength and stability requirements of the structure.

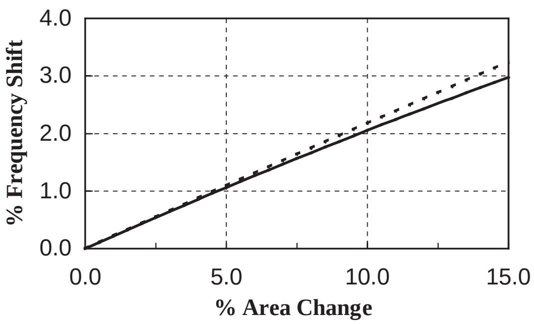

Fig. 9. Comparison of exact method and Method 1 for the first frequency shift (Fig. 8): ——, exact; - - -, approximate.

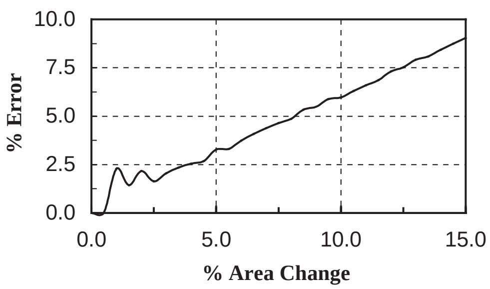

Fig. 10. Percentage error of Method 1 for the first frequency shift (Fig. 8).

Now, from Eq. (14) one has

$$\frac{\partial X}{\partial \alpha} = \sum_i \frac{\partial X}{\partial A_i}\frac{\partial A_i}{\partial \alpha} = \sum_i \frac{\partial X}{\partial A_i}\omega_i \bar{A}_i. \tag{50}$$

Combining Eqs. (13) and (15) yields the following differential equation:

$$\begin{cases} \dfrac{\partial \zeta_1}{\partial \alpha} + \{\mathbf{y}_1\}^{\mathrm{T}} \dfrac{\partial [\mathbf{M}]}{\partial \alpha}\{\mathbf{y}_1\}\zeta_1 - \{\mathbf{y}_1\}^{\mathrm{T}} \dfrac{\partial [\mathbf{K}]}{\partial \alpha}\{\mathbf{y}_1\} = 0 \quad \text{or} \quad \zeta_1' = 705513 - 26.16\zeta_1, \\ \zeta_1|_{\alpha=0} = 15796. \end{cases} \tag{51}$$

The solution of Eq. (51) is

$$\zeta_1 = 26972 - 11176.2\mathrm{e}^{-26.157\alpha}. \tag{52}$$

Fig. 11 shows percentage of approximate and exact frequency shifts versus percentage of area change. Fig. 12 shows percentage of error in the approximate frequency shift versus percentage of area change.

(e) Modifying the cross-sectional areas of members 10 and 16 only, one has

$$\Delta\zeta_1 = 347683(\Delta A_{10} + \Delta A_{16}) = 347683(2 \times \Delta A_{10}) = 695365\Delta A_{10} \tag{53}$$

which gives rise to the following equation:

$$\zeta_1 = 695365(A_{10} - 0.01) + 15796. \tag{54}$$

Figs. 13 and 14 show, respectively, percentage of frequency shift and error versus percentage of area change, computed exactly using ANSYS and approximately using Eq. (54).

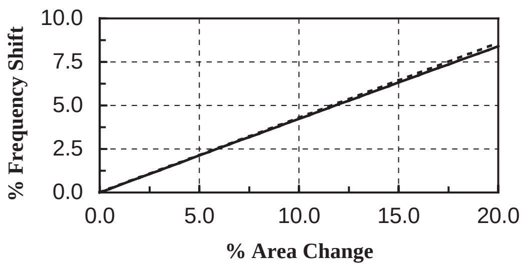

Fig. 11. Comparison of exact method and Method 1(Fig. 8): ——, exact; - - -, approximate.

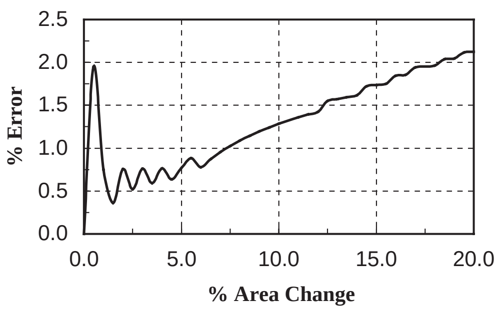

Fig. 12. Percentage error of Method 1 (Fig. 8).

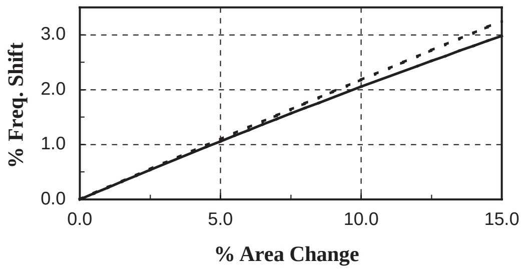

Fig. 13. Comparison of exact method and Method 2 (Fig. 8): ——, exact; - - -, approximate.

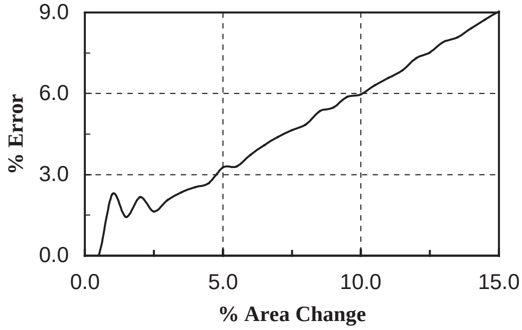

Fig. 14. Percentage error of Method 2 (Fig. 8).

(f) Again, $\omega_i = 1$ $(i = 10, 16)$, and $\omega_i = -1$ $(i = 2, 3, 12, 13, 14)$, and other weights are set equal to zero. Using Eqs. (49) and (50)

$$\zeta_1 = 13525\alpha + 15796 \tag{55}$$

is arrived at.

Similarly, Figs. 15 and 16, respectively, show the percentage of frequency shift and its percentage error using Eq. (55).

(g) Elements 2,3,10,12,13,14, and 16 are selected to be modified, to achieve a shift in the first frequency, such that the variation in mass is a minimum (Method 5). A measure of the structural mass variation can be expressed as

$$\Delta m = \sum_i \{L_i(A_i - \bar{A}_i)^2\}, \quad i = 2, 3, 10, 12, 13, 14, 16, \tag{56}$$

where the density ρ is taken as constant for all the members. It is assumed the structure retains its geometrical symmetry after modification. Therefore, one has

$$\Delta m = 2\{7.5(A_5 - 0.0250)^2 + 7.906(A_{10} - 0.00825)^2 + 8.5(A_{12} - 0.00825)^2 + \tfrac{1}{2}5.0(A_{13} - 0.00825)^2\}. \tag{57}$$

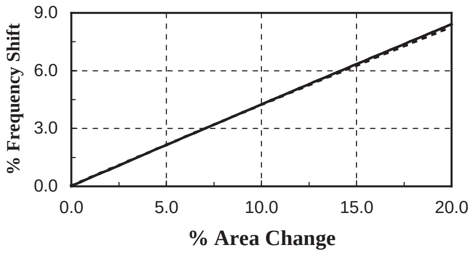

Fig. 15. Comparison of exact method and Method 4 (Fig. 8): ——, exact; - - -, approximate.

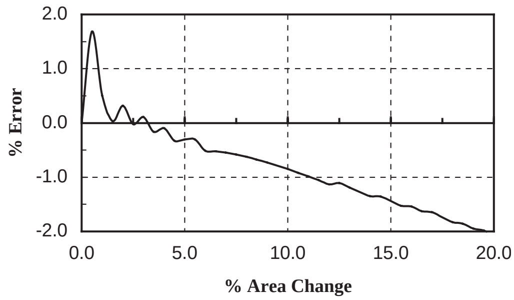

Fig. 16. Percentage error of Method 4 (Fig. 8).

The equation for the eigenvalues shift will be

$$\Delta\zeta_1 = \sum_i \frac{\partial \zeta_1}{\partial A_i} \Delta A_i, \quad i = 2, 3, 10, 12, 13, 14, 16. \tag{58}$$

Hence, using Eqs. (22) and (23), the cross-sectional areas in terms of first frequency shift will be:

$$\begin{cases} A_2 = -9.146 \times 10^{-7} \Delta\zeta_1 + 0.0250, \\ A_{10} = 1.230 \times 10^{-6} \Delta\zeta_1 + 0.00825, \\ A_{12} = -3.276 \times 10^{-7} \Delta\zeta_1 + 0.00825, \\ A_{13} = -3.971 \times 10^{-7} \Delta\zeta_1 + 0.00825, \\ \lambda = 5.638 \times 10^{-11} \Delta\zeta_1. \end{cases} \tag{59}$$

The above results show element 10 as the most sensitive element, as expected. Therefore, Eq. (59) will give approximately the required modified cross-sectional areas in order to achieve the required shift in the first frequency of the structure, with minimum change in the structural mass.

Example 3. As the last example, consider the 2-D arch shown in Fig. 17. The structure consists of 24 plane stress elements with 30 translational degrees of freedom. The initial thickness of the arch is 30 cm and the material and geometric properties are given in the figure. The sensitivity analysis

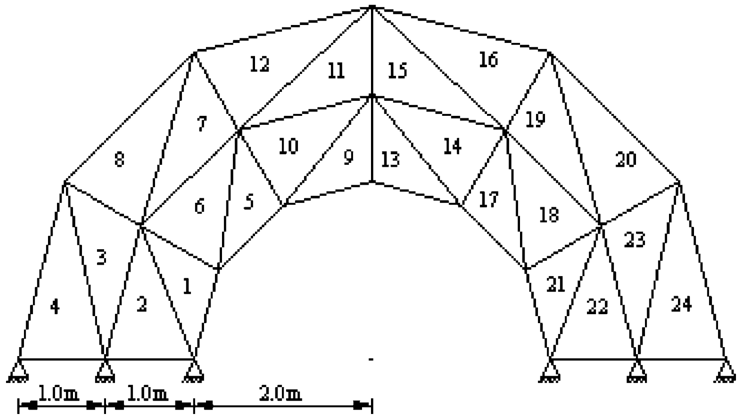

Fig. 17. A 2-D plane stress arch structure: $E = 1E10\ \mathrm{N/m^2}$, $\rho = 2500\ \mathrm{kg/m^3}$, $\nu = 0.25$, $t = 30$ cm.

is again performed on the first frequency of the structure. The consistent mass matrix will be applied for the finite elements. It is necessary to:

(a) Find the members to which the first frequency is more sensitive.

Change the properties of:

(b) the most sensitive members by establishing a first order differential equation for the frequency (Method 1);
(c) the most sensitive members with equal changes using the total differential form (Method 4);
(d) the most sensitive members to increase the second frequency such that the first frequency remains unchanged using the partially constrained total differential form (Method 5);
(e) finally, find a suitable position to add a massless bar element, which makes the most contribution to the first and also the second eigenfrequency. The bar element is assumed to have a modulus of elasticity $E = 2.10\mathrm{E}+11\ \mathrm{N/m^2}$. The position for the following two configurations is found: (1) the bar element added in any position: (2) the bar elements length to lie within an existing element not exceeding the dimensions of any individual existing triangular finite element.

Solution. (a) Using ANSYS program the first five frequencies were identified. These are given in Table 6. Also, Tables 7 and 8 show the sensitivity of the first two frequencies with respect to members' thickness respectively. Fig. 18 shows the first two mode shapes.

(b) Assuming equal relative change in the members thickness, but assigning a negative sign for members with negative $\mathrm{d}\zeta_1/\mathrm{d}t_i$, one has

$$\frac{(t_i - \bar{t}_i)}{\bar{t}_i} = \omega_i \alpha, \quad i = 2, 4, 11, 12, 15, 16, 22, 24, \tag{60}$$

where $\omega_i = 1$ $(i = 2, 4, 22, 24)$, and $\omega_i = -1$ $(i = 11, 12, 15, 16)$.

Therefore, from Eqs. (13) and (14)

$$\begin{cases} \zeta_1' = 79631 + 0.336\zeta_1, \\ \zeta_1|_{\alpha=0} = 198590 \end{cases} \tag{61}$$

Table 6
Natural frequencies

	Mode i				
	1	2	3	4	5
ζ_i	198590	388383	1023320	1372854	2801953
f_i (Hz)	70.93	99.19	161.00	186.48	266.41

Table 7
First eigenvalue sensitivity

	Mem. j							
	2	4	11	12	15	16	22	24
$\mathrm{d}\zeta_1/\mathrm{d}t_j$	61376	81799	−44391	−56270	−44391	−56270	61376	81799

Table 8
Second eigenvalue sensitivity

	Mem. j							
	1	3	11	12	15	16	21	23
$\mathrm{d}\zeta_2/\mathrm{d}t_j$	149351	61969	−109910	−96791	−109910	−96791	149351	61969

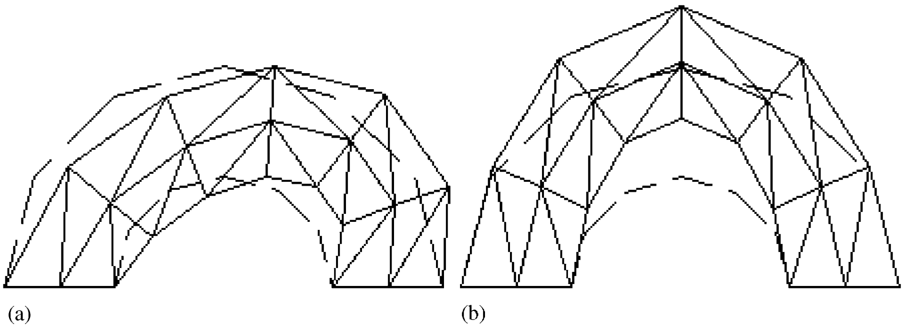

Fig. 18. The two first shape modes: (a) first shape mode; (b) second shape mode.

is arrived at, which gives rise to an equation for eigenvalue change in terms of proportionality factor α:

$$\zeta_1 = -237200 + 435791\mathrm{e}^{0.336\alpha} \tag{62}$$

Figs. 19 and 20 show the comparison and accuracy of the approach.

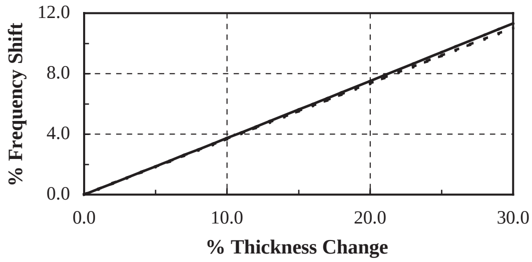

Fig. 19. Comparison of exact method and Method 1 (Fig. 17): ——, exact; - - -, approximate.

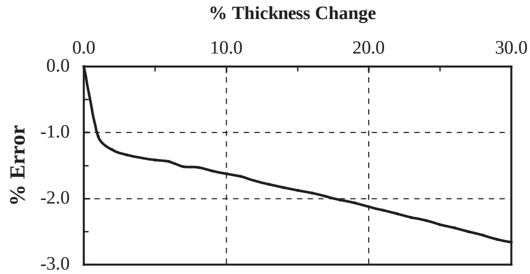

Fig. 20. Percentage error of Method 1 (Fig. 17).

(c) Again, setting $\omega_i = 1$ ($i = 2, 4, 22, 24$), and $\omega_i = -1$ ($i = 11, 12, 15, 16$) with other weights equal to zero, and using Eqs. (49) and (50),

$$\zeta_1 = 146301\alpha + 198590 \tag{63}$$

is arrived at.

Figs. 21 and 22 show the variation and accuracy of the approach.

(d) Noting the values of sensitivities given in Tables 7 and 8, two independent variables α_1 and α_2 are chosen to describe the members' thickness change corresponding to the first and second modes respectively. Table 9 shows the assumed weights for each member and for each mode. The absolute values of weights are chosen according to the contribution of each member to the frequencies shift, and their signs are governed by whether the contribution is positive or negative. Hence, 12 members are chosen to vary in a way that the second frequency increases, while the first frequency is fixed.

The application of Eqs. (30)–(35) will result in the following derivatives:

$$\frac{\partial\zeta_1}{\partial\alpha_1} = 98432, \quad \frac{\partial\zeta_1}{\partial\alpha_2} = 120613, \quad \frac{\partial\zeta_2}{\partial\alpha_1} = 119433, \quad \frac{\partial\zeta_2}{\partial\alpha_2} = 310904. \tag{64}$$

Therefore, Eqs. (56) give rise to the following system of linear equations:

$$\begin{cases} \Delta\zeta_1 = 98432\alpha_1 + 120613\alpha_2, \\ \Delta\zeta_2 = 119433\alpha_1 + 310904\alpha_2. \end{cases} \tag{65}$$

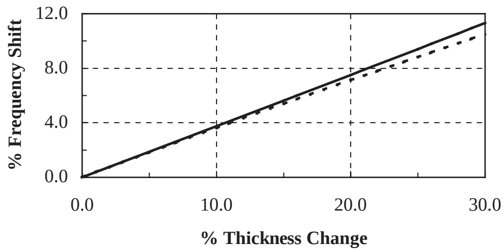

Fig. 21. Comparison of exact method and Method 4 (Fig. 17): —, exact; - - -, approximate.

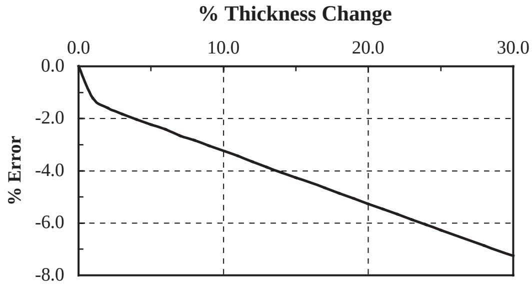

Fig. 22. Percentage error of Method 4 (Fig. 17).

Table 9
Assumed weights for various members and modes

	i											
	1	2	3	4	11	12	15	16	21	22	23	24
ω_{1i}	0.21	0.61	0.20	0.82	−0.44	−0.56	−0.44	−0.56	0.21	0.61	0.20	0.82
ω_{2i}	1.50	0.61	0.62	0.19	−1.10	−0.97	−1.10	−0.97	1.50	0.61	0.62	0.19

Therefore, the unknowns α_s and consequently thickness changes are obtained from

$$\begin{cases} \alpha_1 = 1.919 \times 10^{-5}\Delta\zeta_1 - 7.446 \times 10^{-6}\Delta\zeta_2, \\ \alpha_2 = -7.374 \times 10^{-6}\Delta\zeta_1 + 6.077 \times 10^{-6}\Delta\zeta_2. \end{cases} \tag{66}$$

Now, assuming $\Delta\zeta_1 = 0$, Table 10 shows some of α_s s and the required percentage change of elements' thickness Δt_i, in order to shift the second frequency whilst keeping the first frequency fixed. The table also includes the percentage error of the second frequency shift, *Err* Δf_2, and the percentage of unwanted shift in the first frequency. These results show an acceptable level of accuracy for practical purposes.

It can be seen from these results that a shift of 5.6% in the second frequency can be realized with an error of 4.8% with an induced change in the first frequency of only −0.42%. The maximum and minimum thickness changes occur in elements 1 and 4 respectively. Element

Table 10
Results and errors of partially constrained method (Method 5)

α_1	α_2	$\%\Delta t_1$	$\%\Delta t_2$	$\%\Delta t_3$	$\%\Delta t_4$	$\%\Delta t_{11}$	$\%\Delta t_{12}$	$\%\Delta f_1$	$\%\Delta f_2$	$\%Err\ \Delta f_2$
−0.074	0.061	7.6	−0.8	2.3	−5.0	−3.4	−1.7	−0.03	1.3	1.0
−0.149	0.122	15.1	−1.7	4.6	−9.9	−6.8	−3.4	−0.09	2.5	2.6
−0.186	0.152	18.9	−2.1	5.7	−12.4	−8.5	−4.3	−0.14	3.1	3.0
−0.261	0.213	26.4	−2.9	8.0	−17.3	−11.9	−6.0	−0.26	4.4	4.1
−0.335	0.273	34.0	−3.8	10.3	−22.3	−15.3	−7.8	−0.42	5.6	4.8

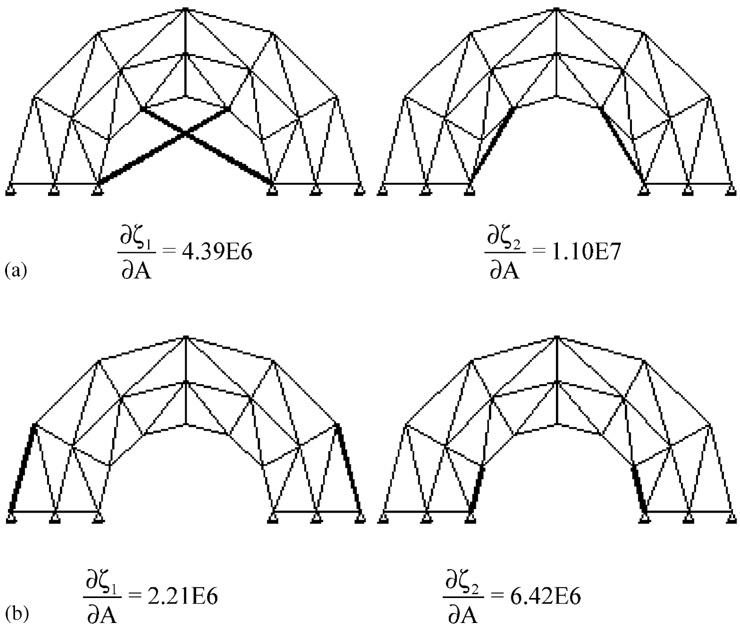

Fig. 23. Additional bar elements for maximum contribution to the first and second frequencies: (a) optimum positions; (b) additional bars adjusted to fit within the existing elements.

number 1 exhibits an increase in thickness of 34%, whilst the thickness of element 4 is decreased by about 22%.

(f) Finally, we locate the optimum position is located for addition of new bar elements into the structure for maximum contribution to the first and second frequencies. To this end, by computing Eq. (7) for every possible configuration, we locate the positions whose sensitivities are maximum for addition of new bars are located, as illustrated in Fig. 23. Fig. 23(a) shows the optimum positions whilst Fig. 23(b) shows the position where the bar elements have been adjusted in order to fit within the boundaries of the existing elements. The values of $\partial\zeta/\partial A$ are shown in the figures for each configuration.

6. Conclusions

The paper presents inverse formulations of eigenvalue problem to optimize the dynamic behaviour of structures. After finding the sensitivity of eigenvalues with respect to design parameters, the desired values of frequency shifts for the structure are determined. This has been done through the establishment of differential equations or linear system of equations, both based on first order expansion approximation of eigenvalues with respect to design variables. The formulations are quite general and applicable to all kinds of finite elements and are suitable for computer code implementation. The generality comes from the fact that the derivatives with respect to design variables are performed at the elemental level. Therefore, as long as the element stiffness and mass matrices derivatives with respect to design variables can be obtained, this method can be used for dynamic optimization. For more complicated elements such as plate or shell elements, the mass and stiffness derivatives with respect to design variables can be numerically calculated. The accuracy of the proposed methods are tested by conducting several examples and validating the results against exact solutions, all of which reveal an acceptable level of accuracy for practical purposes. It is observed that for frequency shifts of about 30 per cent, the error will be less than 9 per cent. One of the advantages of the proposed formulations is that the sensitivity analysis and modification are conducted locally on specific parts of the structure, requiring minimal computational effort. This feature allows the formulations to be used in conjunction with most commercial FEA codes.

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References

[1] V. Belle, Higher order sensitivities in structural design, *American Institute of Aeronautics and Astronautics Journal* 20 (1982) 286–288.
[2] P. Vanhonacker, Differential and difference sensitivities of natural frequencies and mode shapes of natural structures, *American Institute of Aeronautics and Astronautics Journal* 18 (1980) 1511.
[3] J.A. Chen, J.A. Garba, Analytical model improvement using modal test results, *American Institute of Aeronautics and Astronautics Journal* 18 (1980) 684–690.
[4] E. Balmes, A finite element updating procedure using frequency response functions, applications to MIT/SERC interferometer tested, in: *11th International Modal Analysis Conference*, San Diego, USA.
[5] J. Baldwin, S. Hutton, Natural modes of modified structures, *American Institute of Aeronautics and Astronautics Journal* 23 (11) (1985) 1737–1743.
[6] Y.G. Tsuei, E. Yee, A method for modifying dynamic properties of undamped mechanical systems, *Journal of Dynamic Systems, Measurement and Control* 111 (1989) 403–408.
[7] E. Yee, Y.G. Tsuei, Method for shifting natural frequencies of damped mechanical systems, *American Institute of Aeronautics and Astronautics Journal* 29 (11) (1991) 1973–1977.
[8] E. Yee, Y.G. Tsuei, Modification of stiffness for shifting natural frequencies of damped mechanical systems, in: *Modal Analysis, Modelling, Diagnostics and Control—Analytical and Experimental*, DE-Vol. 38, ASME, New York, 1991, pp. 101–106.

[9] Q. Zhang, Ki-ooK. Kim, A review of mass matrices for eigenproblems, *Journal of Computers and Structures* 46 (1993) 1041–1048.
[10] A.J. McMillan, A.J. Keane, Shifting reasonances from a frequency band by applying concentrated masses to a thin rectangular plate, *Journal of Sound and Vibration* 192 (2) (1996) 549–562.
[11] D. Sivan, Y.M. Ram, Mass and stiffness modification to achieve desired natural frequencies, *Communication in Numerical Methods in Engineering* 12 (1996) 531–542.
[12] Y.M. Ram, Enlarging a spectral gap by structural modification, *Journal of Sound and Vibration* 176 (2) (1994) 225–234.
[13] D. Sivan, Y.M. Ram, Optimal construction of mass-spring system with prescribed model and spectral data, *Journal of Sound and Vibration* 201 (3) (1997) 323–334.
[14] K.T. Joseph, Inverse eigenvalue problem in structural design, *American Institute of Aeronautics and Astronautics Journal* 30 (12) (1992) 2890–2896.
[15] G.M.L. Gladwell, Inverse vibration problems for finite element methods, *Inverse Problems* 29 (4) (1997) 421–434.
[16] J.E. Mottershead, On the zeros of structural frequency response functions and their sensitivities, *Mechanics and System and Signal Processing* 12 (5) (1998) 591–597.
[17] J. He, T. Li, Optimal relocation of resonances and anti-resonances of a dynamic system by local modifications, in: *Proceedings of the 15th IMAC,* Orlando, FL, 1997.
[18] A. Djoudi, H. Bahai, I.I. Esat, An inverse eigenvalue formulation for optimising the dynamic behaviour of pin-jointed structures, *Journal of Sound and Vibration* 253 (5) (2002) 1039–1050.
[19] W.C. Mills-Curran, Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues, *American Institute of Aeronautics and Astronautics Journal* 26 (7) (1988) 867–871.
[20] R.L. Dailey, Eigenvectors derivatives with repeated eigenvalues, *American Institute of Aeronautics and Astronautics Journal* 27 (4) (1989) 486–491.