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Mode count and modal density of structural systems: relationships with boundary conditions

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Abstract

The effects of boundary conditions on the mode count and modal density of one- and two-dimensional structural systems, beams and plates, respectively, are investigated by using the wavenumber integration method. Bending vibrations are examined first for a single beam. In this case it is demonstrated that the average mode count is reduced by between 0 and 1 for each boundary constraint, depending on the type of boundary conditions. For more generalised mass and stiffness constraints a frequency-dependent coefficient, which need not lie between 0 and 1, is obtained. The effects of line constraints on the mode count of two-dimensional systems are similar to the equivalent one-dimensional constraints but they are always frequency dependent. Then the mode count of systems of multiple collinear beams and coplanar plates is studied. It is found that an intermediate constraint has the same effect on the average mode count as the same type of constraint applied at an end of the system. The modal density is largely independent of boundary conditions for one-dimensional systems although there are exceptions, while it is dependent on boundary conditions for two-dimensional systems. The results are compared with those from previously published formulae for natural frequencies and with results from finite element method (FEM) analysis. Inclusion of the effect of the boundary conditions in statistical energy analysis (SEA) estimations will result in improved agreements with both analytical and numerical results.
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1. Introduction

The frequency distribution of normal modes in systems of acoustical cavities has been obtained many years ago by various authors [1–6]. This distribution is usually expressed in terms of the mode count, that is the number of the normal modes below a given frequency. The asymptotic expressions for the mode count of an acoustical system were derived by several methods, among

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which the wavenumber space integration, given by Courant and Hilbert [7], is a general technique for finding the distribution of eigenvalues of dynamic systems, not just for acoustical systems. The average number of modes in a unit frequency interval is called the modal density and is clearly related to the mode count. By using the concept of the modal density, the response of acoustic cavities can be treated in a statistical way. This concept has been extended to structure-borne vibrations, and is widely used in statistical energy analysis (SEA), developed amongst others by Lyon [8]. SEA is used for analyses of the vibrations and acoustics of complicated structures at high frequencies.

In order to apply SEA to a complicated structural response problem, it is necessary to know the modal density of the components of the structure under consideration. The evaluation of the mode count and modal density involves the determination of the frequency equation for the structure from the appropriate equation of motion. The resonance frequencies are then summed over all possible modes of vibration. This yields an expression for the mode count in terms of frequency. Differentiation of the expression for the mode count with respect to frequency then yields the expression for the modal density. For a simple vibrating structure such as a rectangular plate, the asymptotic function of the distribution of eigenfrequencies has been analysed by Courant and Hilbert [7] based on the k-space integration technique. However, for a composite structure, this type of mathematical solution is usually impracticable.

The mode count and modal density of basic structural elements such as beams, plates and shells were investigated several decades ago [9–12]. Hart and Shah [11] gave a systematic discussion of the modal density of many basic structural elements. Cremer et al. [13] and Lyon [8] also gave expressions for the mode count and modal density of these basic elements of structures. Langley [14] discussed the modal density of anisotropic structural components. All these results have been used extensively in the applications of SEA for many years. In general, these expressions are based on the forms in which the modal density is taken to be independent of the boundary conditions and is proportional to the size of the system. Hence, the mode count of a structural system is proportional to the length for a one-dimensional structure, to the area for a two-dimensional one and to the volume for a three-dimensional one. For more complicated structures, the simple additive property, that the modal density of the complicated system is equal to sum of the modal densities of its components, is utilised.

The effect of boundary conditions on the mode count and modal density has received comparatively little attention, being seen as of secondary importance. In the expressions for evaluating the mode count of SEA subsystems, presented by Lyon and DeJong [15], the effect of boundary conditions on the mode count and modal density was indicated in terms of a coefficient δ_{BC} that is indicated as usually constant for one-dimensional subsystems and usually assumed to be zero for two- and three-dimensional subsystems. However, for an acoustic cavity with rigid walls, the mode count is given by Morse and Bolt [6]

$$N(f) = \frac{4\pi V}{3c^3} f^3 + \frac{\pi S}{4c^2} f^2 + \frac{L}{8c} f, \tag{1}$$

where V is the volume, S is the total surface area, L is the total length of edges, c is the sound speed in air and f is frequency. This expression was first obtained by Maa [3]. For zero pressure boundary conditions the above expression becomes

$$N(f) = \frac{4\pi V}{3c^3} f^3 - \frac{\pi S}{4c^2} f^2 + \frac{L}{8c} f, \tag{2}$$

which was given by Roe [5]. This demonstrates that the boundary conditions do have effects on the mode count, and from the derivatives of Eqs. (1) and (2), also on the modal density although clearly as frequency increases, $N(f)$ becomes dominated by the first term, proportional to the volume. The types of boundary conditions, which may be present on structural systems, are more diverse and therefore require a more extensive analysis.

More recently, Bogomolny and Hugues [16] and Bertelsen et al. [17] have given expressions for the mode count of a rectangular plate under three standard boundary conditions: free, simple support and clamped on all edges, based on the rigorous analysis by Vasil'ev [18]. In their expressions, there is a perimeter term, which corresponds to δ_{BC} as given in Ref. [15]. However, for one-dimensional systems and the rectangular plate under other combinations of boundary conditions, this perimeter term (or δ_{BC}) is still not generally available.

Some elements of the boundary condition terms for structural systems are therefore already to be found in the literature. This paper presents a systematic approach for the consideration of the effects of boundary conditions on the mode count and modal density of structural systems. It starts with one-dimensional systems, single and multiple beams, and extends this approach to the case of two-dimensional systems, i.e., plates. An approximate method is used based on k-space integration [7]. Since it is clearly impossible to obtain the details for all the various boundary conditions that occur in practice, only the most basic and typical cases will be discussed here. The approximate effects due to various types of boundary condition are initially identified for a single one-dimensional system. The effects due to an intermediate constraint for a multiple collinear beam system are then studied. It is shown, for this system, that an intermediate constraint has the same effect on the average mode count as that type of constraint applied at an end. For two-dimensional systems, the main work carried out concerns a rectangular plate. The effects of intermediate line constraints are also discussed. The conclusions drawn from the studies on the mode count are then used to indicate the effect on the modal density.

Although the boundary conditions of a one-dimensional system are found generally to have negligible effect on the modal density, their inclusion here assists in the interpretation of the two-dimensional case.

2. Average mode count

Before giving a detailed discussion of the mode count, it is instructive to introduce the concept of the *average mode count*. The mode count is the number of modes below a certain frequency. It consists in reality of discrete numbers. If it is plotted against frequency or wavenumber, a "staircase" curve appears as shown in Fig. 1. For frequencies just below the nth mode, the mode count is $n-1$; just above the natural frequency it is n. A continuous function that approximates the average of the staircase function is more useful in practice (see Fig. 1). This average function distributes the mode count along the wavenumber axis (or frequency axis). The average mode

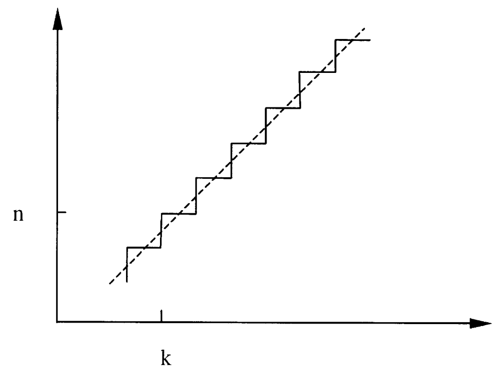

Fig. 1. Illustration of the average mode count (—— staircase function, – – – average mode count $N(k)$).

count can be represented by a curve $N(k)$ that passes through the points

$$N(k_n) = n - \tfrac{1}{2} \tag{3}$$

at the resonance frequencies. The average mode count can also be seen as the average number of modes below a certain frequency occurring within an ensemble of notionally similar structures. The derivative of this average mode count with respect to frequency is the modal density, which is also a statistical quantity. In the remainder of this paper, when a mode count is mentioned, it is normally used to mean the average mode count function, not the discrete mode count.

3. Single one-dimensional system

3.1. Use of phase-closure principle

Lyon and DeJong [15] give an expression for the mode count of the one-dimensional system in terms of wavenumber:

$$N(k) = \frac{kL}{\pi} + \delta_{BC} \tag{4}$$

where δ_{BC} is dependent on the boundary conditions (and, according to Lyon and DeJong, is usually a constant of magnitude less than or equal to 1), L is the length of the one-dimensional system and k is the wavenumber ($= 2\pi$/wavelength), which is related to frequency. For the particular wave type being considered, such as bending or longitudinal waves, the dispersion relation defines the frequency dependence of the wavenumber. No further details are given in Ref. [15] of δ_{BC}; indeed it is eliminated when Eq. (4) is differentiated with respect to k and therefore has no relevance to the modal density. Nevertheless, it is worth exploring how δ_{BC} depends on the type of boundary conditions present.

In a one-dimensional system, the natural modes may be represented as the superposition of equal but opposite-going propagating waves. The natural modes occur when the total phase change as the wave travels one complete circuit around the system is equal to an integral multiple of 2π:

$$2kL + \varepsilon_L + \varepsilon_R = 2n\pi, \tag{5}$$

where ε_L and ε_R are the phase change due to reflections at the boundaries at the left- and right-hand ends, respectively, and n is an integer. This is well known as the *phase-closure* principle. From a knowledge of the wavenumber–frequency relationship as well as the phase change when a propagating wave impinges on each end-boundary of the system, the phase-closure principle can be used to find the natural frequencies [19].

The wavenumber at the nth resonance is given by rearranging (5)

$$k = \frac{n\pi}{L} - \frac{\varepsilon_L + \varepsilon_R}{2L} \tag{6}$$

It can be seen that if ε_L and ε_R are constants, the wavenumber change Δk between two adjacent modes is a constant, equal to π/L. If the natural modes of a one-dimensional system are plotted in wavenumber space, as illustrated in Fig. 2, the boundary effects only influence the distance to the origin. Each mode occupies a length of $\Delta k = \pi/L$ on the k axis. The mode count below a wavenumber k, or a frequency ω, can hence be obtained by finding the number of modes within the length k. This is actually an integration in one-dimensional wavenumber space, which can be expressed as

$$N(k) = \frac{\int_0^k \mathrm{d}k}{\Delta k} - \frac{1}{2}, \tag{7}$$

where $N(k)$ is the average mode count. The term $-1/2$ is the same as that in Eq. (3).

In the following sections, the mode count of a single one-dimensional beam will be derived for various boundary conditions. This will give explicitly the values of δ_{BC}.

3.2. Bending modes in a beam

For the flexural vibrations of a uniform Euler–Bernoulli beam, which have two coupling degrees of freedom, four basic boundary conditions are considered. The phase changes due to boundary conditions are given as follows [13]: for a free boundary condition, the phase change due to reflection is $-\pi/2$, for a simple support, the phase change is π, for a fully fixed boundary, it is again $-\pi/2$, while for a sliding boundary condition, it is 0.

Natural modes can be approximately found using the phase-closure principle when the evanescent waves arriving at the boundaries are ignored. The natural modes of a free–free beam

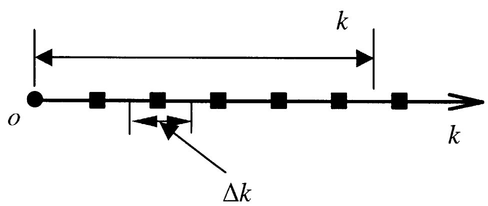

Fig. 2. Illustration of modes in one-dimensional wavenumber space. Each mode occupies a length $\Delta k = \pi/L$ on the k-axis.

are governed by $kL = (n - 3/2)\pi$, where $n = 3, 4, \ldots$ and $n = 1, 2$ correspond to $k = 0$ to include two rigid modes. The average mode count can therefore be given by

$$N = \frac{kL}{\pi} + 1. \tag{8}$$

Changing the boundary condition at one end it is found that this becomes

$$N = \frac{kL}{\pi} + \frac{3}{4}, \quad \text{free–sliding;} \tag{9}$$

$$N = \frac{kL}{\pi} + \frac{1}{4}, \quad \text{free–pinned;} \tag{10}$$

$$N = \frac{kL}{\pi}, \quad \text{free–fixed.} \tag{11}$$

Thus, compared with a free boundary, it can be seen that a sliding boundary adds to the mode count by $-1/4$, a pinned (simple support) boundary adds $-3/4$ and a fixed boundary condition adds -1 for bending vibrations. These results are found to apply to other combinations of boundary conditions at the two ends [20].

The average mode count of one-dimensional systems in flexural vibrations can thus be obtained by considering the mode count of a free–free beam and adding the effects of the boundary conditions at the two ends. This can be given by

$$N = \frac{kL}{\pi} + 1 - \delta_L - \delta_R, \tag{12}$$

where δ_L and δ_R are 0, 1/4, 3/4, and 1 corresponding to the four boundary conditions discussed above. These values are used to find the constants δ_{BC} described in Eq. (4),

$$\delta_{BC} = 1 - \delta_L - \delta_R \tag{13}$$

which is indeed between 1 and -1 for the boundary conditions discussed above [15].

For longitudinal vibrations, it is found that each fixed boundary constraint adds to the mode count by $-1/2$. For a free–free rod $N = kL/\pi + 1/2$ [20].

3.3. General boundary conditions

3.3.1. End spring

When the end of a beam is constrained against transverse displacement by an elastic spring of stiffness K, the amplitude reflection ratio, that is the ratio of the reflected wave amplitude to the incident wave amplitude, can be expressed by [19]

$$r = -\frac{1 - 2\sigma + \mathrm{j}}{1 - 2\sigma - \mathrm{j}} = \mathrm{e}^{\mathrm{j}\theta} \quad \text{where } \theta = \tan^{-1}\left(\frac{2\sigma - 1}{2\sigma(1 - \sigma)}\right), \tag{14}$$

with $-\pi/2 \leqslant \theta \leqslant \pi$, and $\sigma = K/EIk^3$ is a non-dimensional stiffness coefficient, EI being the bending stiffness of the beam. The phase change θ at the boundary is frequency dependent.

Consider a beam that is free at the left-hand end and has a point spring at the right-hand end. The natural modes can be found from Eq. (6) by

$$kL = \left(n - \frac{5}{4}\right)\pi + \frac{\theta}{2} \quad \text{for } n \geqslant 3 \tag{15}$$

when n has been chosen to include two rigid modes, one of which will have a non-zero frequency. For $\sigma \to \infty$, $\theta \to \pi$ and this corresponds to a free–pinned beam. For $\sigma \to 0$, $\theta \to -\pi/2$ and this corresponds to a free–free beam. By comparing Eq. (15) with that of a free–free beam, the effect on the mode count of an end spring can be deduced as

$$\delta_{spring} = \frac{\theta}{2\pi} + \frac{1}{4} \quad \text{for } -\frac{\pi}{2} \leqslant \theta \leqslant \pi \tag{16}$$

δ_{spring} tends to 3/4 at low frequency and to zero at high frequency. This is illustrated in Fig. 3(a).

3.3.2. End mass

When the end of a beam is connected to a point mass, the reflection ratio is

$$r = -\frac{2\mu + 1 + \mathrm{j}}{2\mu + 1 - \mathrm{j}} = \mathrm{e}^{\mathrm{j}\theta} \quad \text{where } \theta = \tan^{-1}\left(\frac{2\mu + 1}{2\mu(\mu + 1)}\right) \tag{17}$$

and $\mu = m\omega^2/EIk^3 = mk/\rho A$ is a non-dimensional coefficient, with ρA the mass per unit length of the beam.

The phase change due to an end point mass is again dependent on frequency. As $f \to 0$, $\mu \to 0$, which corresponds to a free boundary condition. As $f \to \infty$, $\mu \to \infty$, which corresponds to a pinned boundary condition. However, it is found that θ moves through the third quadrant of the complex plane by $-\pi/2$ as frequency increases. This is different from the end spring where the total phase change is $-3\pi/2$.

For a beam that is free at the left-hand end and has a point mass at the right-hand end, similar to the case of an end spring, Eq. (15) again applies. The effect of an end mass on the mode count

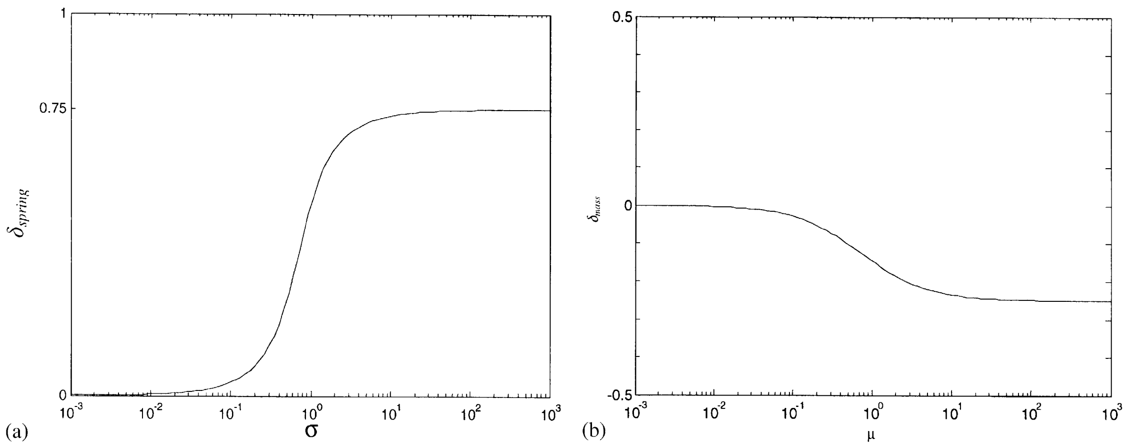

Fig. 3. Illustration of (a) δ_{spring} and (b) δ_{mass}.

can therefore be obtained by

$$\delta_{mass} = \frac{\theta}{2\pi} + \frac{1}{4} \quad \text{for} \; -\pi \leqslant \theta \leqslant -\frac{\pi}{2} \tag{18}$$

δ_{mass} tends to zero at low frequency and tends to $-1/4$ at high frequency. This is illustrated in Fig. 3(b). It should be noted that δ for an end point mass is less than zero. This means that a mass is able to add to the mode count of a beam because it tends to lower the natural frequencies compared with a free end.

These two examples have shown that δ_{BC} need not be constant with frequency.

4. Two-beam system

In this section, the corresponding relationship for a beam of length $2L$ with simple supports at the two ends and an intermediate constraint (Fig. 4) will be discussed.

4.1. A beam with an intermediate fixed constraint

If the extra constraint on the simply supported beam is a fixed condition, the system is then divided exactly into two independent single beams with pinned–fixed boundary conditions. The average mode count for the whole system, N_{total}, can be obtained by adding the two mode counts for the single beams

$$N_{total} = \left(\frac{kL_1}{\pi} - \frac{3}{4}\right) + \left(\frac{kL_2}{\pi} - \frac{3}{4}\right) = \left(\frac{2kL}{\pi} - \frac{1}{2}\right) - 1 = N - 1, \tag{19}$$

where N is the mode count of the original beam of length $2L$ without the extra intermediate constraint. This shows that the mode count of the whole system can be estimated by taking the mode count of the system without the extra constraint and subtracting the coefficient $\delta_{fixed} = 1$ due to the fixed boundary condition. A similar result is found for the trivial case of a "free" intermediate boundary, for which $\delta_{free} = 0$.

4.2. A beam with an intermediate simple support constraint

4.2.1. General solution of natural modes

Consider an intermediate constraint that is a simple support. Define t_m and r_m as the amplitude transmission and reflection coefficients at this middle simple support and write $L_1 = L - l$, $L_2 =$

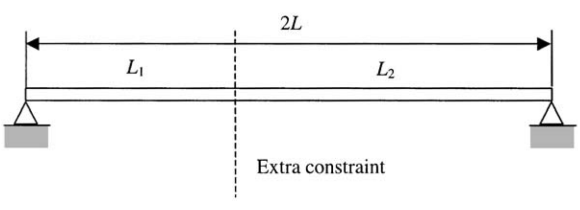

Fig. 4. A simply supported beam of length $2L$ with an extra constraint.

$L+l$, $\alpha = e^{-2kLj}$ and $\beta = e^{-2klj}$. Then it can be shown by a simple analysis that

$$(1 + r_m\alpha/\beta)(1 + r_m\alpha\beta) = t_m^2\alpha^2 \tag{20}$$

t_m and r_m can be obtained by considering the continuity at the middle constraint. By a simple wave analysis, it is found that the amplitude transmission and reflection coefficients of a simple support are given by [21]

$$t_m = \frac{1}{1+j}, \quad r_m = \frac{-1}{1-j}. \tag{21}$$

Now substituting t_m and r_m into Eq. (20) and rearranging it, noting that $1/\beta + \beta = 2\cos(2kl)$, gives

$$\alpha^2 - (1-j)\cos(2kl)\alpha - j = 0. \tag{22}$$

The roots of Eq. (22) have the form

$$\alpha = e^{-2kLj} = \frac{1-j}{2}\sqrt{1-\sin^2(2kl)} \pm \frac{1+j}{2}\sqrt{1+\sin^2(2kl)}. \tag{23}$$

Eq. (23) is an irrational equation in k that gives a general solution for the natural frequencies of the two connected beams with simple support constraints. It can be noted that two sets of modes occur in the system, signified by the $\pm$ sign.

4.2.2. Two identical beams

First, the situation is considered in which the extra constraint is located at the centre. In this case, $l = 0$ and the roots can be expressed as

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_2 = -j. \tag{24}$$

The first root gives $kL = n\pi$. These are exactly the modes of a pinned–pinned beam of length L. In these modes, the two beams vibrate in antiphase. This corresponds to a mode count $N_1 = kL/\pi - \frac{1}{2}$. The second root gives $kL = (n + \frac{1}{4})\pi$. These are the modes of a fixed–pinned beam. In these modes the two beams vibrate in phase. This corresponds to a mode count $N_2 = kL/\pi - \frac{3}{4}$. The total mode count of the system, N_{total}, can be estimated by adding the two sets of modes together

$$N_{total} = N_1 + N_2 = \frac{2kL}{\pi} - \frac{1}{2} - \frac{3}{4} = N - \frac{3}{4}, \tag{25}$$

where N_1 and N_2 are the mode counts of the antisymmetric and symmetric modes respectively and N is the mode count of the beam of length $2L$ without the extra intermediate constraint.

It is thus shown that the mode count of the whole system can be estimated by taking the mode count of the system without the extra constraint and subtracting the coefficient $\delta_{pinned} = 3/4$ due to the simple support boundary condition. This is the same as for the case of the fixed support in Section 4.1.

4.2.3. Asymmetrical simple support

An asymmetrical intermediate simple support represents a more general case. The two roots in Eq. (23) can be expressed as

$$\alpha_1 = \frac{\sqrt{1-\sin^2(2kl)}+\sqrt{1+\sin^2(2kl)}}{2} + \mathrm{j}\left(\frac{-\sqrt{1-\sin^2(2kl)}+\sqrt{1+\sin^2(2kl)}}{2}\right),$$

$$\alpha_2 = \frac{\sqrt{1-\sin^2(2kl)}-\sqrt{1+\sin^2(2kl)}}{2} + \mathrm{j}\left(\frac{-\sqrt{1-\sin^2(2kl)}-\sqrt{1+\sin^2(2kl)}}{2}\right). \tag{26}$$

Note that these two roots are related. Writing $\alpha_1 = a + \mathrm{j}b$, α_2 can be represented as $\alpha_2 = -b - \mathrm{j}a$, where a and b are both positive. These two roots have the same modulus but different phase. It can be readily shown that the modulus is 1 in each case. Therefore the two roots can be represented as $\alpha_1 = \mathrm{e}^{\mathrm{j}\phi_1}$, $\alpha_2 = \mathrm{e}^{\mathrm{j}\phi_2}$, where $0 \leqslant \phi_1 \leqslant \pi/2$ and $-\pi \leqslant \phi_2 \leqslant -\pi/2$. The phases are related by

$$\phi_1 + \phi_2 = -\frac{\pi}{2}. \tag{27}$$

Following the procedure taken in Section 4.2.2, the first root gives $kL = n\pi - \phi_1/2$ and the second root gives $kL = n\pi - \phi_2/2$. The total mode count of the system can be estimated by adding the two sets of modes together

$$N_{total} = \frac{2kL}{\pi} - \frac{1}{2} + \left(\frac{\phi_1+\phi_2}{2\pi}\right) - \frac{1}{2} = N - \frac{3}{4}. \tag{28}$$

This shows that the average mode count of the whole system can be estimated in the same way as in Section 4.2.2. The mode count of the system without the extra constraint is found and the coefficient due to the simple support boundary condition, $\delta_{pinned} = 3/4$, is subtracted. It can be similarly shown that for an intermediate sliding support $N_{total} = N - \delta_{sliding}$, where $\delta_{sliding} = 1/4$.

4.3. A beam with a general intermediate constraint

4.3.1. Intermediate point mass

Suppose that a point mass m is applied at an intermediate position between the two ends of the system. The amplitude transmission and reflection coefficients will become frequency dependent [22]

$$t_m = \frac{(4+\mu)\mathrm{j}}{(4+\mu)\mathrm{j}-\mu}, \quad r_m = \frac{\mu}{(4+\mu)\mathrm{j}-\mu}, \tag{29}$$

where $\mu = m\omega^2/EIk^3 = mk/\rho A$ as before. When frequency is very low, μ tends to zero, $t_m \to 1$ and $r_m \to 0$. This means that there is no constraint applied. Waves will propagate through the mass without reflection. When frequency is very high, μ tends to infinity, $t_m \to 1/(1+\mathrm{j})$ and $r_m \to 1/(\mathrm{j}-1)$. This is equivalent to the case of a simple support constraint (see Section 4.2).

Eq. (20) can be simplified if the two beams are identical in length since $\beta = 1$. In this case,

$$(r_m^2 - t_m^2)\alpha + 2r_m\alpha + 1 = 0. \tag{30}$$

Thus the roots of Eq. (30) are

$$\alpha_1 = -\frac{1}{r_m + t_m} \quad \text{and} \quad \alpha_2 = -\frac{1}{r_m - t_m}. \tag{31}$$

Substituting Eq. (29) in this gives

$$\alpha_1 = \frac{\mu - (4+\mu)\mathrm{j}}{\mu + (4+\mu)\mathrm{j}} \quad \text{and} \quad \alpha_2 = 1. \tag{32}$$

It may be noted from this that one set of modes (the antisymmetric modes) will always be the modes of a pinned–pinned beam of length L whereas the other set of modes will depend on the mass and on frequency. The first root gives

$$\mathrm{e}^{-2kL\mathrm{j}} = \frac{\mu - (4+\mu)\mathrm{j}}{\mu + (4+\mu)\mathrm{j}} = \mathrm{e}^{\,\mathrm{j}\phi} = \mathrm{e}^{-(2n\pi - \phi)\mathrm{j}}, \quad \text{where } \phi = \tan^{-1}\left(\frac{\mu(4+\mu)}{4(\mu+2)}\right) \tag{33}$$

with $\pi \leqslant \phi \leqslant 3\pi/2$. So the natural modes are governed by $kL = n\pi - \phi/2$ and the mode count for this set of modes is (from Eq. (3))

$$N_1 = \frac{kL}{\pi} - \frac{1}{2} + \frac{\phi}{2\pi}. \tag{34}$$

The second root gives

$$N_2 = \frac{kL}{\pi} - \frac{1}{2}. \tag{35}$$

The total mode count of the system can be estimated by adding that for the two sets of modes together

$$N_{total} = N_1 + N_2 = \left(\frac{kL}{\pi} - \frac{1}{2} + \frac{\phi}{2\pi}\right) + \left(\frac{kL}{\pi} - \frac{1}{2}\right) = N + \frac{\phi}{2\pi} - \frac{1}{2}. \tag{36}$$

Therefore, the effect of an intermediate mass on the mode count of the system can be obtained by

$$\delta = \frac{1}{2} - \frac{\phi}{2\pi}, \quad \pi \leqslant \phi \leqslant \frac{3\pi}{2} \tag{37}$$

For $\mu \to 0$, $\phi \to \pi$ and $\delta \to 0$; for $\mu \to \infty$, $\phi \to 3\pi/2$ and $\delta \to -1/4$. This is the same as the case of a mass at the ends of the beams described in Section 3.3. However, it must be indicated that δ for a mass at the ends of the beams is different from that at an intermediate point at a specific frequency, although they have same asymptotic behaviour. It can be shown that δ_{mass} for an intermediate mass is equivalent to δ_{mass} for an end mass in which μ is replaced by $\mu/4$. Note also that θ and ϕ in Eqs. (17), (18) and (33), (37) are not equivalent although δ_{mass} is, apart from this change of μ to $\mu/4$.

The intermediate point mass has a similar effect to that of a point mass applied at the end of a beam, which is discussed in Section 3.3. The difference in the mode count between a beam with and without an intermediate point mass is frequency dependent. The mode count of the system with an intermediate point mass can be expressed by $N_{total} = N - \delta_{mass}$ where N is the mode count of the beam without the intermediate mass and δ_{mass} is a frequency-dependent parameter between 0 and $-1/4$ as described in Eq. (37).

4.3.2. Intermediate point spring

For an intermediate point spring K, the transmission and reflection ratios are readily obtained by considering the continuity conditions as [22]

$$t_m = \frac{(4-\sigma)\mathrm{j}}{(4-\sigma)\mathrm{j}+\sigma}, \quad r_m = \frac{-\sigma}{(4-\sigma)\mathrm{j}+\sigma}, \tag{38}$$

where $\sigma = K/EIk^3$ as before. It can be found that an intermediate point spring has the same effect on the wave propagation as the end point spring discussed in Section 3.3.

The solution of the natural modes can be found as

$$\alpha_1 = \frac{\sigma+(4-\sigma)\mathrm{j}}{\sigma-(4-\sigma)\mathrm{j}} \quad \text{and} \quad \alpha_2 = 1. \tag{39}$$

It can be seen from this that, in a similar fashion to that of the mass constraint, one set of modes will always be the modes of a pinned–pinned beam whereas the other set of modes will depend on the stiffness and on frequency.

The total mode count of the system can be found to correspond to Eq. (36) but in this case, $\phi = \tan^{-1}(\sigma(4-\sigma)/4(\sigma-2))$ with $-\pi/2 \leqslant \phi \leqslant \pi$. The phase change is from $-\pi/2$ at low frequencies to π at high frequencies. Therefore, the effect of an intermediate spring on the mode count of the system can be obtained by

$$\delta = \frac{1}{2} - \frac{\phi}{2\pi}, \quad -\frac{\pi}{2} \leqslant \phi \leqslant \pi. \tag{40}$$

For $\sigma \to \infty$, $\phi \to -\pi/2$ and $\delta \to 3/4$; for $\sigma \to 0$, $\phi \to \pi$ and $\delta \to 0$. This is the same as the case of a spring at the ends of the beams described in Section 3.3. As for the mass discussed above, there is some difference in δ between the case at an end and at an intermediate position. The two expressions for δ_{spring} can be shown to be equivalent if K is replaced by $K/4$ in δ_{spring} for the end spring.

Based on the analysis described in Sections 4.1–4.3, it can be concluded that the mode count of a beam with an extra intermediate constraint is equal to the mode count of this beam without any extra constraint, modified by a coefficient that depends only on the type of the constraint. These coefficients are normally the same as those applying for constraints at the ends of the beams described in Section 3. For a mass or a spring, the coefficients have a frequency-dependent effect, which is the same as for an end constraint apart from a constant factor in the non-dimensionalised mass or stiffness property.

5. Multi-beam system

The study is now briefly extended to the case of a multiple collinear beam system, with arbitrary intermediate supports.

5.1. Intermediate fixed constraints

For the case of fully fixed intermediate constraints, the conclusions for the two-beam system can readily be applied. If there are two simple supports at the ends and $m-1$ fixed intermediate constraints are applied, the system can be regarded as two pinned–fixed beams and $m-2$ fixed–fixed beams all of length L. The mode count of the whole system can be calculated by

$$N_{total} = N_{pinned\text{–}fixed,1} + \sum_{i=2}^{m-1} N_{fixed\text{–}fixed,i} + N_{pinned\text{–}fixed,m} \tag{41}$$

with

$$N_{pinned\text{–}fixed} = \frac{kL}{\pi} - \frac{3}{4}, \quad N_{fixed\text{–}fixed} = \frac{kL}{\pi} - 1, \tag{42}$$

where $N_{pinned\text{–}fixed}$ is the mode count of a pinned–fixed beam, $N_{fixed\text{–}fixed}$ is the mode count of a fixed–fixed beam and m is the number of beam segments of the system. This yields

$$N_{total} = \frac{kL}{\pi} - \frac{1}{2} - (m-1) = N - (m-1)\delta_{fixed}, \tag{43}$$

where N is the mode count of the long beam, length $L = \sum L_i$, with simple supports at the two ends and $\delta_{fixed} = 1$ as before.

Since fixed constraints separate the system into exactly uncoupled segments, the mode count for the whole system can be represented by the mode count of a long beam minus the product of the number of constraints $(m-1)$ and the constraint coefficient δ.

5.2. Intermediate simple supports

To provide an example of a multi-span beam with intermediate simple supports, a 10 segment beam has been generated in a finite element (FE) model. The modal frequencies have then been calculated using the FE method. The lengths of the segments were 20, 22, 18, 17, 23 20, 19, 25, 15 and 21 cm, giving a total length of 2 m. Fig. 5 shows the mode count of this system. The mode count difference between the random multi-beam system and the average result for the long beam is calculated and presented in Fig. 6. The average value is 6.8 which corresponds to $9\delta_{pinned}$, with $\delta_{pinned} = 3/4$.

This and other examples given in Ref. [20] therefore demonstrates, although without analytical proof, that the mode count of a system of m arbitrary length multiple collinear beams can be estimated by

$$N_{random} = N - (m-1)\delta_{pinned}. \tag{44}$$

As for the two-beam systems, an intermediate constraint has the same effect on the average mode count as the same type of constraint applied at an end.

6. Mode count of rectangular plates

The number of modes of a rectangular plate with wavenumber less than a given value of k is given by Hart and Shah [11] and Cremer et al. [13] as

$$N = \frac{k^2 S}{4\pi}, \tag{45}$$

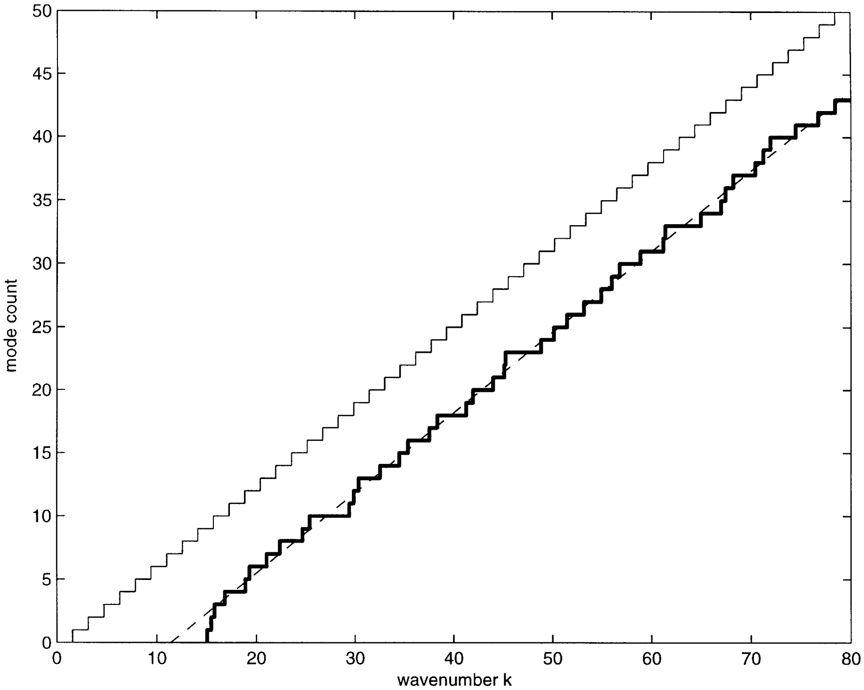

Fig. 5. Mode counts of multi-beam systems (—— 2 m beam, thick line: 2 m beam with 10 arbitrary length spans, – – – average mode count).

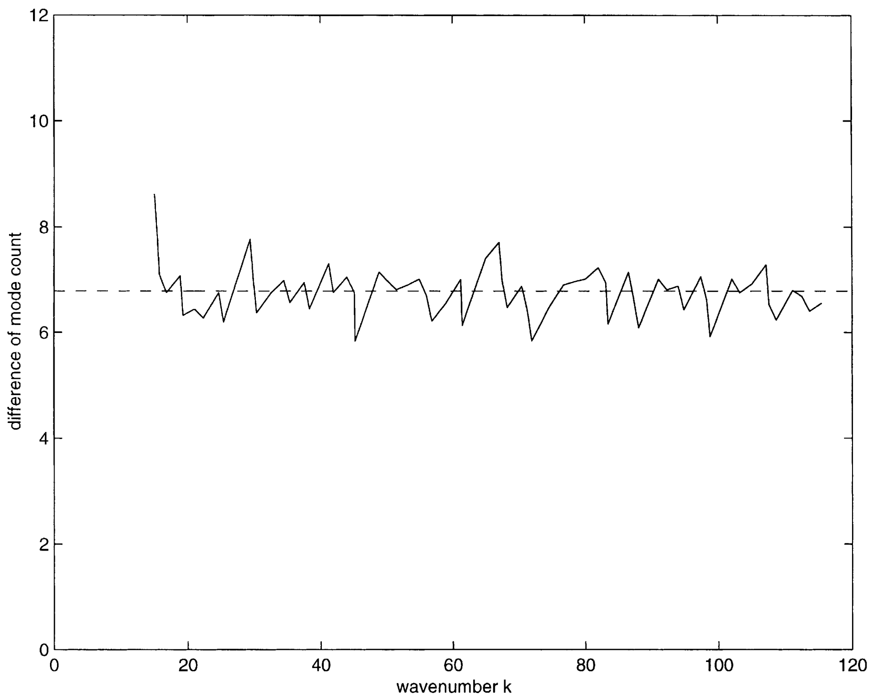

Fig. 6. Mode count difference between the random multi-beam system and the result for the long beam (——). The average value is 6.8 (– – –).

where N is the mode count, k is wavenumber and S is the area of the plate under consideration. Lyon and DeJong [15] extend this by writing

$$N \simeq \frac{k^2 S}{4\pi} + \Gamma_{BC} P k, \tag{46}$$

where P is the perimeter length and Γ_{BC} depends on the boundary conditions. The modal density of the system can be obtained by the differentiation of Eq. (46). This yields a term of Γ'_{BC} in the expression of the modal density. It is suggested in Ref. [15] that although the quantity Γ'_{BC} can often be determined for an isolated system, it is best to assume it to be zero for connected systems because the effective boundary conditions change with frequency. However, in this section an approximate expression for Γ'_{BC} will be obtained.

6.1. Natural modes and mode count

The flexural wave equation of motion of a plate of thickness h is given by

$$B\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial^2 x\, \partial^2 y} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = 0, \tag{47}$$

where $B = Eh^3/12(1 - v^2)$ is the flexural rigidity with E, Young's modulus and v, the Poisson ratio, ρ is the density and w is the out-of-plane displacement. The harmonic plane wave solution has the form

$$w(x, y, t) = \mathrm{e}^{-\mathrm{j}k_x x}\mathrm{e}^{-\mathrm{j}k_y y}\mathrm{e}^{\,\mathrm{j}\omega t}, \tag{48}$$

where k_x and k_y are the trace wavenumbers in the x and y directions.

For a finite plate, natural modes will occur due to the wave reflections at the boundaries. Considering a rectangular plate of dimensions a and b as shown in Fig. 7, as for the case of a beam, the phase-closure principle can be applied to it to find the natural modes. By ignoring the effect of nearfield waves across the plate, i.e., assuming that a nearfield wave generated at one edge

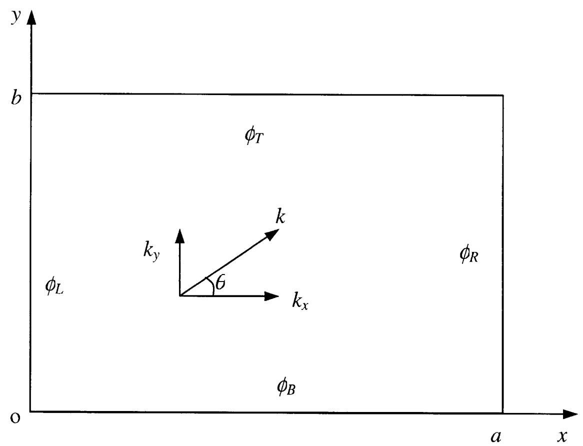

Fig. 7. Illustration of the rectangular plate under consideration.

will have negligible effect at another, the natural modes will occur when

$$2k_x a + \phi_L + \phi_R = 2m\pi, \tag{49}$$

$$2k_y b + \phi_T + \phi_B = 2n\pi, \tag{50}$$

where ϕ_L, ϕ_R are the phase changes at the boundaries for the x direction and ϕ_T, ϕ_B for the y direction, and m, n are integers. This corresponds to Bolotin's asymptotic method [23], see also Ref. [24]. The natural mode (m,n) can be found from

$$k^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a} - \frac{\phi_L + \phi_R}{2a}\right)^2 + \left(\frac{n\pi}{b} - \frac{\phi_T + \phi_B}{2b}\right)^2, \tag{51}$$

where $k = (\rho h\omega^2/B)^{1/4}$.

These natural modes can also be plotted in the wavenumber domain, 'k-space', as illustrated in Fig. 8. Each point corresponds to a mode (m,n). The trace or component wavenumbers in the two directions are given by

$$k_x = \frac{m\pi}{a} - \frac{\phi_L + \phi_R}{2a}, \quad k_y = \frac{n\pi}{b} - \frac{\phi_T + \phi_B}{2b}. \tag{52}$$

It can be seen that the variation of the trace wavenumber from one mode to the next is constant, namely π/a and π/b for constant ϕ_L, ϕ_R, etc. The boundary conditions effectively only influence the distance to the axes, not the separation between points. This characteristic is very useful, allowing the use of k-space integration to calculate the mode count of a rectangular plate. These observations are, however, only approximate due to the neglect of nearfield waves across the plate. In an exact analysis the boundary conditions would distort the grid of points in k-space as well as shift them relative to the axes.

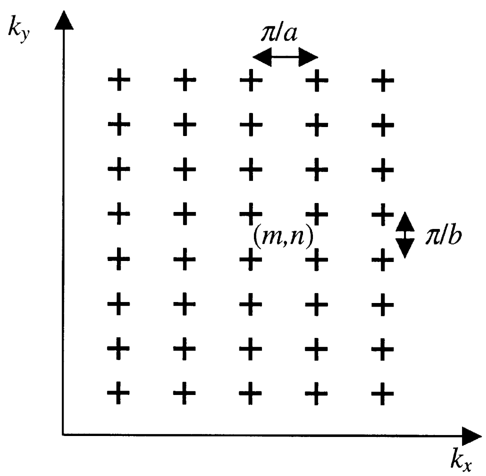

Fig. 8. Illustration of the k-space of the natural modes of a rectangular plate.

6.1.1. Simply supported plate

If all four edges of the plate are simply supported, the phase change at each edge is π. The phase-closure principle gives

$$k_x a = m\pi;$$

$$k_y b = n\pi. \tag{53}$$

The natural frequencies are given by

$$\omega_{mn} = \left(\frac{B}{\rho h}\right)^{1/2} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right], \tag{54}$$

which in this case is exact.

The k-space plot of the modes is presented in Fig. 9(a). The mode count below a particular frequency can be obtained by finding the number of modes located within the quarter circle of radius k. This is given by

$$N(k) = \frac{\int_S \mathrm{d}k_x \mathrm{d}k_y}{\Delta k_x \Delta k_y}, \tag{55}$$

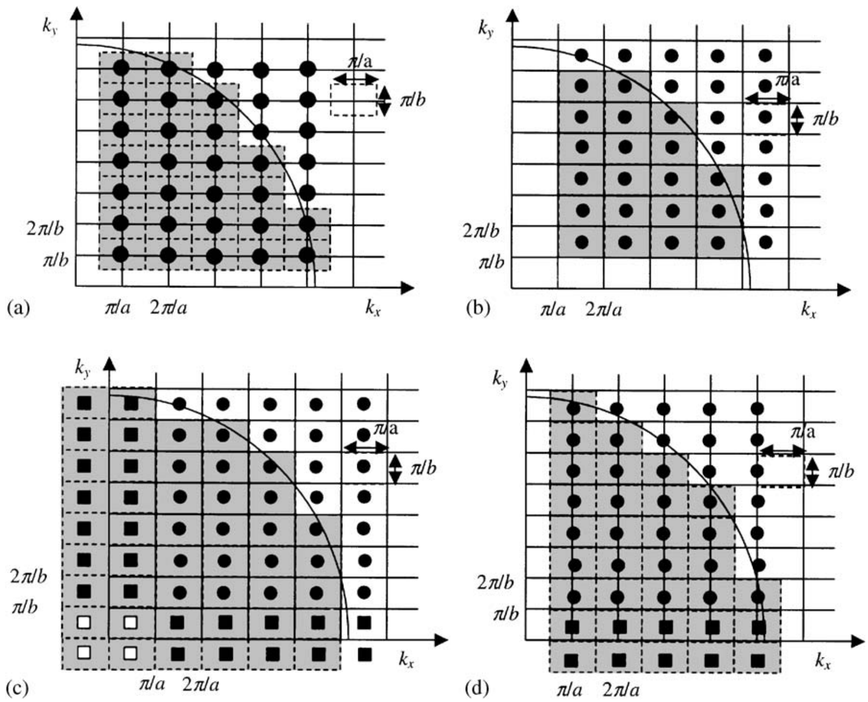

Fig. 9. The modes of a rectangular plate shown in k-space: (a) simply supported, (b) fully fixed, (c) free and (d) two opposite edges simply supported (solid square: beam-mode like mode, open square: rigid mode, solid circle: plate mode).

where S is the area of integration, Δk_x and Δk_y are the changes in trace wavenumber from one mode to the next. Eq. (55) can be evaluated as

$$N(k) = \frac{\int_0^k \int_0^{\pi/2} k\,\mathrm{d}\theta\,\mathrm{d}k}{\Delta k_x \Delta k_y} = \frac{\frac{1}{4}\pi k^2}{\Delta k_x \Delta k_y}. \tag{56}$$

In considering the average area occupied by each mode in k-space, it can be noted that each mode occupies an area of $\pi/a \times \pi/b$. However, the area of two strips along the axes should not be taken into account by integration of Eq. (56). This is illustrated in Fig. 9(a) by the shaded region. Therefore, the average mode count below wavenumber k is given by a modification of Eq. (56)

$$N(k) = \frac{\left[\frac{1}{4}\pi k^2 - k\left(\frac{\pi}{2a} + \frac{\pi}{2b}\right) + \frac{\pi}{2a}\frac{\pi}{2b}\right]}{\Delta k_x \Delta k_y}, \tag{57}$$

where Δk_x and Δk_y are given by π/a and π/b. Hence, the mode count for a simply supported rectangular plate can be given by

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{2}k\left(\frac{a+b}{\pi}\right) + \frac{1}{4} \tag{58}$$

or

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{4\pi}kP + \frac{1}{4}, \tag{59}$$

where $P = 2(a+b)$ is the perimeter of the plate and $S = ab$ is its area. The first term in Eq. (59) is that often used in the literature to estimate the mode count for a plate. It can be seen that the result of the mode count for a simply supported plate from Eq. (59) is less than that from Eq. (45) due to the inclusion of a perimeter term.

Bogomolny and Hugues [16] give this perimeter term as

$$N_p = \beta \frac{L}{4\pi} k, \tag{60}$$

where $\beta = -1$ for the simply supported plate. It can be seen that Eq. (59) gives the same results as those from Eq. (60).

6.1.2. *Fully fixed plate*

The natural modes of a fully fixed rectangular plate occur approximately when the wavenumbers satisfy

$$k_x a = \left(m + \tfrac{1}{2}\right)\pi, \quad k_y b = \left(n + \tfrac{1}{2}\right)\pi, \tag{61}$$

where $m, n = 1, 2, 3, \ldots$. The k-space plot of the natural modes is presented in Fig. 9(b). Based on Eq. (56) and the concept of the average mode count, the mode count for a fully fixed rectangular plate can be given by

$$N(k) = \frac{\left[\frac{1}{4}\pi k^2 - k\left(\frac{\pi}{a} + \frac{\pi}{b}\right) + \frac{\pi}{a}\frac{\pi}{b}\right]}{\frac{\pi}{a}\frac{\pi}{b}}$$

or

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{2\pi} kP + 1. \tag{62}$$

In terms of Eq. (60), Bogomolny and Hugues [16] give $\beta = -1.7628$ for a fully fixed plate. From Eq. (62), the present approximate analysis has given $\beta = -2$.

6.1.3. Free plate

For a plate with four free edges, three rigid modes should be included. These consist of translation, rotation about the x direction and rotation about y direction. A fourth low frequency mode (not a bending mode) exists in which the plate flexes with opposite corners moving in phase. The remaining natural modes should satisfy approximately

$$k_x a = \left(m - \tfrac{3}{2}\right)\pi, \quad k_y b = \left(n - \tfrac{3}{2}\right)\pi \tag{63}$$

where $m, n = 3, 4, \ldots$.

Corresponding to a rigid mode in one direction, either translation or rotation, a set of beam-like modes occur in the other direction. These beam modes have a similar modal characteristic to a one-dimensional system except for a small effect of the Poisson ratio. The plate modes, therefore, should include two sets of beam modes in each direction corresponding to the rigid modes in the other direction. This can be shown in a k-space plot as in Fig. 9(c). The two sets of beam modes, and the four rigid modes, have been plotted for convenience at $k_x = \pm\pi/2a$ or $k_y = \pm\pi/2b$, although their wavenumbers are actually zero.

The average mode count hence can be given by evaluating the shaded area, i.e.,

$$N(k) = \frac{\left[\frac{1}{4}\pi k^2 + k\left(\frac{\pi}{a} + \frac{\pi}{b}\right) + \frac{\pi}{a}\frac{\pi}{b}\right]}{\frac{\pi}{a}\frac{\pi}{b}} \tag{64}$$

or

$$N(k) = \frac{k^2 S}{4\pi} + \frac{1}{2\pi} kP + 1. \tag{65}$$

In terms of Eq. (60), Bertelsen et al. [17] give $\beta = 1.7126$ for a free plate. From Eq. (65), β is approximately given as 2.

6.1.4. Plate with two opposite edges simply supported

Considering a plate having only two opposite edges $x = 0$ and a with simple supports, the other two edges, $y = 0$ and b, being free, the natural modes should satisfy approximately

$$k_x a = m\pi, \quad k_y b = (n - 3/2)\pi, \tag{66}$$

where m and n are integers starting from 1 and 3, respectively.

Since the two edges $y = 0$ and b are free, two rigid modes will occur in the y direction, $n = 1, 2$. These two rigid modes correspond to two sets of the simply supported beam modes for the x direction of the plate. This is shown in a k-space plot, Fig. 9(d). All the shaded area must be taken into account for calculating the mode count. The expression of the average mode count is given by

$$N(k) = \frac{\left[\frac{1}{4}\pi k^2 - k\frac{\pi}{2a} + k\frac{\pi}{b} - \frac{\pi}{2a}\frac{\pi}{b}\right]}{\frac{\pi}{a}\frac{\pi}{b}},$$

thus

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{2}k\frac{b}{\pi} + k\frac{a}{\pi} - \frac{1}{2}. \tag{67}$$

This shows that the free edges of length a increase the mode count, while the simply supported edges of length b reduce it, compared with Eq. (45).

6.2. Verification

To verify the result of Eqs. (59), (62), (65) and (67), an aluminium plate of dimensions $0.4 \times 0.3 \times 0.002\ \mathrm{m}^3$ is considered. The natural modes are calculated based on equations from Leissa [25] and then the staircase function of the mode count is plotted. The average mode counts from Eq. (45) are also calculated and plotted for the purpose of comparison.

Fig. 10 presents the case of the simply supported plate. The differences between the results from Eq. (59), (45) and the actual mode count are plotted in Fig. 11. The values of the actual mode count are given at each resonance frequency by the top stair point minus 0.5 in each case (see Section 2). The average difference between the results from Eq. (59) and the actual value is

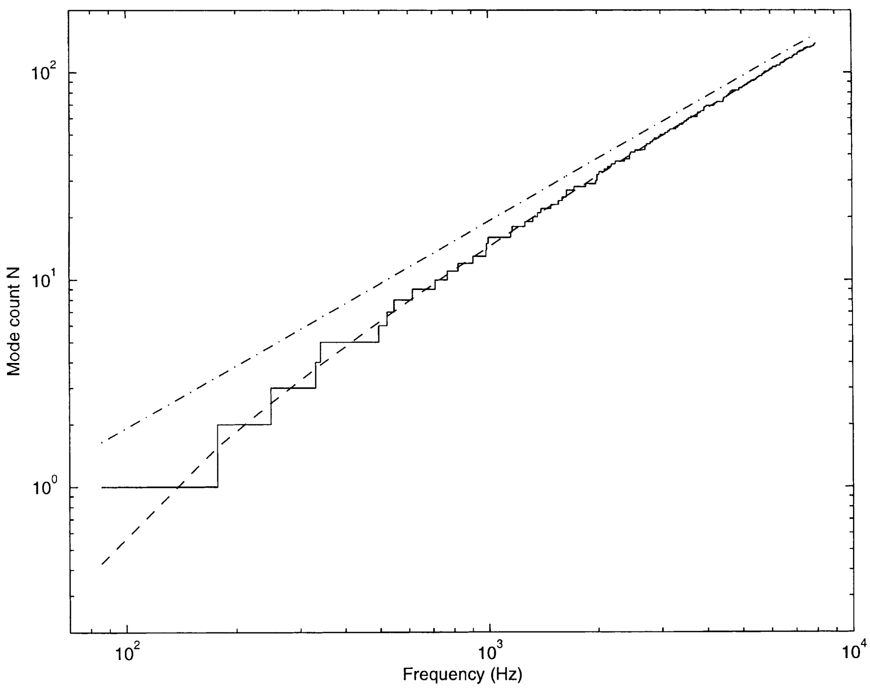

Fig. 10. The mode count for a simply supported plate (—, stair-case; – – –, including perimeter term, Eq. (59); – · –, excluding perimeter term, Eq. (45)).

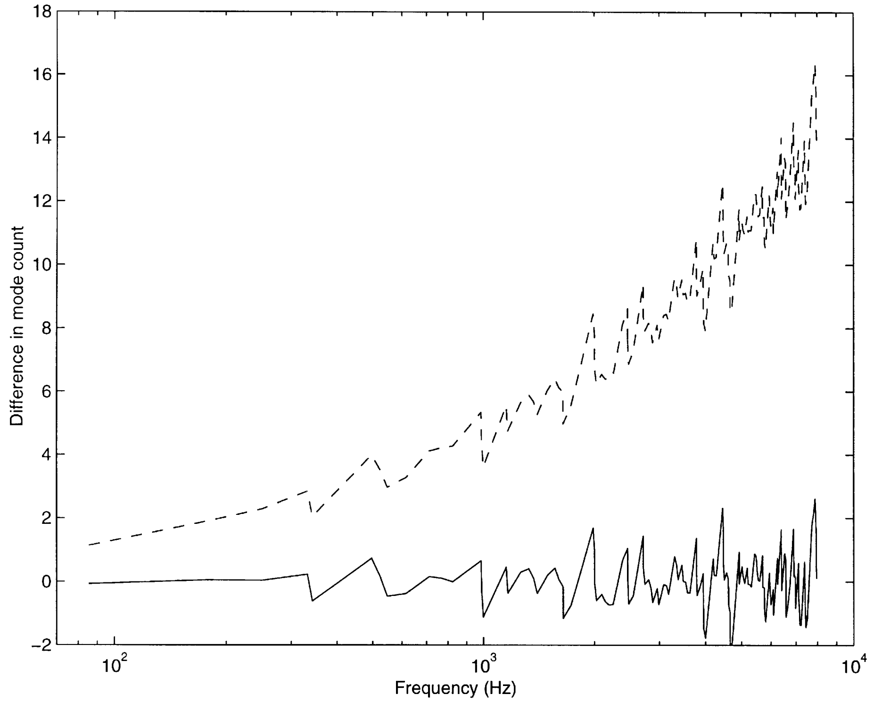

Fig. 11. Difference between the estimated mode count and the actual one for a simply supported plate (—, difference between estimate from Eq. (59) and actual one; – – –, corresponding difference omitting perimeter terms, based on Eq. (45)).

only −0.043. However, Eq. (45) has a systematic error increasing as frequency increases (although the relative error reduces as seen in Fig. 10).

Figs. 12 and 13 present the case of the fully fixed plate. It can be seen that Eq. (62) gives much better agreement with the staircase function than Eq. (45). The errors of Eq. (62) are caused by approximations made in the phase-closure principle from which it is derived. Due to ignoring the nearfield waves in using the phase-closure principle to obtain the natural modes, the solutions of the first few modes and all modes with $m = 1$ or $n = 1$ tend to have larger values than those from the exact solution. This eventually causes an underestimate of the mode count Eq. (62) by 2 or 3 at high frequencies. The result based on $\beta = -1.7628$, given by Bogomolny and Hugues [16], Eq. (60) gives a better agreement at high frequencies.

Figs. 14 and 15 present the results for the free plate and the plate with two opposite edges simply supported. These also support the above analysis, although the approximate result derived for the free plate slightly overestimates the mode count.

6.3. Relationship between mode count and boundary conditions

From Section 6.2 above, it has been seen that the mode count of a plate is not independent of boundary conditions. Using Eq. (45) to evaluate the mode count of a plate may cause a

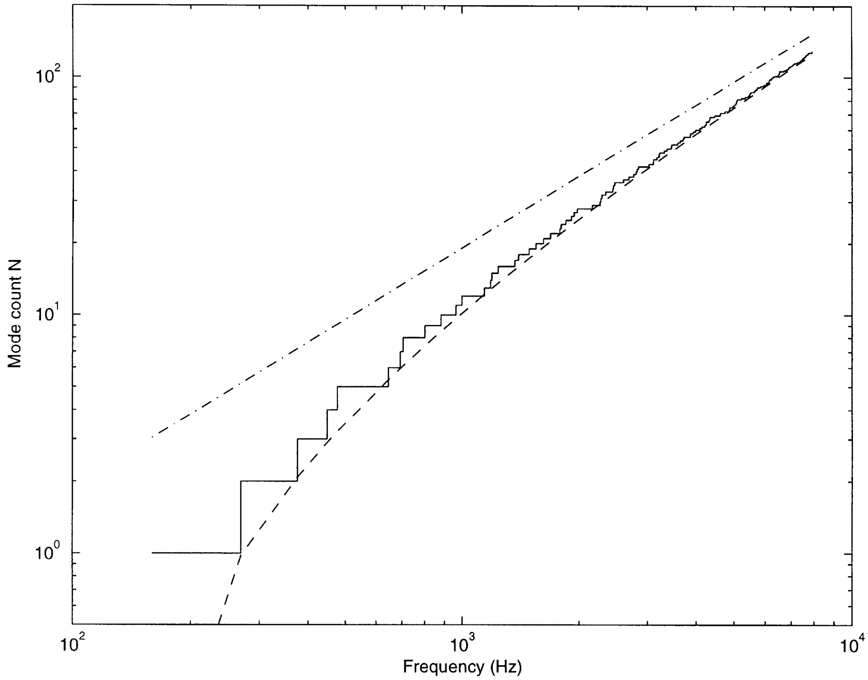

Fig. 12. The mode count for a fully fixed plate (—, stair-case; – – –, including perimeter term, Eq. (62); – · –, excluding perimeter term, Eq. (45)).

considerable error. The formulae based on the phase-closure principle and the k-space plot for the four combinations of boundary conditions, although approximate, have presented a more accurate result. It can also be seen that the k-space plot for the mode count of a rectangular plate, under any combination of boundary conditions, is effectively a shift of the mode lattice that depends only on the dimensions. By observing Eqs. (58), (62), (65) and (67), the general expression for the mode count of a rectangular plate can be written as

$$N(k) = \frac{k^2 S}{4\pi} + \delta_x \frac{kb}{\pi} + \delta_y \frac{ka}{\pi} - \Delta, \tag{68}$$

where δ_x and δ_y are the boundary effects for waves propagating in the x and y directions and Δ is a small constant.

Actually δ_x and δ_y include the effects from two opposite edges in their own directions. Because all examples used previously are symmetric in this respect, it is reasonable to anticipate that half the value of δ_x or δ_y will be the boundary effect of one edge. Following the method used for the beam and considering the free plate as a basis for comparison, the effect of each boundary can be obtained. The constant term Δ will be ignored because it is very small. The approximate mode count for a free rectangular plate is hence given by

$$N(k) = \frac{k^2 S}{4\pi} + \frac{kb}{\pi} + \frac{ka}{\pi}. \tag{69}$$

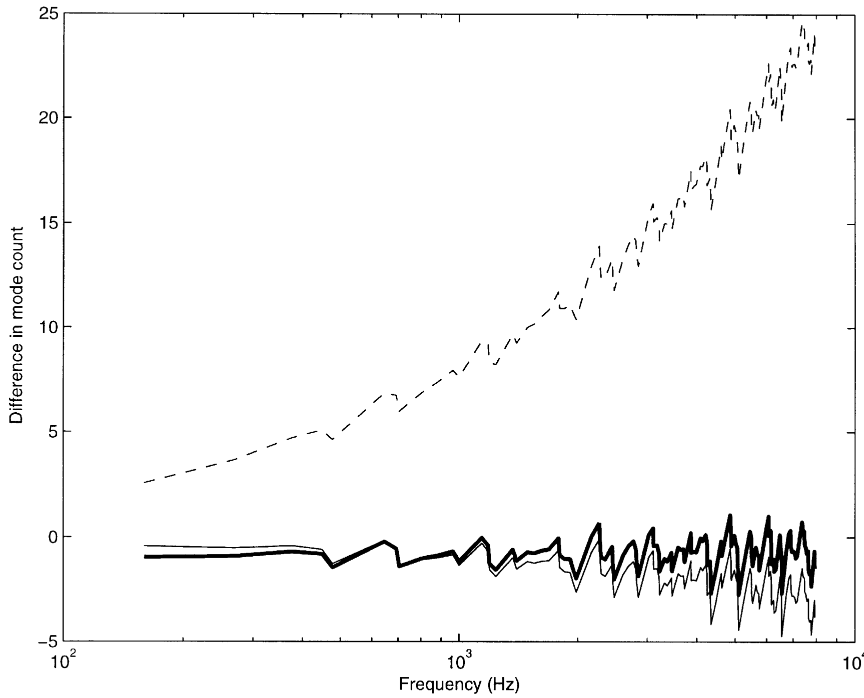

Fig. 13. Difference between the estimated mode count and the actual one for a fully fixed plate (—, difference between estimate from Eq. (62) and actual one; thick line, corresponding difference using Eq. (60); – – –, corresponding difference omitting perimeter terms, based on Eq. (45)).

Comparing Eqs. (67) and (69), the effect due to the simple supports on the each edge $x = 0$ and a is given by

$$\delta_{x\text{-}pinned} = -\frac{3}{4}\frac{kb}{\pi}, \tag{70}$$

where b is the length of the edge concerned. It is noted that the coefficient in the above expression (3/4) is equal to the constant effect of a simple support on the mode count of one-dimensional systems. By further comparing the case of a plate simply supported on four edges with the free plate, the effect due to a simple support on one edge $y = 0$ and b can be given as

$$\delta_{y\text{-}pinned} = -\frac{3}{4}\frac{ka}{\pi}. \tag{71}$$

Similarly, by comparing the case of a fully fixed plate with that of a free plate, the effect of a fixed condition on one edge can be given as

$$\delta_{x\text{-}fixed} = -\frac{kb}{\pi}, \quad \delta_{y\text{-}fixed} = -\frac{ka}{\pi}. \tag{72}$$

The constant coefficients in Eq. (72) also correspond to the result for a fixed condition for one-dimensional systems (1).

Based on the above derivations, it is straightforward to conclude that the effect of a line constraint on the mode count of a rectangular plate is equal to the product of the constant

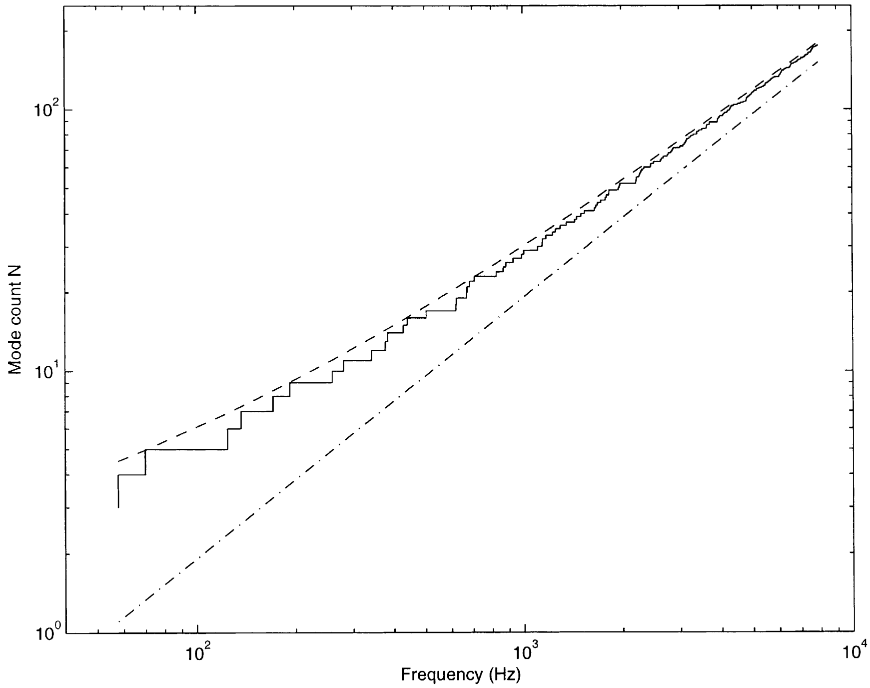

Fig. 14. The mode count for a free plate (—, stair-case; – – –, including perimeter term, Eq. (65); – · – ·, excluding perimeter term, Eq. (45)).

effect of the same type of constraint in a one-dimensional beam and a frequency-dependent term, which depends on the dimensions and the dispersion relation of the plate. This can be represented by

$$\delta_{2\text{-}D} = \delta_{1\text{-}D}\frac{kL_{edge}}{\pi}, \tag{73}$$

where $\delta_{1\text{-}D}$ corresponds to the boundary effect in one-dimensional systems and L_{edge} is the length of the line constraint.

Summarising, the mode count of a rectangular plate can be given by

$$N(k) = \frac{k^2 S}{4\pi} + (1 - \delta_{x\text{-}left} - \delta_{x\text{-}right})\frac{kb}{\pi} + (1 - \delta_{y\text{-}top} - \delta_{y\text{-}bottom})\frac{ka}{\pi}, \tag{74}$$

where $\delta_{x\text{-}left}$, $\delta_{x\text{-}right}$, $\delta_{y\text{-}top}$ and $\delta_{y\text{-}bottom}$ are the one-dimensional boundary effect terms corresponding to the boundary conditions of the four edges. This expression has also been verified for other combinations of boundary conditions.

7. A plate with intermediate line constraints

Following the conclusion in Sections 4 and 5 that, for one-dimensional systems, an intermediate constraint has the same effect on the average mode count as the same type of constraint applied at an end, it can be anticipated that an intermediate line constraint will have the same effect as the

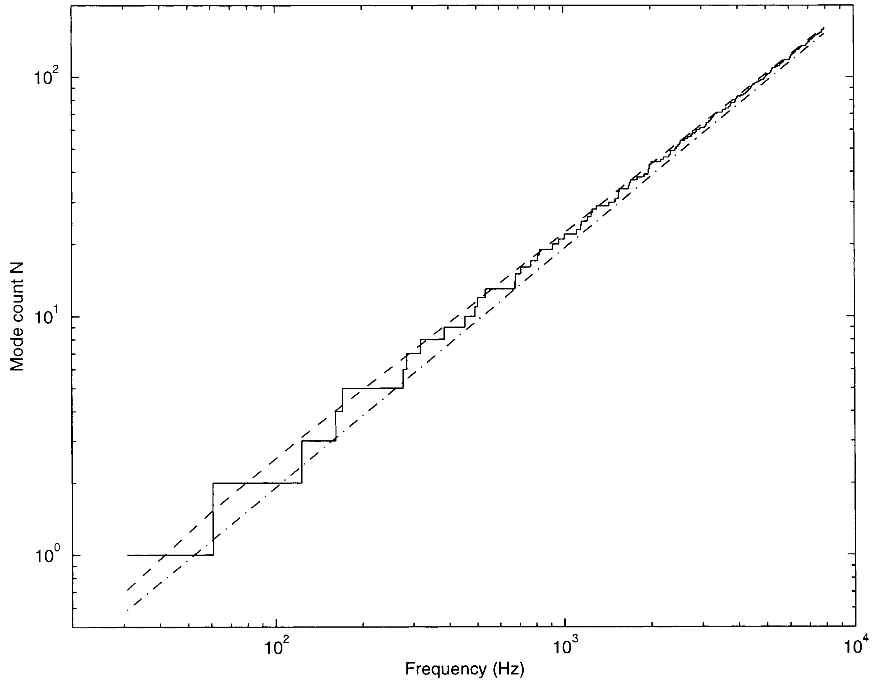

Fig. 15. The mode count for a plate with two opposite edges free and others simply supported plate (—, stair-case; – – –, including perimeter term, Eq. (67); – · – ·, excluding perimeter term, Eq. (45)).

same type of line constraint applied at an edge for two-dimensional systems. If a simply supported rectangular plate is considered with one or more intermediate line simple supports applied, the average mode count of such a system can be given by

$$N_{total}(k) = N(k) - m\delta_{BC}, \tag{75}$$

where $N_{total}(k)$ is the average mode count of the whole system, $N(k)$ is the average mode count without considering the intermediate constraints, given by Eq. (74), m is the number of applied intermediate constraints and δ_{BC} is the line boundary effect of a simple support on the mode count that is determined by Eq. (73).

To verify Eq. (75), two examples are studied. The first is a simply supported plate of dimensions $0.4 \times 0.3 \times 0.002\ \mathrm{m}^3$ with another line simple support applied along the y direction at $x = 0.16$ m. The second is the same plate with two line simple supports applied at $x = 0.16$ and 0.27 m. The position of the line constraint in both cases is arbitrarily chosen. The average mode count of the first example should be given by

$$N_{total}(k) = N(k) - \delta_{x\text{-}pinned}, \tag{76}$$

where $N(k)$ is given by Eq. (58) and $\delta_{x\text{-}pinned} = \frac{3}{4}kb/\pi$. The average mode count of the second example should be given by

$$N_{total}(k) = N(k) - 2\delta_{x\text{-}pinned}. \tag{77}$$

Since analytical solutions of the natural frequencies for such systems are not available, an FE modal analysis is used to obtain numerical results. Then the estimated results from Eqs. (76) and

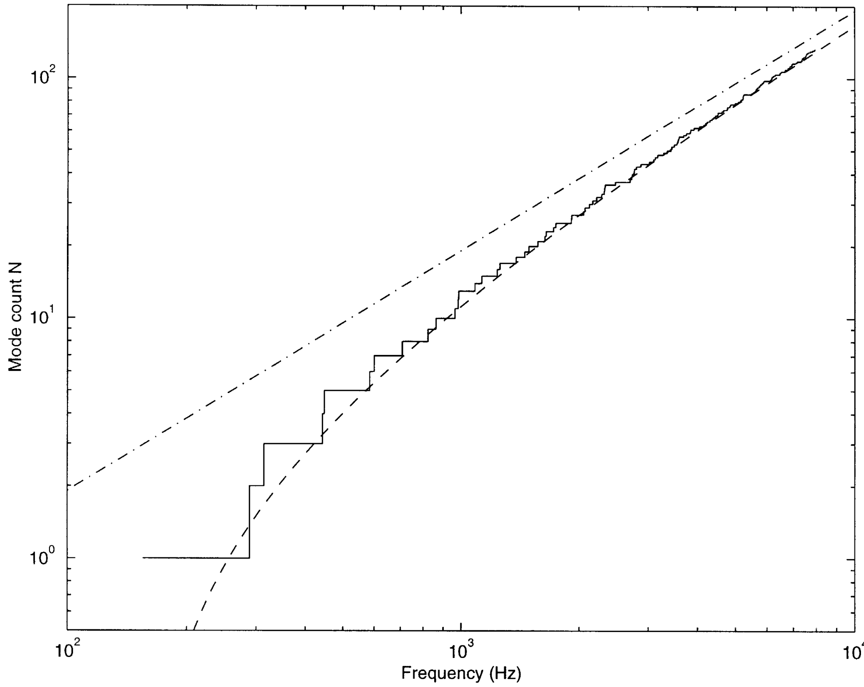

Fig. 16. The mode count for a simply supported plate with one intermediate simple support (—, stair-case from FE model; – – –, estimate from Eq. (76); – · –, result without intermediate support, $N(k)$).

(77) are compared with those from the FE analysis, as shown in Figs. 16 and 17. It can be seen that the estimated values have a good agreement with the results from the FE analysis.

8. Modal density

8.1. Modal density of one-dimensional systems

The asymptotic modal density of a one-dimensional system can be found from the derivative of the average mode count function. For a single one-dimensional system, the derivative of Eq. (4) gives

$$n(\omega) = \frac{\mathrm{d}N}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{L}{\pi c_g} - \frac{\mathrm{d}\delta_{BC}}{\mathrm{d}\omega}, \tag{78}$$

where c_g is the group velocity.

For basic boundary conditions, δ_{BC} is constant, as shown in Section 3.2 so that the modal density is independent of the boundary conditions. This also applies to multiple collinear beam systems. In general, therefore, the modal density of one-dimensional systems is proportional to the length of the systems and is independent of boundary conditions, provided that δ_{BC} is constant. However, it will be noted that mass or stiffness boundary conditions have a frequency-dependent δ_{BC} and therefore an influence on the modal density.

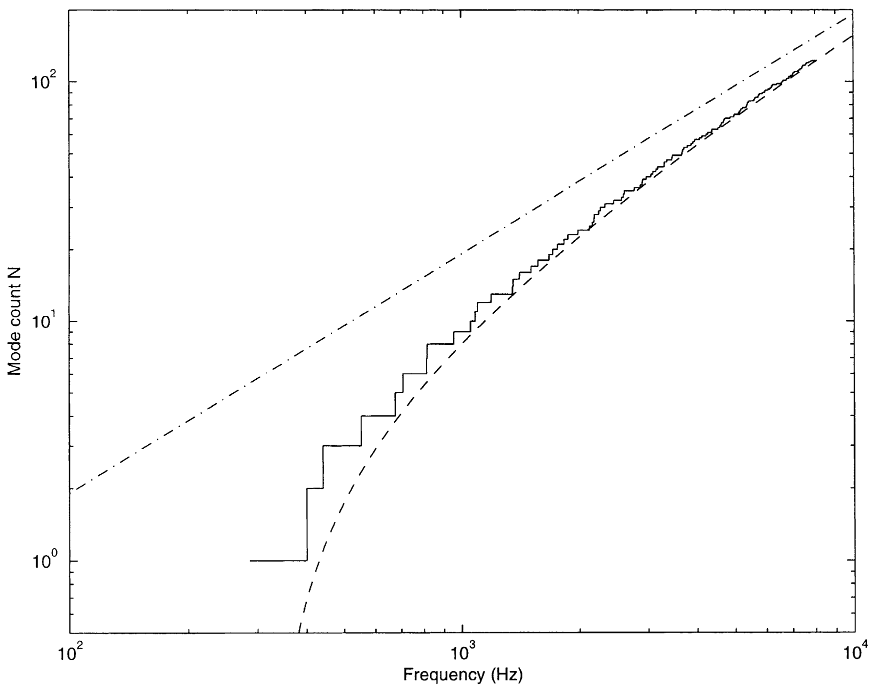

Fig. 17. The mode count for a simply supported plate with one intermediate simple support (—, stair-case from FEM; – – –, estimate from Eq. (77); – · – ·, result without intermediate support, $N(k)$).

8.2. Modal density of rectangular plates

For a two-dimensional system, the modal density can be similarly obtained through the derivative of the average mode count. Hence, in terms of frequency, the modal density of a rectangular plate is given by

$$n(\omega) = \frac{\partial N(\omega)}{\partial \omega} = \frac{S}{4\pi}\sqrt{\frac{m''}{B}} + \frac{1}{2}\left(\frac{m''}{B}\right)^{1/4} \times \left[\frac{(1-\delta_{x\text{-}left}-\delta_{x\text{-}right})b}{\pi} + \frac{(1-\delta_{y\text{-}top}-\delta_{y\text{-}bottom})a}{\pi}\right]\omega^{-1/2} \tag{79}$$

where $m'' = \rho h$. It can be seen from Eq. (79) that the modal density of the plate is frequency dependent. The first term is the constant term that is given in most literature

$$n(\omega) = \frac{S}{4\pi}\sqrt{\frac{m''}{B}}. \tag{80}$$

This depends only on the material, thickness and area of the plate under consideration. The second term in Eq. (79) is frequency dependent and contains information on the geometric characteristics of the plate. It becomes smaller and less important as frequency increases so that the modal density tends to a constant at high frequency.

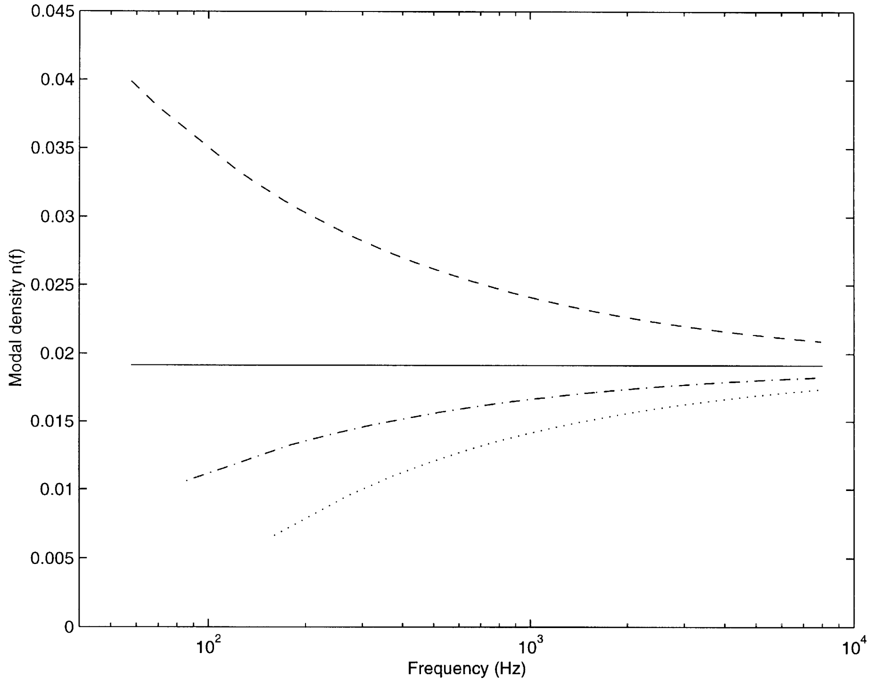

Fig. 18. Modal density of a plate 0.4 × 0.3 × 0.002 m^3 (solid: Eq. (80); – · –, simple support; – – –, free; ···, fixed).

According to Eq. (79), the modal density of a simply supported plate can be given by

$$n(\omega) = \frac{S}{4\pi}\sqrt{\frac{m''}{B}} - \frac{1}{4}\left(\frac{m''}{B}\right)^{1/4}\left(\frac{a+b}{\pi}\right)\omega^{-1/2} \tag{81}$$

the modal density of a free plate can be given by

$$n(\omega) = \frac{S}{4\pi}\sqrt{\frac{m''}{B}} + \frac{1}{2}\left(\frac{m''}{B}\right)^{1/4}\left(\frac{a+b}{\pi}\right)\omega^{-1/2} \tag{82}$$

and the modal density of a fully fixed plate can be given by

$$n(\omega) = \frac{S}{4\pi}\sqrt{\frac{m''}{B}} - \frac{1}{2}\left(\frac{m''}{B}\right)^{1/4}\left(\frac{a+b}{\pi}\right)\omega^{-1/2}. \tag{83}$$

From the above three equations it can be noted that the modal density of the plate can be either larger or smaller than the constant value from Eq. (80). Taking the plate investigated in Section 6.2 as an example, the results of the above equations for the modal densities of a simply supported, a fully fixed and a free plate are shown in Fig. 18 (N.B. results are plotted as $n(f) = 2\pi n(\omega)$). It can be seen that Eq. (80) can only represent the true modal density well at high frequencies. Although the first mode of this simply supported plate occurs below 100 Hz, even at 1000 Hz the error can be as much as 13%. For the cases of the free and fully fixed plate, the corresponding errors at 1000 Hz are 26%.

The modal density can also be obtained directly by counting the number of modes in a band from the analytical results. Fig. 19 shows the results counted in overlapping octave bands for a

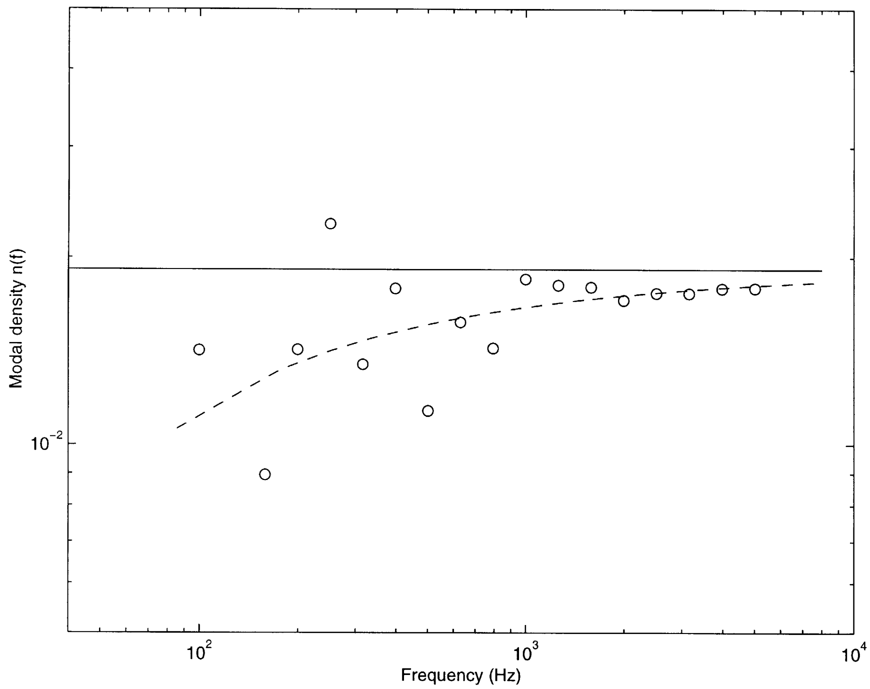

Fig. 19. Modal density of the simply supported plate $0.4 \times 0.3 \times 0.002\ \mathrm{m}^3$ (solid: Eq. (80) omitting boundary effects; – – – Eq. (81) including boundary effects; circle, counted from analytical results in overlapping octave bands).

simply supported plate. It can be seen that the result from octave bands agrees better with the curve from Eq. (81) and converges to the constant determined by Eq. (80) only at high frequencies. Fig. 20 presents equivalent results for the free plate. The same phenomenon as for the simply supported plate can be observed.

9. Conclusions

The mode count of one- and two-dimensional structural systems has been investigated. A simple relationship has been shown between the mode count and the boundary conditions for one-dimensional systems. For bending vibrations, a sliding constraint adds to the mode count by $-1/4$, a simple support condition by $-3/4$ and a fixed boundary constraint by -1 compared with a free boundary. For longitudinal vibrations, a fixed boundary constraint adds to the mode count by $-1/2$. For more general boundary conditions, in particular a point mass and a point spring, the boundary condition effect on the mode count is frequency dependent.

For multi-beam systems in a single line, the mode count of the system can be estimated by the mode count of a long beam without any extra constraints minus the sum of the constraint coefficients. An intermediate constraint has the same effect on the average mode count of a one-dimensional system as the same type of constraint applied at an end.

Line constraints have systematic effects on the mode count of a two-dimensional system. The effect depends on the type of the boundary condition, as well as the geometric and material

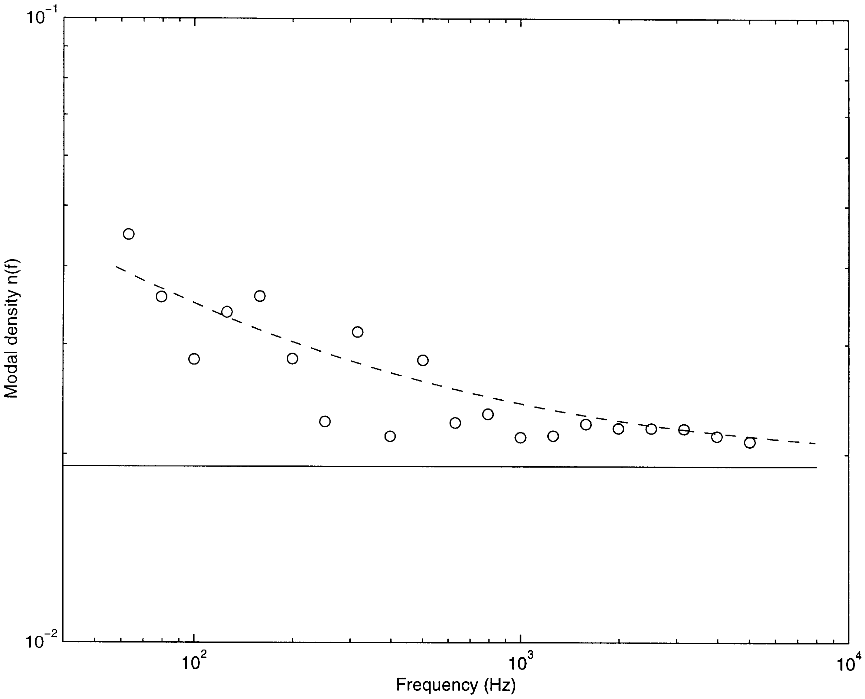

Fig. 20. Modal density of the free plate $0.4 \times 0.3 \times 0.002\ \mathrm{m}^3$ (solid: Eq. (80) omitting boundary effects; – – –, Eq. (82) including boundary effects; circle, counted from analytical results in overlapping octave bands).

properties of the system. The results follow on from those for the same type of boundary in a one-dimensional system. Approximate theoretical expressions used to estimate the mode count of a two-dimensional system have been obtained. The results from these estimated formulae have shown the limitation of the commonly used formula, in which the effects of the boundary conditions are neglected.

For a composite two-dimensional system, an intermediate line constraint has the same effect on the mode count as the equivalent constraint applied on an edge. The average mode count of such a composite system can be estimated by that of the system without intermediate constraints minus the product of the number of the constraints, the constraint effect δ_{BC} and the term kL/π, where k is the wavenumber and L is the length of the constrained edges.

Theoretical expressions have been obtained for the modal density of a two-dimensional system including boundary effects. The modal density of the rectangular plate is a frequency-dependent parameter, which depends on geometric information and the dispersion relation of the plate under consideration. However, at high enough frequency, the modal density tends to a constant value, which is determined only by the area of the plate and dispersion relation and is independent of the boundary conditions.

Although in practice it is not possible to analyse in detail all the various combinations of boundary conditions, the analyses for the most basic boundary conditions presented in this paper are expected to provide enough general insight to permit some sorts of complicated structure comprising many small beams or plates to be dealt with in applications of SEA.

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