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Oscillations of captured spherical drop of frictionless liquid

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Abstract

The natural frequencies and mode shapes have been investigated for a frictionless spherical liquid globule where part of the free surface is embedded in a spherical cap of the same radius. Five different liquid systems have been considered. With increasing cap angle α and higher modes n, the natural frequencies increase. The different mode shapes exhibit a drastic change of frequencies. In addition, the response of the captured spherical drop due to harmonically forced translational excitation of the rigid cap has been determined for the two major directions.

1. Introduction

A space station in orbit offers an environment providing a weightless condition for some unique experiments and manufacturing processes in material science and technology. The drastic reduction of gravity makes surface tension the major force influencing the motion of liquids. As the force of gravity approaches zero the equilibrium position of a liquid globule assumes to be a perfect sphere. Not only in mechanical sciences and metallurgy, the behavior of a captured or freely floating liquid system is of interest, it also finds application in chemical engineering, nuclear fission, as well as in geological and mechanical engineering. The dynamical behavior of such systems is of importance to solidification processes and especially to these of partially containerless processings of liquified materials. Partly captured spheres are those of which part of the free liquid surface is replaced by a solid, in our case here, by a solid spherical cap of the same radius as the liquid sphere. The liquid sphere is embedded in the cap and exhibits due to the reduced free surface area increased natural frequencies and drastically changed mode shapes as

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well as different responses to harmonically forced translational excitation of the solid cap in various directions.

A remarkable series of experiments on these and related problems have been performed by Plateau [1] in the years from 1843 to 1869. In 1879, Lord Rayleigh [2,3] investigated the vibrations of a liquid mass of spherical configuration and determined the natural frequencies of modes being symmetric to an axis of the sphere. Later Lamb [4] gave a slightly generalized result by supposing that a sphere of liquid with density ρ_2 is surrounded by an infinite liquid mass of density ρ_1. For a small viscosity of the liquid, Lamb [5] presented its influence upon the frequency of oscillation. Since 1973, a numerical treatment of the problem [6] has been performed. For a simple spherical drop, the natural frequencies have been presented in Ref. [7]. A large variety of spherical configurations has been treated for frictionless liquids by Bauer [8]. The natural frequencies and stability of some basic spherical liquid systems have also been presented for frictionless and viscous liquids in Ref. [9], in which also in addition to cases considered in Ref. [10], a few more cases of immiscible spherical liquid arrangements have been shown. But also for non-linear motion of capillary surface waves results have been presented in Refs. [11,12]. Some investigations have also been performed for freely floating liquid spheres consisting of visco-elastic material [13–15]. No results are known, if the spherical drop is captured, i.e., embedded partly in a rigid spherical cap.

The following paper investigates the oscillatory behavior of a liquid sphere consisting of frictionless liquid and being partly embedded in a rigid cap of the same radius as the liquid sphere. The change of the natural frequencies due to the presence and the magnitudes of a rigid spherical cap shall be investigated and the change of the mode shapes shall be presented. Five different liquid systems have been considered. For harmonically forced translational excitation in the two major directions, i.e., the z and x direction, the response of the captured spherical liquid drop is determined.

2. Basic equations

A spherical drop of volume $V_0 = (4\pi/3)a^3$ or an annular drop of volume $V_0 = (4\pi/3)a^3(1-k^3)$ ($k = b/a$) is in a zero-gravity environment and is partly captured by a spherical wall of the same radius a, covering a certain given ranges in the meridian co-ordinate ϑ (Fig. 1). If subjected to a

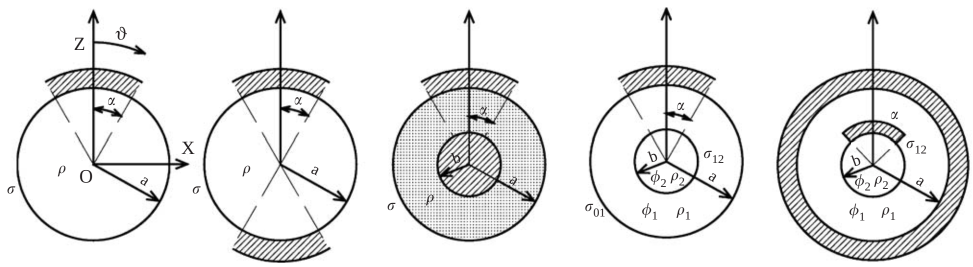


Fig. 1. Captured spherical drop system: (a) simple liquid sphere; (b) spherical drop captured at both poles; (c) annular spherical liquid system; (d) immiscible spherical liquid system; (e) immiscible liquids in a rigid sphere.

disturbance, the captured liquid globule will perform oscillations. The following shall investigate oscillations of such systems for a frictionless liquid.

If the liquid is incompressible and frictionless, exhibiting the density ρ performing small oscillations in irrotational motion, the velocity $\vec{v} = u\vec{e}_r + v\vec{e}_\theta + w\vec{e}_\varphi$ may be represented as a gradient of a velocity potential $\phi(r, \vartheta, \varphi, t)$, yielding the Laplace equation

$$\Delta\phi = \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2} = 0, \tag{1}$$

which has to be satisfied with the appropriate boundary conditions, i.e., a vanishing radial velocity $\partial\phi/\partial r = 0$ at the ϑ ranges covered by a spherical wall, and the free surface condition

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\sigma}{\rho a^2}\left\{2\frac{\partial\phi}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial^2\phi}{\partial r\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^3\phi}{\partial\varphi^2\partial r}\right\} = 0, \tag{2}$$

at the free surface ranges of the liquid globule.

3. Method of solution

In the following, we shall investigate first the natural frequencies of some captured frictionless liquid drops. The geometry and co-ordinate system used are presented in Fig. 1. The results are based on axisymmetric motion of the system ($\partial/\partial\varphi \equiv 0,\ v \equiv 0$), and shall yield the approximate lower natural frequencies for systems of various magnitudes α.

3.1. Free oscillations

The solution of the Laplace equation (1) is given by

$$\phi(r, \vartheta, t) = \sum_{m=0}^{m\leqslant n}\sum_{n=1}^{\infty} e^{i\omega_{mn}t}\left[A_{mn}\left(\frac{r}{a}\right)^n + B_{mn}\left(\frac{r}{a}\right)^{-(n+1)}\right]P_n^m(\cos\vartheta)\cos m\varphi, \tag{3}$$

in which $P_n^m(\cos\vartheta)$ represent the associate Legendre function, A_{mn} and B_{mn} are unknown integration constants, and ω_{mn} are the yet unknown natural frequencies of the captured spherical liquid system.

3.1.1. Simple liquid sphere

If the liquid sphere is imbedded in a spherical cap of radius a, covering the ranges $0 \leqslant \vartheta \leqslant \alpha$ (Fig. 1(a)), the above velocity potential exhibits $B_{mn} \equiv 0$ and has to satisfy at the rigid cap

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{4}$$

and at the free surface

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\sigma}{\rho a^2}\left\{2\frac{\partial\phi}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial^2\phi}{\partial r\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^3\phi}{\partial r\partial\varphi^2}\right\} = 0,$$
$$\text{at } r = a \text{ in the range } \alpha < \vartheta \leqslant \pi. \tag{5}$$

Both these conditions have to be satisfied at $r = a$, but also in this respective ϑ range. This yields the expression for a given angular mode m.

$$\sum_{n=1}^{\infty} nA_{mn}P_n^m(\cos\vartheta) = 0 \quad \text{in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{6}$$

and

$$\sum_{n=1}^{\infty} A_{mn}\left\{\frac{\sigma}{\rho a^3} n(n-1)(n+2) - \omega^2\right\} P_n^m(\cos\vartheta) = 0 \quad \text{in the range } \alpha < \vartheta \leqslant \pi. \tag{7}$$

Satisfying these two equations at a finite number of points on the surface $r = a$ in their particular ranges requires $(N_1 + 1)$ points in the range $0 \leqslant \vartheta \leqslant \alpha$ and N_2 points in the range $\alpha < \vartheta \leqslant \pi$. This yields the homogeneous algebraic equations

$$\sum_{n=1}^{N_1+N_2+1} nA_{mn}P_n^m\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right] = 0 \quad \text{for } n_1 = 0, 1, 2, \ldots, N_1, \tag{8}$$

and

$$\sum_{n=1}^{N_1+N_2+1} A_{mn}\left\{\frac{\sigma n}{\rho a^3}(n-1)(n+2) - \omega^2\right\} P_n^m\left[\cos\left(\alpha + \frac{(\pi-\alpha)n_2}{N_2}\right)\right] = 0$$

$$\text{for } n_2 = 1, 2, \ldots, N_2. \tag{9}$$

These are $(N_1 + N_2 + 1)$ equations in $A_{m1}, A_{m2}, \ldots, A_{m(N_1+N_2+1)}$, of which the vanishing coefficient determinant represents the natural frequency equation for the determination of the lower natural frequencies ω_{mn}.

If the free liquid surface is not obstructed by a spherical cap, i.e., if the liquid globule is free of any solids covering it, we shall have the free oscillation frequency of a liquid sphere [7,8].

$$\omega_{mn}^2 = \frac{\sigma}{\rho a^3} n(n-1)(n+2). \tag{10}$$

3.1.2. Spherical drop captured at both poles

For a liquid drop, which is as in free fall experiments often used, captured at both poles the boundary conditions are (Fig. 1(b)) at the rigid caps

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at } r = a \text{ in the ranges } 0 \leqslant \vartheta \leqslant \alpha,\ \pi - \alpha \leqslant \vartheta \leqslant \pi, \tag{11}$$

while the free surface condition (2) has to be satisfied at $r = a$ and the range $\alpha < \vartheta < \pi - \alpha$. This yields in a similar way as above for a given m mode

$$\sum_{n=1}^{N_1+N_2+N_3+1} nA_{mn}P_n^m\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right] = 0 \quad \text{for } n_1 = 0, 1, 2, \ldots, N_1, \tag{12}$$

$$\sum_{n=1}^{N_1+N_2+N_3+1} nA_{mn}P_n^m\left[\cos\left(\pi - \frac{n_2}{N_2}\alpha\right)\right] = 0 \quad \text{for } n_2 = 0, 1, 2, \ldots, N_2, \tag{13}$$

and at the free liquid surface

$$\sum_{n=1}^{N_1+N_2+N_3+1} A_{mn}\left\{\frac{\sigma}{\rho a^3}n(n-1)(n+2)-\omega^2\right\}P_n^m\left[\cos\left(\alpha+\frac{(\pi-2\alpha)n_3}{N_3}\right)\right]=0,$$

$$\text{for } n_3=1,2,\ldots,N_3-1. \tag{14}$$

These are $(N_1+N_2+N_3+1)$ homogeneous algebraic equations of which the vanishing coefficient determinant represents the approximate natural frequency equation.

3.1.3. Annular spherical liquid system

If an annular liquid of volume $V_0=(4\pi/3)a^3(1-k^3)$ $(k=b/a)$ is placed around a spherical center core of radius b and in addition captured at $r=a$ in the range $0\leqslant\vartheta\leqslant\pi$, the boundary conditions are (Fig. 1(c)) at the rigid cap

$$\frac{\partial\phi}{\partial r}=0 \quad \text{at } r=a \text{ in the range } 0\leqslant\vartheta\leqslant\alpha, \tag{15}$$

at the rigid center core

$$\frac{\partial\phi}{\partial r}=0 \quad \text{at } r=b \quad (\text{for all } \vartheta), \tag{16}$$

while the free surface condition is presented by Eq. (5). With the equation of the velocity potential (3), we obtained from the boundary condition (16),

$$B_{mn}=\frac{n}{(n+1)}k^{2n+1}A_{mn}. \tag{17}$$

From the boundary condition (15), we find finally

$$\sum_{n=1}^{N_1+N_2+1} nA_{mn}(1-k^{2n+1})P_n^m\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right]=0 \quad \text{for } n_1=0,1,2,\ldots,N_1, \tag{18}$$

and the free surface condition (5) yields

$$\sum_{n=1}^{N_1+N_2+1} A_{mn}\left\{\frac{\sigma n}{\rho a^3}(n^2-1)(n+2)(1-k^{2n+1})-\omega^2(n+1+nk^{2n+1})\right\}$$
$$\times P_n^m\left[\cos\left(\alpha+\frac{(\pi-\alpha)n_2}{N_2}\right)\right]=0 \quad \text{for } n_2=1,2,\ldots,N_2. \tag{19}$$

These are (N_1+N_2+1) homogeneous algebraic equations of which the vanishing coefficient determinant represents the natural frequency equation for the determination of the lower approximate natural frequencies of the captured annular liquid system. If the range $\alpha\to 0$, then we obtain the well-known natural frequencies [7]

$$\omega_{mn}^2=\frac{\sigma}{\rho a^3}\frac{n(n^2-1)(n+2)(1-k^{2n+1})}{(n+1+nk^{2n+1})}. \tag{20}$$

3.1.4. Immiscible spherical liquid system

For an immiscible spherical liquid system consisting of a center sphere of density ρ_2 and radius b and an annular sphere of density ρ_1 and radius a with ϑ range $0 \leqslant \vartheta \leqslant \alpha$ in which the outer free surface is captured, the boundary conditions are given (Fig. 1(d)) at the inner free surface by

$$\frac{\partial \phi_1}{\partial r} = \frac{\partial \phi_2}{\partial r} \quad \text{at } r = b \quad (\text{for all } \vartheta \text{ value}), \tag{21}$$

at the rigid cap by

$$\frac{\partial \phi_1}{\partial r} = 0 \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{22}$$

and at the free surface and interface, i.e.,

$$\rho_1 \frac{\partial^2 \phi_1}{\partial t^2} - \frac{\sigma_{01}}{a^2}\left\{2\frac{\partial \phi_1}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial^2 \phi_1}{\partial r \partial \vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^3 \phi_1}{\partial r \partial \varphi^2}\right\} = 0,$$

$$\text{at } r = a \text{ in the range } \alpha < \vartheta \leqslant \pi \tag{23}$$

and

$$\rho_1 \frac{\partial^2 \phi_1}{\partial t^2} - \rho_2 \frac{\partial^2 \phi_2}{\partial t^2} + \frac{\sigma_{12}}{b^2}\left\{2\frac{\partial \phi_2}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial^2 \phi_2}{\partial r \partial \vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^3 \phi_2}{\partial r \partial \varphi^2}\right\} = 0,$$

$$\text{at } r = b \quad (\text{for all } \vartheta \text{ value}). \tag{24}$$

The solution of the Laplace equation in the first region (annular sphere) is given by

$$\phi_1(r, \vartheta, \varphi, t) = \sum_{m=0}^{m \leqslant n} \sum_{n=1}^{\infty} \left[A_{mn}\left(\frac{r}{a}\right)^n + B_{mn}\left(\frac{r}{a}\right)^{-(n+1)}\right] P_n^m(\cos\vartheta)\cos m\varphi \mathrm{e}^{\mathrm{i}\omega t}, \tag{25}$$

and in the spherical region $0 \leqslant r \leqslant b$ by

$$\phi_2(r, \vartheta, \varphi, t) = \sum_{m=0}^{m \leqslant n} \sum_{n=1}^{\infty} C_{mn}\left(\frac{r}{a}\right)^n P_n^m(\cos\vartheta)\cos m\varphi \mathrm{e}^{\mathrm{i}\omega t}. \tag{26}$$

With the boundary condition (21), we obtain

$$C_{mn} = A_{mn} - \frac{(n+1)}{n} k^{-(2n+1)} B_{mn}, \tag{27}$$

and from Eq. (24) we obtain

$$B_{mn} = -\frac{k^{2n+1}\left[\omega^2(\rho_1 - \rho_2) + \frac{\sigma_{12}}{b^3} n(n-1)(n+2)\right]}{\omega^2\left[\rho_1 + \frac{(n+1)}{n}\rho_2\right] - \frac{\sigma_{12}}{b^3}(n^2-1)(n+2)} A_{mn} \equiv -\chi_n A_{mn}, \tag{28}$$

which may be introduced into the above velocity potentials. Eq. (22) yields then in the range $0 \leqslant \vartheta \leqslant \alpha$ the expression (for fixed given angular mode m):

$$\sum_{n=1}^{N_1+N_2+1} A_{mn}[n + (n+1)\chi_n] P_n^m\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right] = 0 \quad \text{for } n_1 = 0, 1, 2, \ldots, N_1, \tag{29}$$

and Eq. (26)

$$\sum_{n=1}^{N_1+N_2+1} A_{mn}\left\{\frac{\sigma_{01}}{\rho_1 a^3}(n-1)(n+2)[n+(n+1)\chi_n]-\omega^2(1-\chi_n)\right\} \times P_n^m\left[\cos\left(\alpha+\frac{(\pi-\alpha)n_2}{N_2}\right)\right]=0, \tag{30}$$

for $n_2 = 1, 2, \ldots, N_2$. These are $(N_1 + N_2 + 1)$ homogeneous algebraic equations in A_{mn} of which the vanishing coefficient determinant represents the natural frequency equation. If the inner liquid is represented as a gas bubble, the density ρ_2 vanishes.

3.1.5. Immiscible liquids in a rigid sphere

If there are two immiscible liquids of density ρ_1 and ρ_2 in a rigid sphere of radius a, and the interface at $r = b$ is captured by a spherical cap of radius b (Fig. 1(e)), the velocity potentials are given by Eqs. (25) and (26). The boundary conditions are given at the outer rigid sphere by

$$\frac{\partial\phi_1}{\partial r}=0 \quad \text{at } r=a \quad (\text{for all } \vartheta \text{ value}), \tag{31}$$

at the inner rigid cap by

$$\frac{\partial\phi_1}{\partial r}=\frac{\partial\phi_2}{\partial r}=0 \quad \text{at } r=b \text{ in the range } 0\leqslant\vartheta\leqslant\alpha, \tag{32}$$

and the interface surface conditions by

$$\frac{\partial\phi_1}{\partial r}=\frac{\partial\phi_2}{\partial r} \quad \text{at } r=b \text{ in the range } \alpha<\vartheta\leqslant\pi, \tag{33}$$

$$\rho_1\frac{\partial^2\phi_1}{\partial t^2}-\rho_2\frac{\partial^2\phi_2}{\partial t^2}+\frac{\sigma_{12}}{b^2}\left\{2\frac{\partial\phi_2}{\partial r}+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial^2\phi_2}{\partial r\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^3\phi_2}{\partial r\partial\varphi^2}\right\}=0,$$

$$\text{at } r=b \text{ in the range } \alpha<\vartheta\leqslant\pi. \tag{34}$$

Eq. (31) yields with the velocity potential (3)

$$B_{mn}=\frac{n}{(n+1)}A_{mn}, \tag{35}$$

while remaining boundary conditions are given by

$$\sum_{n=1}^{N_1+N_2+1}\{A_{mn}n[k^{n-1}-k^{-(n+2)}]-C_{mn}nk^{n-1}\}P_n^m\left[\cos\left(\alpha+\frac{(\pi-\alpha)n_2}{N_2}\right)\right]=0,$$

$$\text{for } n_2=0,1,2,\ldots,N_2, \tag{36}$$

$$\sum_{n=1}^{N_1+N_2+1} C_{mn}nk^{n-1}P_n^m\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right]=0 \quad \text{for } n_1=0,1,2,\ldots,N_1, \tag{37}$$

and

$$\sum_{n=1}^{N_1+N_2+1} A_{mn} n[k^{n-1} - k^{-(n+2)}] P_n^m \left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right] = 0 \quad \text{for } n_1 = 0, 1, 2, \ldots, N_1, \tag{38}$$

$$\sum_{n=1}^{N_1+N_2+1} \left\{ C_{mn} k^n \rho_2 \omega^2 - A_{mn} \left[k^n + \frac{n}{(n+1)} k^{-(n+1)}\right] \rho_1 \omega^2 - \frac{\sigma_{12} n}{b^3}(n-1)(n+2) k^n C_{mn} \right\}$$
$$\times P_n^m \left[\cos\left(\alpha + \frac{(\pi - \alpha) n_2}{N_2}\right)\right] = 0 \quad \text{for } n_2 = 0, 1, 2, \ldots, N_2. \tag{39}$$

These are $2(N_1 + N_2 + 1)$ homogeneous algebraic equations of which the vanishing coefficient determinant represents the natural frequency equation. For a gas bubble in the center the density ρ_2 must vanish.

3.2. Forced oscillations

If the spherical wall part is harmonically excited, the captured drop (Fig. 1(a)) performs forced oscillations. The response of such an excited oscillation may be determined as the magnification function presenting the magnitude of the free surface amplitude as a function of the forcing frequency Ω. We shall investigate here the two basic harmonically excited translations of the system.

3.2.1. Translational excitation in x direction

If the spherical wall is harmonically excited in x direction by $X_0 e^{i\Omega t}$ where X_0 is the excitation amplitude and Ω the forcing frequency, the Laplace equation

$$\Delta\phi = \frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2} = 0, \tag{40}$$

has to be solved with the boundary conditions

$$\frac{\partial\phi}{\partial r} = X_0 i\Omega e^{i\Omega t} \sin\vartheta \cos\varphi \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{41}$$

and with the free surface condition

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\sigma}{\rho a^2}\left\{ 2\frac{\partial\phi}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial^2\phi}{\partial\vartheta\partial r}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^3\phi}{\partial r\partial\varphi^2} \right\} = 0$$
$$\text{at } r = a \text{ in the range } \alpha < \vartheta \leqslant \pi. \tag{42}$$

With the transformation

$$\phi(r, \vartheta, \varphi, t) = X_0 e^{i\Omega t} i\Omega \cos\varphi [\psi(r, \vartheta) + r\sin\vartheta] \tag{43}$$

we obtain

$$\Delta\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial r} = 0 \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha \tag{44}$$

and

$$\frac{\sigma}{\rho a^2}\left\{2\frac{\partial\psi}{\partial r}+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial^2\psi}{\partial r\partial\vartheta}\right)-\frac{1}{\sin^2\vartheta}\frac{\partial\psi}{\partial r}\right\}+\Omega^2\psi=-r\Omega^2\sin\vartheta$$

$$\text{at } r=a \text{ in the range } \alpha<\vartheta\leqslant\pi. \tag{45}$$

The solution of the Laplace equation $\Delta\psi=0$ is obtained to be

$$\psi(r,\vartheta)=\sum_{n=1}^{\infty}A_n\left(\frac{r}{a}\right)^n P_n^1(\cos\vartheta), \tag{46}$$

where $P_n^1(\cos\vartheta)$ represents the associated Legendre function and A_n are yet unknown values. The two boundary conditions in ψ have to be satisfied and yield at $r=a$

$$\sum_{n=1}^{\infty} nA_nP_n^1(\cos\vartheta)=0 \quad \text{in the range } 0\leqslant\vartheta\leqslant\alpha, \tag{47}$$

and

$$\sum_{n=1}^{\infty} A_n\left\{\frac{\sigma n}{\rho a^3}(n-1)(n+2)-\Omega^2\right\}P_n^1(\cos\vartheta)=a\Omega^2\sin\vartheta \quad \text{in the range } \alpha<\vartheta\leqslant\pi. \tag{48}$$

Satisfying these two equations at a finite number of points on surface $r=a$ in their particular given range requires (N_1+1) points in the range $0\leqslant\vartheta\leqslant\alpha$ and N_2 points in the range $\alpha\leq\vartheta\leqslant\pi$. This yields the inhomogeneous algebraic system for the determination $A_n(\Omega)$ for $n=1,2,\dots,(N_1+N_2+1)$. It is with $\vartheta=(n_1/N_2)\alpha$ for the first range, and $\vartheta=\alpha+(\pi-\alpha)n_2/N_2$ for the second range, i.e.,

$$\sum_{n=1}^{N_1+N_2+1} nA_nP_n^1\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right]=0 \quad \text{for } n_1=0,1,2,\dots,N_1, \tag{49}$$

and

$$\begin{aligned}\sum_{n=1}^{N_1+N_2+1} A_n\left\{\frac{\sigma n}{\rho a^3}(n-1)(n+2)-\Omega^2\right\}P_n^1\left[\cos\left(\alpha+\frac{(\pi-\alpha)n_2}{N_2}\right)\right]\\ = a\Omega^2\sin\vartheta\left[\alpha+\frac{(\pi-\alpha)n_2}{N_2}\right] \quad \text{for } n_2=1,2,\dots,N_2.\end{aligned} \tag{50}$$

These are (N_1+N_2+1) equations in $A_1,A_2,\dots,A_{N_1+N_2+1}$ for the determination of the response values $A_j(\Omega)$. The velocity potential is therefore presented by

$$\phi(r,\vartheta,\varphi,t)=X_0\mathrm{i}\Omega e^{\mathrm{i}\Omega t}\left\{r\sin\vartheta+\sum_{n=1}^{N_1+N_2+1}A_n\left(\frac{r}{a}\right)^n P_n^1(\cos\vartheta)\right\}, \tag{51}$$

of which the free surface response $\zeta(\Omega)$ may be determined from $\zeta(\Omega)=(1/\mathrm{i}\Omega)(\partial\phi/\partial r)|_{r=a}$.

3.2.2. Translational excitation in z direction

If the spherical wall is harmonically excited with $Z_0 e^{i\Omega t}$ in z direction, the Laplace equation $(\partial/\partial\varphi = 0)$ has to be solved with the boundary condition

$$\frac{\partial\phi}{\partial r} = Z_0 i\Omega e^{i\Omega t}\cos\vartheta \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{52}$$

and the above free surface condition (observing $\partial/\partial\varphi = 0$). With the transformation

$$\phi(r,\vartheta,t) = Z_0 i\Omega e^{i\Omega t}[\psi(r,\vartheta) + r\cos\vartheta], \tag{53}$$

we obtain

$$\Delta\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial r} = 0 \quad \text{at } r = a \text{ in the range } 0 \leqslant \vartheta \leqslant \alpha, \tag{54}$$

and

$$\frac{\sigma}{\rho a^2}\left\{2\frac{\partial\psi}{\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial^2\psi}{\partial r\partial\vartheta}\right)\right\} + \Omega^2\psi = -\Omega^2 a\cos\vartheta \quad \text{in the range } \alpha < \vartheta \leqslant \pi. \tag{55}$$

The solution of the Laplace equation $\Delta\psi = 0$ yields the expression

$$\psi(r,\vartheta) = \sum_{n=1}^{\infty} B_n\left(\frac{r}{a}\right)^n P_n^0(\cos\vartheta), \tag{56}$$

where $P_n^0(\cos\vartheta)$ are the Legendre polynomials. Satisfying the two boundary conditions in the respective range yields

$$\sum_{n=1}^{N_1+N_2+1} nB_n P_n^0\left[\cos\left(\frac{n_1}{N_1}\alpha\right)\right] = 0 \quad \text{for } n_1 = 0,1,2,\ldots,N_1, \tag{57}$$

and

$$\sum_{n=1}^{N_1+N_2+1} B_n\left\{\frac{\sigma n}{\rho a^3}(n-1)(n+2) - \Omega^2\right\} P_n^0\left[\cos\left(\alpha + \frac{(\pi-\alpha)n_2}{N_2}\right)\right]$$
$$= a\Omega^2\cos\vartheta\left\{\alpha + \frac{(\pi-\alpha)n_2}{N_2}\right\} \quad \text{for } n_2 = 1,2,\ldots,N_2. \tag{58}$$

The solution of the algebraic system renders $B_j(\Omega)$, $j = 1,2,\ldots,(N_1+N_2+1)$. Therefore, the velocity potential is given by

$$\phi(r,\vartheta,t) = i\Omega Z_0 e^{i\Omega t}\left[r\cos\vartheta + \sum_{n=1}^{N_1+N_2+1} B_n\left(\frac{r}{a}\right)^n P_n^0(\cos\vartheta)\right], \tag{59}$$

from which the response of the free surface displacement $\zeta(\Omega) = (1/i\Omega)(\partial\phi/\partial r)|_{r=a}$ may be obtained.

4. Numerical evaluations

Some of the above obtained analytical results have been determined numerically. The results of a simple liquid sphere captured in a spherical cap of the range $0<\alpha<\pi$ have been evaluated and the axisymmetric ($m=0$) natural frequencies have been determined for various cap magnitudes $\bar{\alpha}=\alpha/\pi=0, 0.1, \ldots$ (0.1), … 0.9 for the lower modes $n=1,2,3,4$. The natural frequency ratios (see also Table 1) $\Omega_{0n}=\omega_{0n}/\sqrt{\sigma/\rho a^3}$ are presented in Fig. 2 for increasing constraint angle $\bar{\alpha}$. It may be noticed that the natural frequency of the liquid sphere increases with increasing cap magnitude $\bar{\alpha}$ and mode number n. Fig. 3 shows the vibration modes for $\bar{\alpha}=0$ (first line), $\bar{\alpha}=0.1, 0.2, 0.3$ to 0.9. For a freely floating sphere, the natural frequencies are given by Eq. (10), i.e.,

Table 1
Non-dimensional circular frequency Ω_{0n} of a simple liquid drop system

$\bar{\alpha}$	$n=1$	$n=2$	$n=3$	$n=4$
0	0.000	2.828	5.477	8.485
0.1	0.404	3.081	5.992	9.349
0.2	0.768	3.647	7.073	11.046
0.3	1.192	4.524	8.703	13.544
0.4	1.754	5.834	11.117	17.207
0.5	2.585	7.867	14.857	22.847
0.6	3.966	11.276	21.136	32.276
0.7	6.635	17.789	33.171	50.254
0.8	13.204	33.467	62.348	93.504
0.9	40.826	97.139	183.062	269.858

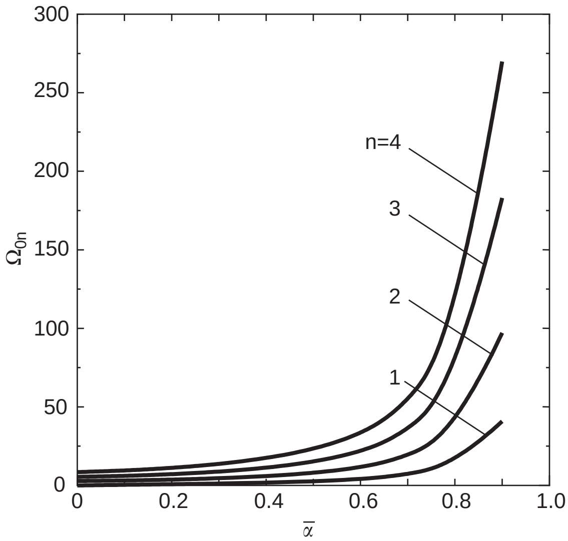


Fig. 2. Natural frequencies $\Omega_{0n}=\omega_{0n}/\sqrt{\sigma/\rho a^3}$ as a function of the solid cap angle $\bar{\alpha}$.

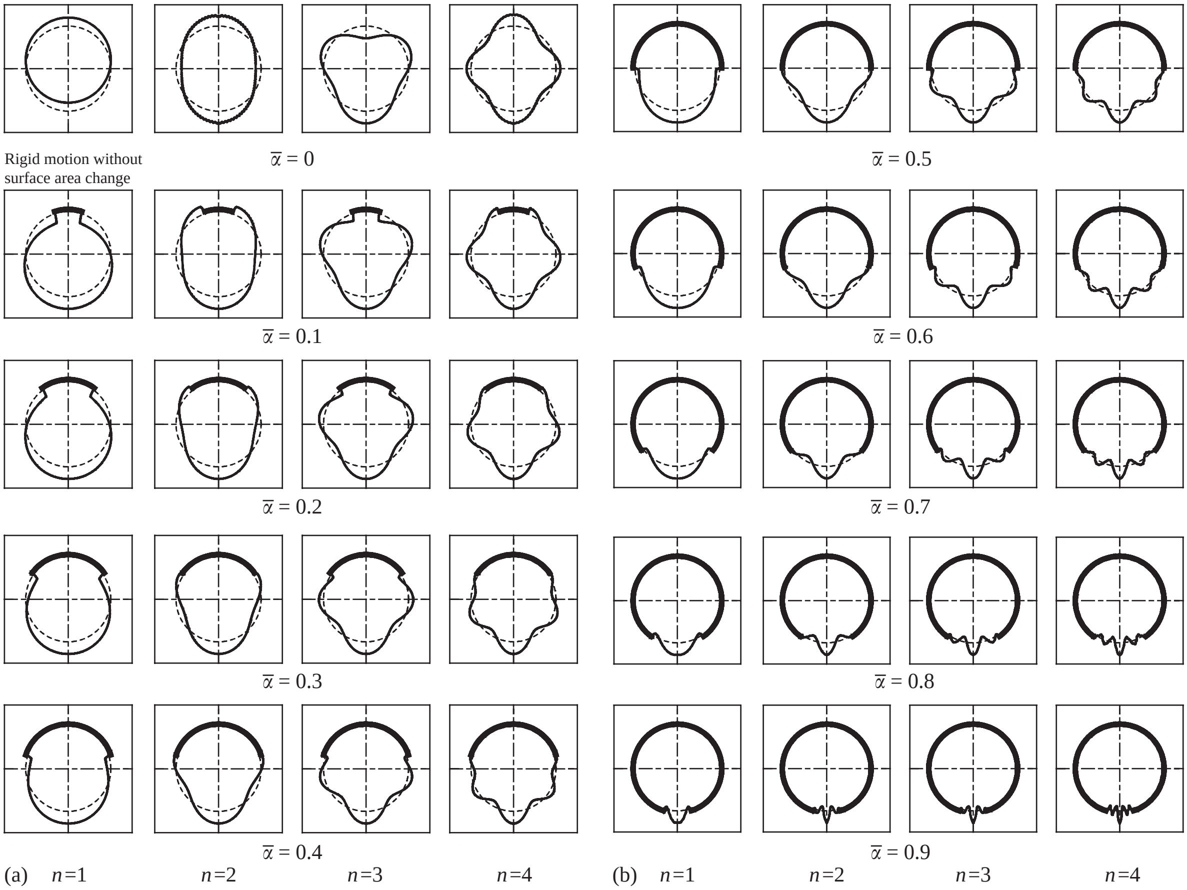


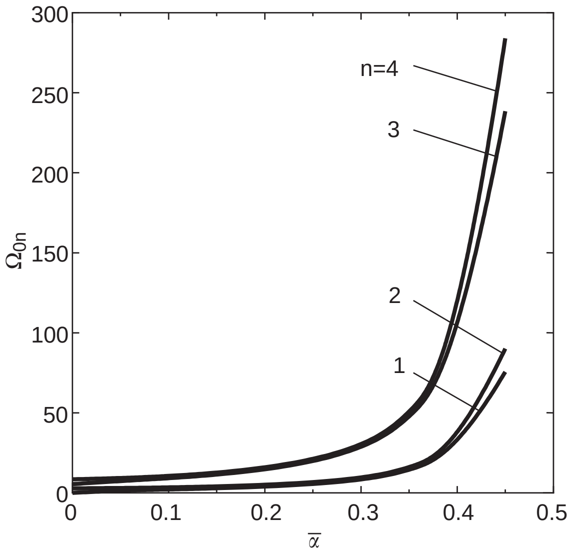
Fig. 3. Vibration modes for simple captured liquid system for various $\bar{\alpha}$ and mode numbers n.

$\omega_{0n}^2 = \sigma n(n-1)(n+2)/\rho a^3$. For $\bar{\alpha} = 0$, we notice the free floating liquid sphere, where the first mode shape represents the case of translational undisturbed (rigid) surface motion of the liquid sphere. The dashed lines show the undisturbed spherical form of the liquid. Corresponding modes for $\bar{\alpha} \neq 0$ are presented below each other and exhibit the change of the mode shapes according to the magnitude of the rigid spherical cap. Since the continuity equation is satisfied, the displacement of the liquid at the pole ($\vartheta = \pi$) is considerably larger than that outside of it. Table 1 represents the non-dimensional circular frequency ratio Ω_{0n} for $\bar{\alpha} = 0, \ldots$ (0.1), $\ldots$ 0.9 for the modes $n = 1, 2, 3, 4$.

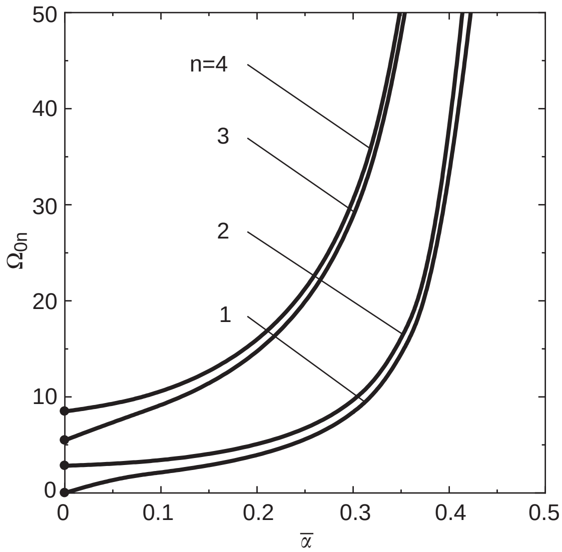
For a spherical drop captured by two rigid pole areas of equal magnitude $\bar{\alpha}$ (Fig. 1(b)), the natural frequency ratio Ω_{0n} for $\bar{\alpha} = 0, 0.05, \ldots$ (0.05), $\ldots$ 0.45 is presented in Table 2 for the four lower vibration modes $n = 1, 2, 3, 4$. The natural frequencies increase with increasing $\bar{\alpha}$ and increasing mode number n. At $\bar{\alpha} = 0$, we observe the results of a freely floating (not captured) liquid sphere, for which $\Omega_{01} = 0$, $\Omega_{02} = 2.83$, $\Omega_{03} = 5.48$ and $\Omega_{04} = 8.49$. For $\bar{\alpha} = 0.5$, the liquid sphere would be captured inside a closed rigid sphere of radius a (Fig. 4). Table 2 also indicates the number ratio of points at which conditions (12)–(14) are satisfied. Here $N_1 = N_2$ are the equal

Table 2
Non-dimensional circular frequency Ω_{0n} of a spherical system captured at both poles: $m = 0$, $N_1 = N_2$

$\bar{\alpha}$	$N_1 : N_3$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0		0.00	2.83	5.48	8.49
0.05	4:90	1.54	3.01	7.40	9.16
0.1	9:80	2.11	3.38	9.04	10.43
0.15	14:70	2.84	4.01	11.26	12.52
0.2	19:60	3.86	4.99	14.46	15.68
0.25	24:50	5.41	6.56	19.37	20.68
0.3	29:40	8.01	9.23	27.57	29.15
0.35	34:30	13.02	14.51	43.26	45.80
0.4	39:20	25.20	27.72	81.24	87.27
0.45	44:10	75.62	90.09	238.50	284.00



(a) Over all



(b) Magnified

Fig. 4. Natural frequencies Ω_{0n} for a liquid system captured at both poles.

amounts of points for the rigid caps, while N_3 represents the amount of points on the free surface. The mode shapes are shown in Fig. 5 for $\bar{\alpha} = 0, 0.05, 0.1$ and 0.2 for the lower vibration modes $n = 1, 2, 3, 4$.

For the unrestraint ($\bar{\alpha} = 0$) annular spherical liquid system, consisting of an annular spherical liquid body around a solid concentric inner sphere (Fig. 1(c) with $\bar{\alpha} = 0$), the natural frequencies are given by Eq. (20), i.e.,

$$\omega_{0n}^2 = \frac{\sigma}{\rho a^3} \frac{n(n^2 - 1)(n + 2)(1 - k^{2n+1})}{(n + 1 + nk^{2n+1})}.$$

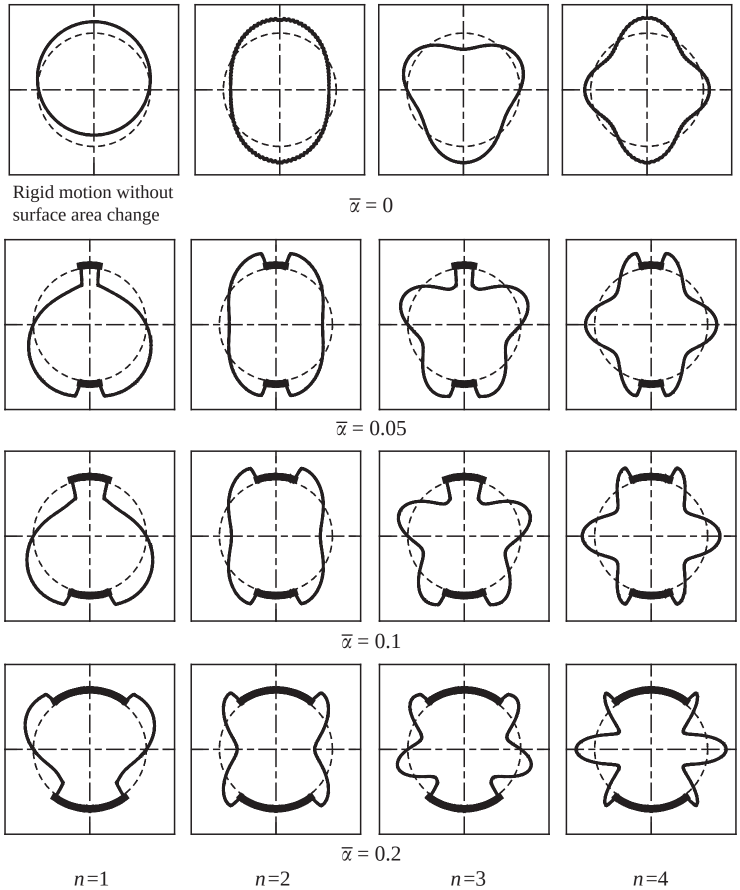


Fig. 5. Vibration modes for a liquid system captured at both poles.

For this annular spherical liquid system placed around a rigid sphere of radius b and captured by an additional spherical cap of magnitude $\bar{\alpha}$ at $r = a$, the natural frequencies are presented in Table 3 for different diameter ratios $k = 0.1, \ldots (0.1), \ldots 0.9$ and various $\bar{\alpha}$ values. The shaded horizontal areas represent the natural frequencies of the freely floating annular liquid system (Eq. (20)), while the vertically shaded results are those of the full spherical drop ($k = 0$) with a cap $\bar{\alpha}$ (see Table 1). In addition, Fig. 6 exhibits the natural frequencies Ω_{0n} for $k = 0.1, 0.8$ and 0.9 as a function of the cap angle $\bar{\alpha}$. Again we notice a strong increase of frequencies with the increase of $\bar{\alpha}$ and mode number n. Fig. 7 represents Ω_{0n} as a function of the diameter ratio $k = b/a$ for various $\bar{\alpha} = 0, 0.1, 0.2$ and 0.3, where we notice that the natural frequencies decrease with increasing k magnitude. The mode shapes are presented in Fig. 8 for different values of $\bar{\alpha}$ and k. The remaining geometrical configurations (Fig. 1(d) and (e)) have not been evaluated numerically. They would present no difficulty and are for reasons of limited space omitted.

If the spherical cap in which the drop is captured is moved harmonically in forced translational excitation in z direction, the free liquid surface displacement will exhibit a response presented in Fig. 9. It is shown at various locations $\vartheta = \pi/2$ (——), $3\pi/4\pi$ (- - ·) and π (– · –). $\bar{\Omega}$ is the

Table 3
Non-dimensional circular frequency Ω_{0n} for a captured annular spherical liquid system: $m = 0$

$\bar{\alpha}$	$N_1 : N_2$	k = 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
(a) $n = 1$											
0		0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.10	9:90	0.40	0.40	0.40	0.40	0.38	0.37	0.34	0.30	0.25	0.18
0.20	19:80	0.77	0.77	0.76	0.75	0.73	0.70	0.65	0.58	0.48	0.34
0.30	29:70	1.19	1.19	1.19	1.17	1.14	1.09	1.02	0.91	0.76	0.54
0.40	39:60	1.75	1.75	1.75	1.73	1.73	1.62	1.52	1.36	1.14	0.82
0.50	49:50	2.58	2.58	2.58	2.55	2.51	2.42	2.28	2.07	1.74	1.26
0.60	59:40	3.97	3.97	3.96	3.93	3.88	3.78	3.60	3.30	2.82	2.05
0.70	69:30	6.64	6.63	6.63	6.61	6.56	6.45	6.24	5.83	5.08	3.75
0.80	79:20	13.20	13.20	13.20	13.18	13.14	13.06	12.85	12.35	11.21	8.61

$\bar{\alpha}$	k = 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
(b) $n = 2$										
0	2.83	2.83	2.83	2.82	2.80	2.76	2.65	2.45	2.10	1.53
0.10	3.08	3.08	3.08	3.07	3.05	3.00	2.89	2.67	2.29	1.68
0.20	3.65	3.65	3.65	3.64	3.62	3.56	3.43	3.19	2.75	2.01
0.30	4.52	4.52	4.52	4.51	4.49	4.43	4.29	4.01	3.49	2.57
0.40	5.83	5.83	5.83	5.82	5.82	5.74	5.60	5.28	4.65	3.47
0.50	7.87	7.87	7.86	7.85	7.83	7.77	7.63	7.30	6.54	4.96
0.60	11.28	11.28	11.27	11.26	11.23	11.17	11.05	10.72	9.85	7.66
0.70	17.79	17.79	17.78	17.77	17.74	17.68	17.56	17.26	16.35	13.31
0.80	33.47	33.47	33.46	33.45	33.42	33.36	33.23	32.96	32.14	28.20
(c) $n = 3$										
0	5.48	5.48	5.48	5.48	5.47	5.44	5.34	5.09	4.53	3.39
0.10	5.99	5.99	5.99	5.99	5.98	5.95	5.85	5.58	4.97	3.74
0.20	7.07	7.07	7.07	7.07	7.06	7.03	6.92	6.64	5.96	4.51
0.30	8.70	8.70	8.70	8.70	8.69	8.65	8.55	8.27	7.52	5.77
0.40	11.12	11.12	11.11	11.11	11.11	11.06	10.97	10.70	9.89	7.74
0.50	14.86	14.86	14.85	14.85	14.83	14.79	14.71	14.47	13.65	10.99
0.60	21.14	21.14	21.13	21.12	21.10	21.06	20.99	20.79	20.02	16.80
0.70	33.17	33.17	33.17	33.16	33.13	33.09	33.00	32.83	32.22	28.58
0.80	62.35	62.35	62.34	62.33	62.31	62.26	62.16	61.96	61.48	58.18
(d) $n = 4$										
0	8.49	8.49	8.49	8.49	8.48	8.47	8.41	8.18	7.50	5.80
0.10	9.35	9.35	9.35	9.35	9.35	9.33	9.27	9.03	8.30	6.45
0.20	11.05	11.05	11.05	11.04	11.04	11.02	10.96	10.72	9.96	7.83
0.30	13.54	13.54	13.54	13.54	13.53	13.51	13.45	13.24	12.46	9.99
0.40	17.21	17.21	17.20	17.20	17.20	17.17	17.11	16.93	16.18	13.31
0.50	22.85	22.85	22.84	22.84	22.82	22.80	22.74	22.59	21.92	18.68
0.60	32.28	32.28	32.27	32.26	32.25	32.21	32.16	32.03	31.49	28.03
0.70	50.25	50.25	50.25	50.24	50.22	50.18	50.11	49.98	49.58	46.34
0.80	93.50	93.50	93.50	93.49	93.47	93.42	93.32	93.15	92.76	90.44

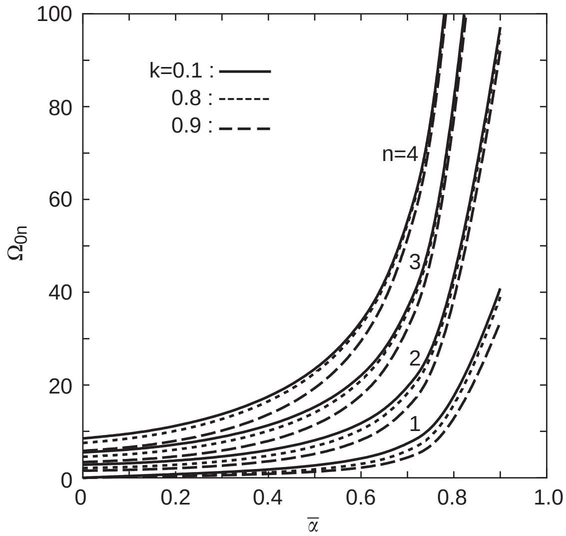


Fig. 6. Natural frequencies Ω_{0n} for an annular liquid system for various diameter ratio k as a function of $\bar{\alpha}$.

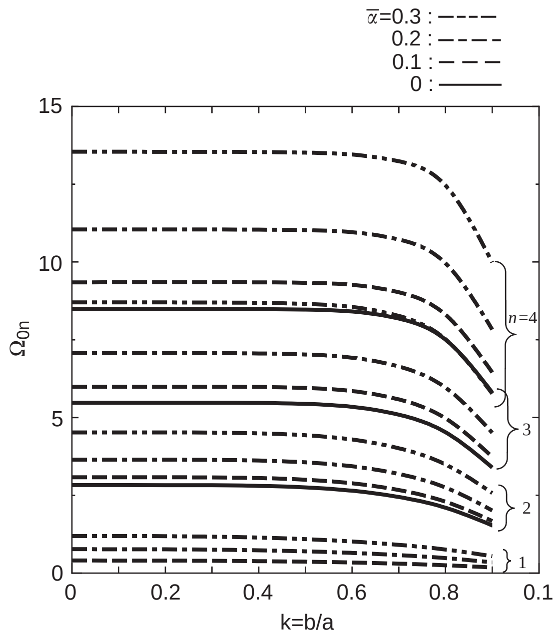


Fig. 7. Natural frequencies Ω_{0n} for an annular liquid system for various $\bar{\alpha}$ values as a function of k.

non-dimensional forcing frequency $\bar{\Omega} = \Omega/\sqrt{\sigma/\rho a^3}$. Fig. 9 shows the response $\bar{\zeta} = |\zeta^*/Z_0|$ for $\bar{\alpha} = 0.2$, while Fig. 10 exhibits the liquid surface displacement for $\bar{\alpha} = 0.5$, i.e., a drop of which half its surface is covered by a semi-spherical cap. We see that at the resonances the displacement of the liquid becomes as expected for frictionless liquid infinite. The geometrical forms of the

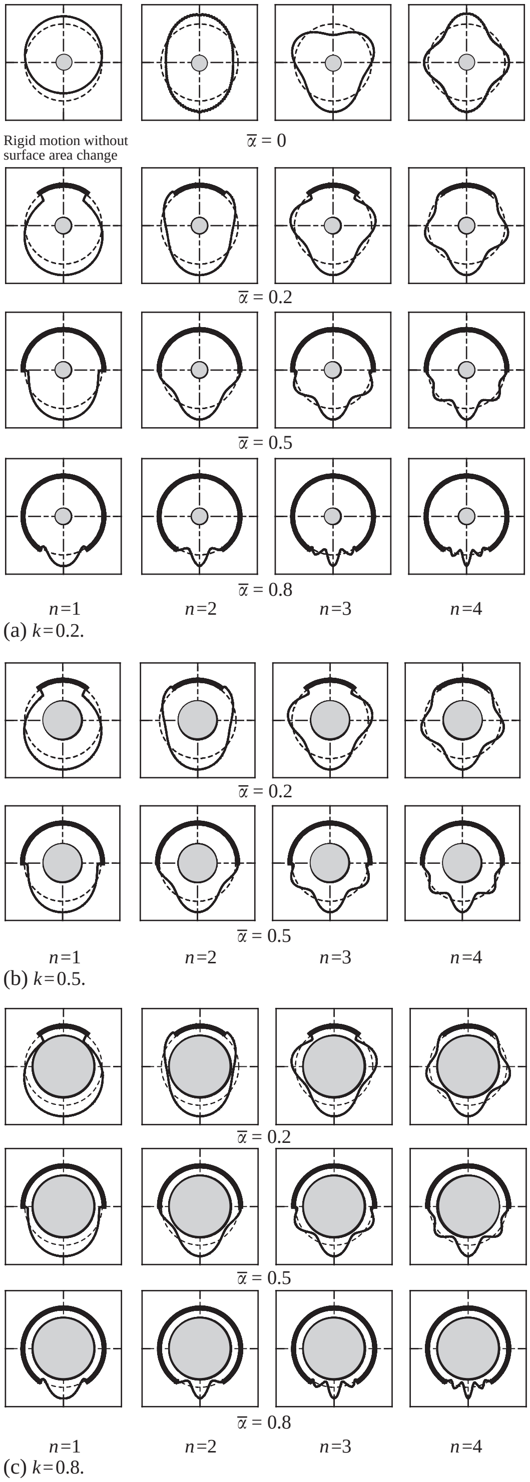


Fig. 8. Mode shapes of captured annular liquid system; (a) $k = 0.2$; (b) $k = 0.5$; (c) $k = 0.8$.

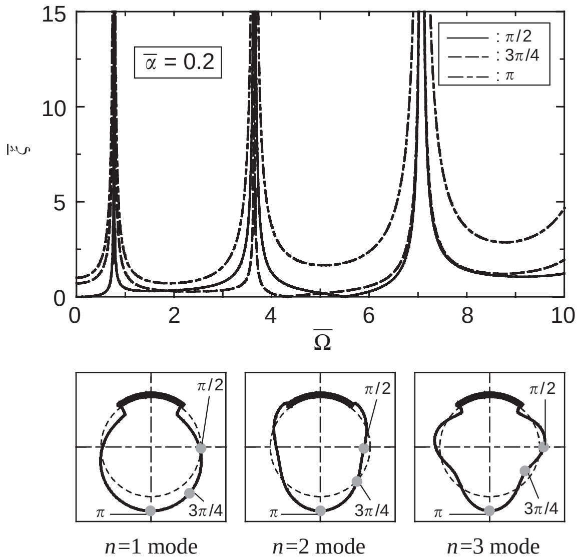


Fig. 9. Response of the free surface of a simple captured liquid system under translational excitation in z direction for $\bar{\alpha} = 0.2$.

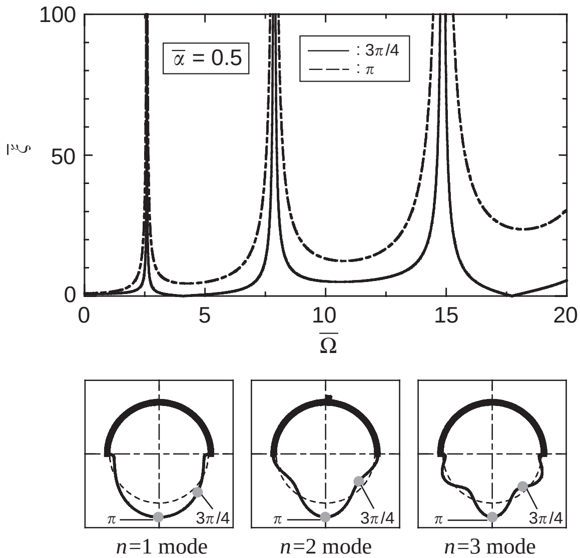
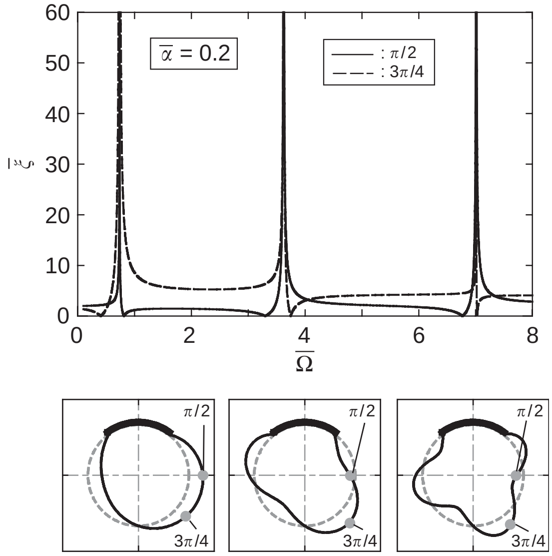


Fig. 10. Response of the free surface of a simple captured liquid system under translational excitation in z direction for $\bar{\alpha} = 0.5$.

captured liquid drop are also presented in Figs. 9 and 10 for reference. These are natural vibration modes obtained from free vibration analysis.

For translational harmonic excitation in x direction, the free surface response is presented in Fig. 11 at the location $\vartheta = \pi/2$ and $3\pi/4$ for a cap angle $\bar{\alpha} = 0.2$. The non-dimensional circular



n=1 mode *n*=2 mode *n*=3 mode

Fig. 11. Response of the free liquid surface for translational excitation in x direction for $\bar{\alpha} = 0.2$.

Table 4
Non-dimensional circular frequency $\Omega_{1n} = \omega_{1n}/\sqrt{\sigma/\rho a^3}$ for a simple captured liquid sphere: $m = 1$, $N = N_1 + N_2 + 1 = 100$

$\bar{\alpha}$	First	Second	Third	Fourth
0	0	2.828	5.477	8.485
0.1	0.374	3.059	5.944	9.278
0.2	0.736	3.623	7.011	10.970
0.3	1.150	4.499	8.619	13.458
0.4	1.692	5.810	10.998	17.105
0.5	2.486	7.846	14.674	22.717
0.6	3.793	11.271	20.834	32.102
0.7	6.284	17.828	32.603	50.011
0.8	12.308	33.654	60.962	93.138
0.9	37.015	98.395	177.032	269.714

frequencies $\bar{\Omega}_{1n} = \omega_{1n}/\sqrt{\sigma/\rho a^3}$ are presented for different cap angle $\bar{\alpha}$ in Table 4. For an angle $\bar{\alpha} = 0.4$, the response and modal behavior of the free liquid surface are shown in Fig. 12.

Finally, the results of the free surface response for a cap angle $\bar{\alpha} = 0.5$ are presented in Fig. 13.

5. Conclusion

(a) A spherical liquid globule captured partly in a rigid cap of equal radius exhibits with increasing cap angle $\bar{\alpha}$ increasing natural frequencies. The higher modes n show the same behavior.

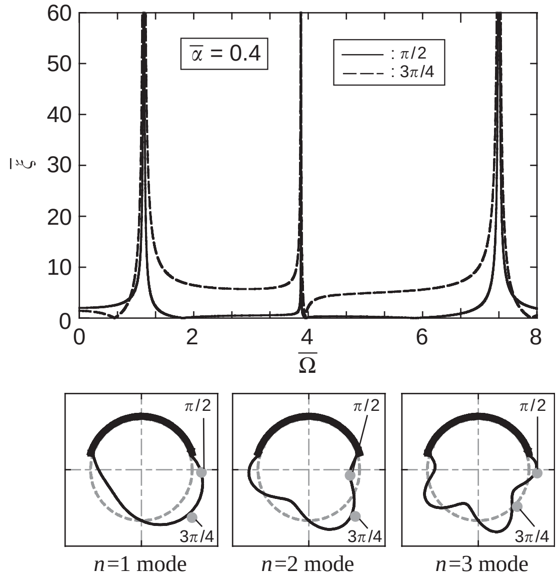


Fig. 12. Response of the free liquid surface for translational excitation in x direction for $\bar{\alpha} = 0.4$.

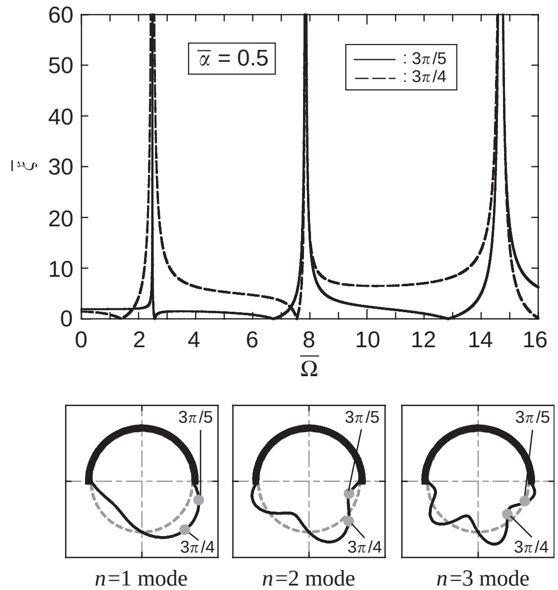


Fig. 13. Response of the free liquid surface for translational excitation in x direction for $\bar{\alpha} = 0.5$.

(b) A spherical liquid system captured partly in both pole areas by rigid caps show even larger increased natural frequencies, for increasing cap angle $\bar{\alpha}$ as well as for increased mode number n.

(c) A spherical annular liquid system around a rigid sphere exhibits with the increase of the cap area $\bar{\alpha}$ an increase of its natural liquid frequencies. With the increase of the diameter ratio $k = b/a$ (i.e., thinner layer), we find a decrease of the natural frequencies.

Acknowledgements

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Appendix A. Nomenclature

A_{mn}, B_{nm}	coefficients of Eq. (3)
A_n, B_n	coefficients of Eqs. (46) and (56)
a	radius of drop
b	radius of core or center sphere
C_{mn}	coefficient of Eq. (26)
i	imaginary unit
k	radius ratio for annular drop, $\equiv b/a$
r, ϑ, φ	co-ordinate system
m	angular mode number
t	time
$P_n^m(\cos\vartheta)$	associate Legendre function
u, υ, w	velocity components of drop
V_0	drop volume
x, y, z	co-ordinate system
X_0	amplitude of excitation force in x direction
Z_0	amplitude of excitation force in z direction
α	covering area of drop: $0<\alpha<\pi$ ($\bar{\alpha}=\alpha/\pi$)
ζ	free surface displacement ($\bar{\zeta}=\lvert\zeta^*/X_0\rvert$ or $\lvert\zeta^*/Z_0\rvert$)
ζ^*	amplitude of free surface displacement: $\zeta=\zeta^* e^{i\Omega t}$
ρ	density of drop
σ	surface tension
ϕ, ψ	velocity potentials
Ω	forcing circular frequency ($\bar{\Omega}=\Omega/\sqrt{\sigma/\rho a^3}$)
ω_{mn}	natural circular frequency ($\Omega_{mn}=\omega_{mn}/\sqrt{\sigma/\rho a^3}$)

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